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Yau's resolution of Calabi's conjecture

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Abstract

This thesis deals with Yau's theorem (see [Yau77], [Yau78]) about the solution of Calabi's conjecture (see [Cal54]). We state both of them below:

Theorem (Calabi's Conjecture, 1954). Let M^m be a compact and connected Kähler manifold with Kähler form ω. Then for every closed real $(1,1)$-form $\rho \in 2\pi[C_1(M)]$ there exists a unique Kähler form $\Omega \in [\omega]$ such that $\mathrm{Ric}(\Omega) = \rho$.

Theorem (Yau, 1978). Let that M^m be a compact Kähler manifold with Kähler form ω. Let $F \in C^\infty(M)$ and $C > 0$ such that $C \int_M e^F \, dV_g = \mathrm{vol}(M)$. Then there is $\varphi \in C^\infty(M)$ such that

$$\omega + \sqrt{-1}\, \partial\overline{\partial}\varphi$$

defines a Kähler form on M and

$$(\omega + \sqrt{-1}\, \partial\overline{\partial}\varphi)^m = e^{CF}\omega^m$$

The goal of this work is to explore the relation between these results and to discuss each step of Yau's proof in detail. In order to achieve this objective, a brief (yet as much self-contained as possible) introduction to the theory of Kähler manifolds will be made, starting from the setting of complex geometry.

Contents

Introduction

The concept of Kähler manifolds emerged from the intersection of complex geometry, Riemannian geometry, and symplectic geometry. The origins of Kähler geometry lie in the study of complex manifolds, which are smooth manifolds equipped with a complex structure. In the early 20^{th} century, mathematicians such as Élie Cartan and Henri Poincaré made significant contributions to the understanding of complex structures and their associated metrics.

The formal notion of what we now call a Kähler manifold was introduced by the German mathematician Erich Kähler in 1933 in his seminal paper [Käh32]. Kähler's work focused on the special properties of complex manifolds equipped with a Hermitian metric whose fundamental $(1,1)$-form is closed. This structure led to significant simplifications in the study of both the metric and the complex structure, laying the groundwork for what would later become a central concept in differential geometry.

In the decades following Kähler's initial work, mathematicians further explored the properties of Kähler manifolds, discovering a wealth of examples and applications. In particular, Kähler geometry became deeply connected with questions about the curvature of these manifolds and their cohomological properties.

One of the most influential developments in this field was the conjecture proposed by Eugenio Calabi in the 1950s. Calabi theorized the existence of a Kähler metric with prescribed Ricci curvature on a compact Kähler manifold. Specifically, it suggested that for any compact Kähler manifold with Kähler form Ω, and for any closed real $(1,1)$-form in the cohomology class of its Ricci form, there exists a unique Kähler metric with a Kähler form in the cohomology class of Ω that has the specified form as its Ricci form.

This conjecture represented a pivotal moment in the study of Kähler manifolds, as it connected the geometric properties of the manifold to its topological data, given by its cohomology class. However, proving the conjecture posed a formidable challenge and remained an open problem for more than two decades.

A major step toward resolving the Calabi Conjecture was made by Thierry Aubin in the 1970s: through new analytical techniques, he was able to prove the existence of Kähler-Einstein metrics with negative first Chern class, hence clearing a specific case of the conjecture.

The resolution of the Calabi Conjecture by Shing-Tung Yau in 1978 marked a turning point in Kähler geometry. Yau's groundbreaking proof, using methods from nonlinear partial differential equations and geometric analysis, established the existence and uniqueness of Kähler metrics satisfying Calabi's conditions. This result not only resolved the conjecture but also had profound implications for the methods applied to find the solution.

The work of this thesis is organized as follows. The first chapter is devoted to establishing the general framework of complex geometry. We introduce the key objects that will be central to this thesis, such as complex manifolds, holomorphic functions, and the complexified structures of a manifold.

In the second chapter, we develop the concept of Kähler manifolds, examining the interplay between the structures introduced earlier. Special focus is given to the intrinsic geometry of Kähler manifolds, culminating in introduction of the first Chern class.

Finally, the third chapter is dedicated to the study of Yau's resolution of the Calabi Conjecture. The proof will be broken down into its essential steps to facilitate a clear understanding of the arguments and techniques involved. Particular emphasis will be placed on the analytic methods and the geometric insights that form the core of Yau's approach.

Chapter 1

Complex Manifolds

1.1 Basics of Complex Geometry

In this section, we lay the foundation for our work, presenting some notions from complex analysis and introducing the framework of complex manifolds.

1.1.1 Euclidean setting

Definition 1 (Complex structure on $\mathbb{R}^{2k}$). For any $k \geq 1$, we can identify $\mathbb{C}^k \cong \mathbb{R}^{2k}$ via the $\mathbb{R}$-linear isomorphism

$$(z_1, \ldots, z_k) \longleftrightarrow (x_1, \ldots, x_k, y_1, \ldots, y_k)$$

where $z_i = x_i + \sqrt{-1}\, y_i$. Under this identification, the multiplication by $\sqrt{-1}$ on $\mathbb{C}^k$ corresponds to the endomorphism j_k of $\mathbb{R}^{2k}$ represented in the standard basis by the matrix

$$\begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}$$

where I_k is the identity matrix of order k. The map j_k is called the *canonical complex structure* on $\mathbb{R}^{2k}$, and it satisfies $j_k^2 = -Id_{\mathbb{R}^{2k}}$.

Under the identification used in Definition 1, a map $f\colon U \overset{open}{\subseteq} \mathbb{C}^n \to \mathbb{C}^m$ can be uniquely paired with a map $F\colon U \subseteq \mathbb{R}^{2n} \to \mathbb{R}^{2m}$, where if $f = (f_1, \ldots, f_m)$ and $F = (F_1, \ldots, F_{2m})$ then for all i

$$f_i = F_i + \sqrt{-1} F_{m+i}$$

We say that f *is of class* C^k *(resp. smooth) on* U *in the real sense* if F is of class C^k (resp. smooth) on U. When $m = 1$, we will denote $f \in C^k(U)$, $k \in \mathbb{N} \cup \{\infty\}$.

Definition 2 (Holomorphic map on the complex space). Let $U \overset{open}{\subseteq} \mathbb{C}^n$. A function $f \in C^1(U)$ is called *holomorphic* at $p \in U$ if

$$\frac{\partial f}{\partial \bar{z}^i}(p) = 0 \quad \forall\, i = 1,\, \ldots,\, n$$

where the $\frac{\partial}{\partial \bar{z}^i}$'s are the *anti-holomorphic Wirtinger operators*

$$\frac{\partial}{\partial \bar{z}^i} := \frac{1}{2}\left(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}\right)$$

f is called *holomorphic* on U if it is holomorphic at q for all $q \in U$. A map $g = (g_1,\, \ldots,\, g_m)\colon U \to \mathbb{C}^m$ of class C^1 on U is called *holomorphic* if its components $g_i \in C^1(U)$ are holomorphic.

For $f = (f_1, \ldots, f_m)$ we compute

$$\frac{\partial f_i}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial F_i}{\partial x^j} - \frac{\partial F_{m+i}}{\partial y^j} + \sqrt{-1}(\frac{\partial F_{m+i}}{\partial x^j} + \frac{\partial F_i}{\partial y^j})\right)$$

Consequently, the condition $\frac{\partial f_i}{\partial \bar{z}^j} = 0$ reads

$$\frac{\partial F_i}{\partial x^j} = \frac{\partial F_{m+i}}{\partial y^j}\ , \qquad \frac{\partial F_{m+i}}{\partial x^j} = -\frac{\partial F_i}{\partial y^j}$$

These are called the *Cauchy-Riemann equations*. In the next lemma, we discuss how these equations connect the canonical complex structure and the holomorphic condition.

Lemma 1. Let $f : U \overset{open}{\subseteq} \mathbb{C}^n \to \mathbb{C}^m$ be of class C^1 on U. Then f is holomorphic on U if and only if $\forall\, p \in U$ the differential of F at p satisfies

$$j_m \circ (F)_{*p} = (F)_{*p} \circ j_n$$

Proof. Let $J(F)(p)$ be the Jacobian matrix of F at $p \in U$, i.e.

$$J(F)(p) = \left(\begin{array}{c|c} \frac{\partial F_i}{\partial x^j}(p) & \frac{\partial F_i}{\partial y^j}(p) \\ \hline \frac{\partial F_{m+i}}{\partial x^j}(p) & \frac{\partial F_{m+i}}{\partial y^j}(p) \end{array} \right)_{i=1,\dots,m,\ j=1,\dots,n}$$

By Definition 2, f is holomorphic at p if

$$\begin{aligned}
&\frac{\partial f_i}{\partial \bar{z}^j}(p) = 0 \quad \forall\, i,\ j \iff \\
&\qquad \iff \frac{\partial F_i}{\partial x^j}(p) = \frac{\partial F_{m+i}}{\partial y^j}(p)\,, \quad \frac{\partial F_{m+i}}{\partial x^j}(p) = -\frac{\partial F_i}{\partial y^j}(p) \quad \forall\, i,\ j \\
&\qquad \iff \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} J(F)(p) = J(F)(p) \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \\
&\qquad \iff j_m \circ (F)_{*p} = (F)_{*p} \circ j_n
\end{aligned}$$

□

Remark 1. It is straightforward from Lemma 1 that the sum and composition of holomorphic functions is holomorphic (for the latter, use the real chain rule).

Introduce the *holomorphic Wirtinger operators*

$$\tfrac{\partial}{\partial z^i} := \tfrac{1}{2}\left(\tfrac{\partial}{\partial x^i} - \sqrt{-1}\tfrac{\partial}{\partial y^i}\right)$$

which act on $f = (f_1, \dots, f_m)$ as

$$\tfrac{\partial f_i}{\partial z^j} = \tfrac{1}{2}\left(\tfrac{\partial F_i}{\partial x^j} + \tfrac{\partial F_{m+i}}{\partial y^j} + \sqrt{-1}(\tfrac{\partial F_{m+i}}{\partial x^j} - \tfrac{\partial F_i}{\partial y^j})\right)$$

We define the *complex Jacobian* of f at p to be the matrix

$$J_{\mathbb{C}}(f)(p) := \left(\tfrac{\partial f_i}{\partial z^j}(p)\right)$$

Here is a crucial relation between a holomorphic map and its real counterpart.

Lemma 2. Let $f\colon U \overset{open}{\subseteq} \mathbb{C}^n \to \mathbb{C}^n$ be holomorphic. Then $\forall\, p \in U$

$$\det(J(F)(p)) = |\det(J_{\mathbb{C}}(f)(p))|^2_{\mathbb{C}}$$

where $|\cdot|_{\mathbb{C}}$ denotes the complex modulus.

Proof. Adding $\sqrt{-1}$-times the bottom blocks to the top and using the Cauchy-Riemann equations, we get

$$\det(J(F)) = \det\left(\begin{array}{c|c} \frac{\partial F_i}{\partial x^j} + \sqrt{-1}\frac{\partial F_{m+i}}{\partial x^j} & -\frac{\partial F_{m+i}}{\partial x^j} + \sqrt{-1}\frac{\partial F_i}{\partial x^j} \\ \hline \frac{\partial F_{m+i}}{\partial x^j} & \frac{\partial F_i}{\partial x^j} \end{array}\right)$$

Then adding $-\sqrt{-1}$-times the left blocks to the right yields

$$\det(J(F)) = \det\left(\begin{array}{c|c} \frac{\partial F_i}{\partial x^j} + \sqrt{-1}\frac{\partial F_{m+i}}{\partial x^j} & 0 \\ \hline * & \frac{\partial F_i}{\partial x^j} - \sqrt{-1}\frac{\partial F_{m+i}}{\partial x^j} \end{array}\right)$$

Now, since f is holomorphic, by the Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial f_i}{\partial z^j} &= \frac{1}{2}\left(\frac{\partial F_i}{\partial x^j} + \frac{\partial F_i}{\partial x^j} + \sqrt{-1}\left(\frac{\partial F_{m+i}}{\partial x^j} + \frac{\partial F_{m+i}}{\partial x^j}\right)\right) = \\ &= \frac{\partial F_i}{\partial x^j} + \sqrt{-1}\frac{\partial F_{m+i}}{\partial x^j} \end{aligned}$$

Consequently, we compute

$$\begin{aligned} \det(J(F)) = \det\left(\begin{array}{c|c} J_{\mathbb{C}}(f) & 0 \\ \hline * & \overline{J_{\mathbb{C}}(f)} \end{array}\right) &= \\ = \det\left(J_{\mathbb{C}}(f)\right)\det\left(\overline{J_{\mathbb{C}}(f)}\right) &= \\ = \det\left(J_{\mathbb{C}}(f)\right)\overline{\det\left(J_{\mathbb{C}}(f)\right)} &= |\det(J_{\mathbb{C}}(f))|^2_{\mathbb{C}} \end{aligned}$$

□

Using Lemma 2, many results valid in the real setting can be proven in the complex one. In order to address them, we need to study the main properties of the Wirtinger operators.

Lemma 3. Let $f,\ g \in C^1(U)$. Then

- $\overline{\frac{\partial f}{\partial z^i}} = \frac{\partial \overline{f}}{\partial \overline{z}^i}\ ,\quad \overline{\frac{\partial f}{\partial \overline{z}^i}} = \frac{\partial \overline{f}}{\partial z^i}$
- $\frac{\partial \alpha f+\beta g}{\partial z^i} = \alpha\frac{\partial f}{\partial z^i} + \beta\frac{\partial g}{\partial z^i}\ ,\quad \frac{\partial \alpha f+\beta g}{\partial \overline{z}^i} = \alpha\frac{\partial f}{\partial \overline{z}^i} + \beta\frac{\partial g}{\partial \overline{z}^i}\quad \forall\, \alpha,\ \beta \in \mathbb{C}$
- $\frac{\partial fg}{\partial z^i} = \frac{\partial f}{\partial z^i}g + f\frac{\partial g}{\partial z^i}\ ,\quad \frac{\partial fg}{\partial \overline{z}^i} = \frac{\partial f}{\partial \overline{z}^i}g + f\frac{\partial g}{\partial \overline{z}^i}$

If $f,\ g$ are of class C^1 on open subsets of $\mathbb{C}^n$ and $g \circ f$ exists, then

- $\frac{\partial (g\circ f)_i}{\partial z^j} = \sum\limits_k \big(\frac{\partial g_i}{\partial z^k} \circ f\big)\frac{\partial f_k}{\partial z^j} + \big(\frac{\partial g_i}{\partial \overline{z}^k} \circ f\big)\frac{\partial \overline{f_k}}{\partial z^j}$
  $\frac{\partial (g\circ f)_i}{\partial \overline{z}^j} = \sum\limits_k \big(\frac{\partial g_i}{\partial z^k} \circ f\big)\frac{\partial f_k}{\partial \overline{z}^j} + \big(\frac{\partial g_i}{\partial \overline{z}^k} \circ f\big)\frac{\partial \overline{f_k}}{\partial \overline{z}^j}$

Proof. Straightforward computations from the properties of $\frac{\partial}{\partial x^i},\ \frac{\partial}{\partial y^i}$. ☐

As anticipated, here is a complex version of the Inverse Function Theorem.

Theorem 1 (Holomorphic IFT). Let $U,\ V \overset{open}{\subseteq} \mathbb{C}^n$ with $0 \in U$. Suppose $f\colon U \to V$ is holomorphic with $\det(J_{\mathbb{C}}(f)(0)) \neq 0$. Then f is bijective in (a possibly smaller) neighborhood of 0 and f^{-1} is holomorphic.

Proof. By Lemma 2: $\det(J(F)(p)) \neq 0$, so we can apply the Inverse Function Theorem to obtain a smooth inverse of F near 0, that is, f has a smooth inverse near 0. For all $i,\ j$, by Lemma 3 we compute near 0

$$\begin{aligned} 0 = \frac{\partial z_i}{\partial \overline{z}^j} &= \frac{\partial (f^{-1}\circ f)_i}{\partial \overline{z}^j} = \\ &= \sum_k \big(\frac{\partial (f^{-1})_i}{\partial z^k} \circ f\big)\frac{\partial f_k}{\partial \overline{z}^j} + \big(\frac{\partial (f^{-1})_i}{\partial \overline{z}^k} \circ f\big)\frac{\partial \overline{f_k}}{\partial \overline{z}^j} = \\ &= \sum_k \big(\frac{\partial (f^{-1})_i}{\partial \overline{z}^k} \circ f\big)\overline{\frac{\partial f_k}{\partial z^j}} \end{aligned}$$

where we used that f is holomorphic. Hence, near 0 the matrix product

$$\left(\frac{\partial (f^{-1})_i}{\partial \overline{z}^j} \circ f\right)\overline{J_{\mathbb{C}}(f)}$$

is zero, and since $\overline{J_{\mathbb{C}}(f)}$ is non-degenerate near 0 by continuity of the determinant, we conclude that

$$\tfrac{\partial(f^{-1})_i}{\partial \bar{z}^j} = 0 \quad \forall\, i,\ j$$

so f^{-1} is holomorphic. □

1.1.2 Manifolds setting

In the following, we identify $\mathbb{C}^k \cong \mathbb{R}^{2k}$ via the $\mathbb{R}$-linear isomorphism

$$(z_1, \ldots, z_k) \longleftrightarrow (x_1, y_1, \ldots, x_k, y_k)$$

where $z_i = x_i + \sqrt{-1}\, y_i$.

Definition 3 (Complex manifold). Let M be a smooth manifold. Assume there exists an atlas $\mathcal{U} = \{(U_\alpha,\ \phi_\alpha)\}_{\alpha \in A}$ of M, where

$$\phi_\alpha \colon U_\alpha \to \phi_\alpha(U_\alpha) \overset{open}{\subseteq} \mathbb{C}^n$$

and $\forall\, \alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$: $(U_\alpha,\ \phi_\alpha),\ (U_\beta,\ \phi_\beta)$ are *holomorphically compatible*, i.e. the transition maps

$$\phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

are holomorphic. Then, M is called a *complex manifold of complex dimension* n.

Let us establish some terminology related to Definition 3. The atlas $\mathcal{U}$ and the charts $(U_\alpha,\ \phi_\alpha)$ are called *holomorphic*. The components z^i of ϕ over $\mathbb{C}^n$ are called *holomorphic coordinates*.

As in the smooth case, a holomorphic atlas uniquely determines another holomorphic atlas that contains it and all its holomorphically compatible charts. This kind of atlas is called a *holomorphic structure* on the manifold.

Definition 4 (Holomorphic map on a complex manifold). Fix a holomorphic atlas $\mathcal{U} = \{(U_\alpha,\ \phi_\alpha)\}_{\alpha\in A}$ on a complex manifold M. A map $f\colon M \to \mathbb{C}^m$ is called *holomorphic* if $\forall\, \alpha \in A$

$$f \circ \phi_\alpha^{-1}\colon \phi_\alpha(U_\alpha) \to \mathbb{C}^m \text{ is holomorphic}$$

If N is another complex manifold and $\mathcal{V} = \{(V_\beta,\ \psi_\beta)\}_{\beta\in B}$ is a holomorphic atlas on N, a map $g\colon M \to N$ is called *holomorphic* if $\forall\, \alpha \in A,\ \forall\, \beta \in B$

$$\psi_\beta \circ g \circ \phi_\alpha^{-1}$$

is holomorphic whenever it is defined (i.e. when $g(U_\alpha) \subseteq V_\beta$) .

The following observations stem from Definition 4 in a manner analogous to the case of smooth maps on a smooth manifold.

Remark 2. i) The concepts introduced in Definition 4 are inherently local and independent of the choice of holomorphic atlas within a fixed holomorphic structure.

ii) By definition, any coordinate system derived from a holomorphic chart is, in particular, a holomorphic map on the manifold.

iii) The composition of holomorphic maps is a holomorphic map.

iv) It is always possible to choose a holomorphic chart ϕ around a point p that is centered at that point, i.e. $\phi(p) = 0$ (just apply a translation).

1.2 Complexification

Useful information about a smooth manifold can be obtained by pairing it with linear-algebraic structures, such as the tangent space, the tangent and cotangent bundles, etc.
The same is true for complex manifolds. In this section, we extend to the complex setting, in a natural way, the linear-algebraic structures associated to a smooth manifold.

1.2.1 Linear setting

We begin in the easier context of linear algebra, exploring the concept of "linear" complexification. We will always deal with vector spaces of finite dimension.

> **Lemma 4 (Complexification of a vector space).** Let V be a real vector space. The following real vector spaces equipped with their respective products on $\mathbb{C}$:
>
> - $V \oplus V\ ,\ (x + \sqrt{-1}\, y)(v,\ w) := (xv - yw,\ yv + xw)$
> - $V \otimes_{\mathbb{R}} \mathbb{C}\ ,\ \lambda(v \otimes z) := v \otimes \lambda z$
>
> are complex vector spaces. Moreover, there exists a unique $\mathbb{C}$-isomorphism $\phi_V : V \oplus V \to V \otimes_{\mathbb{R}} \mathbb{C}$ such that the below diagram commutes:
>
> $$\begin{array}{ccc} & V & \\ \swarrow & & \searrow \\ V \oplus V & \xrightarrow{\ \phi_V\ } & V \otimes_{\mathbb{R}} \mathbb{C} \end{array}$$
>
> where the arrows from V are the *standard embeddings*
>
> $$V \to V \oplus V :\ v \mapsto (v,\ 0)\ , \qquad V \to V \otimes_{\mathbb{R}} \mathbb{C} :\ v \mapsto v \otimes 1$$
>
> Up to isomorphism, the *complexification of V* is defined as the above complex vector space and it is referred to as $V_{\mathbb{C}}$.

Proof. It is a routine check from the definition that the above-defined products give a complex vector space structure to the underlying space.
Furthermore, assuming ϕ_V exists, we compute by $\mathbb{C}$-linearity and commutativity of the diagram

$$\begin{aligned}\phi_V(v,w) &= \phi_V((v,0)+\sqrt{-1}(w,0)) = \\ &= \phi_V(v,0)+\sqrt{-1}\phi_V(w,0) = \\ &= v\otimes 1+\sqrt{-1}(w\otimes 1) = \\ &= v\otimes 1 + w\otimes\sqrt{-1}\end{aligned}$$

which means ϕ_V is unique. Now define ϕ_V by

$$\phi_V(v,w) := v\otimes 1 + w\otimes\sqrt{-1}$$

Clearly, ϕ_V makes the diagram commute, and it is $\mathbb{R}$-linear by the identities of the tensor product of vectors. To see that it is actually $\mathbb{C}$-linear, we compute

$$\begin{aligned}\phi_V(\sqrt{-1}(v,w)) &= \phi_V(-w,v) = \\ &= -w\otimes 1 + v\otimes\sqrt{-1} = \\ &= \sqrt{-1}(w\otimes\sqrt{-1}+v\otimes 1) = \\ &= \sqrt{-1}\phi_V(v,w)\end{aligned}$$

Finally, we construct the inverse of ϕ_V. Consider the $\mathbb{R}$-bilinear map

$$V\times\mathbb{C}\to V\oplus V:\ (v,z)\mapsto z(v,0)$$

By the universal property of the tensor product there is a unique $\mathbb{R}$-linear map $\varphi_V: V\otimes_{\mathbb{R}}\mathbb{C}\to V\oplus V$ such that $\forall\,(v,z)\in V\times\mathbb{C}$

$$\varphi_V(v\otimes z) = z(v,0)$$

In particular φ_V is $\mathbb{C}$-linear since

$$\begin{aligned}\varphi_V(\sqrt{-1}(v\otimes z)) &= \varphi_V(v\otimes\sqrt{-1}z) = \\ &= \sqrt{-1}z(v,0) = \sqrt{-1}\varphi_V(v\otimes z)\end{aligned}$$

and it satisfies

$$\begin{aligned}\cdot\ (\varphi_V \circ \phi_V)(v,w) &= \varphi_V(v\otimes 1 + w\otimes\sqrt{-1}) = \\ &= (v,0)+\sqrt{-1}(w,0) = (v,w)\\ \cdot\ (\phi_V\circ\varphi_V)(v\otimes z) &= \phi_V(z(v,0)) = \\ &= z(v\otimes 1) = v\otimes z\end{aligned}$$

which implies $\phi_V^{-1} = \varphi_V$. □

Remark 3 (Complex decomposition). We observed in Lemma 4 that $\forall\, v\in V$

$$\sqrt{-1}\,(v,\ 0) = (0,\ v)$$

In particular, any $z\in V_{\mathbb{C}}$ admits a (unique) decomposition as $v+\sqrt{-1}\,w$. We call $\mathrm{Re}(z) := v,\ \mathrm{Im}(z) := w$ the *real* and *imaginary part* of z, respectively.

Lemma 5 (Basis for the complexification). Let V be a real vector space, and $\{e_j\}_j$ be a $\mathbb{R}$-basis for V. Then $\{e_j\otimes 1\}_j \equiv \{(e_j,\ 0)\}_j$ is a $\mathbb{C}$-basis for $V_{\mathbb{C}}$. In particular $\dim_{\mathbb{C}}(V_{\mathbb{C}}) = \dim_{\mathbb{R}}(V)$.

Proof. Pick $(v,w)\in V_{\mathbb{C}}$. Denote by $v_i, w_i \in \mathbb{R}$ the components of v,w with respect to $\{e_j\}_j$. By Remark 3

$$\begin{aligned}(v,w) &= (v,0)+\sqrt{-1}(w,0) = \\ &= \sum_i v_i(e_i,0) + \sqrt{-1}\sum_i w_i(e_i,0) = \sum_i (v_i+\sqrt{-1}w_i)(e_i,0)\end{aligned}$$

so $(v,w)\in \mathrm{span}_{\mathbb{C}}(\{e_j\}_j)$. Moreover, if $\sum_i z_i(e_i,0) = (0,0)$ for some $z_i\in\mathbb{C}$, the already done computation yields

$$(0,0) = (\sum_i \mathrm{Re}(z_i)e_i,\ \sum_i \mathrm{Im}(z_i)e_i)$$

thus by linear independence of $\{e_j\}_j$: $\mathrm{Re}(z_i) = \mathrm{Im}(z_i) = 0$ for all i, that is, $\{(e_j,0)\}_j$ is linearly independent. □

Lemma 6 (Complexification of a linear map). Let $f\colon V \to W$ be a $\mathbb{R}$-linear homomorphism between real vector spaces. There is a unique $\mathbb{C}$-linear map $g\colon V \oplus V \to W \oplus W$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ \downarrow & & \downarrow \\ V \oplus V & \xrightarrow{\ g\ } & W \oplus W \end{array}$$

where the vertical maps are the standard embeddings. Moreover, the following diagram also commutes:

$$\begin{array}{ccc} V \oplus V & \xrightarrow{\ g\ } & W \oplus W \\ {\scriptstyle \phi_V}\downarrow & & \downarrow{\scriptstyle \phi_W} \\ V \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{f \otimes Id_{\mathbb{C}}} & W \otimes_{\mathbb{R}} \mathbb{C} \end{array}$$

Up to natural isomorphisms, the *complexification of* f is defined as the above map and it is referred to as $f_{\mathbb{C}}\colon V_{\mathbb{C}} \to W_{\mathbb{C}}$.

Proof. First, assume that such g exists. By Remark 3 and commutativity of the first diagram, $\forall\, v, w \in V$

$$\begin{aligned} g(v,w) &= g(v,0) + \sqrt{-1}g(w,0) = \\ &= (f(v,0)) + \sqrt{-1}(f(w,0)) = (f(v), f(w)) \end{aligned}$$

which means g is unique. Now define g by

$$g(v,w) = (f(v), f(w))$$

Since f is $\mathbb{R}$-linear, the same holds for g. Moreover, g clearly makes the first diagram commutative. In addition, g is $\mathbb{C}$-linear: indeed

$$\begin{aligned} g(\sqrt{-1}(v,w)) &= g(-w,v) = (f(-w), f(v)) = \\ &= (-f(w), f(v)) = \sqrt{-1}(f(v), f(w)) = \sqrt{-1}g(v,w) \end{aligned}$$

and finally, we compute $\forall\, v, w \in V$

$$\begin{aligned}(f \otimes Id_{\mathbb{C}} \circ \phi_V)(v,w) &= f \otimes Id_{\mathbb{C}}(v \otimes 1 + w \otimes \sqrt{-1}) = \\ &= f(v) \otimes 1 + f(w) \otimes \sqrt{-1} = \\ &= \phi_W(f(v), f(w)) = (\phi_W \circ g)(v,w)\end{aligned}$$

that is, g makes the second diagram commute. □

The uniqueness statement in Lemma 6 can be used to easily prove some properties of the complexification on linear maps.

Corollary 1 (Functorial properties of the complexification). Let $U,\ V,\ W$ be real vector spaces.

1. $(Id_V)_{\mathbb{C}} = Id_{V_{\mathbb{C}}}$.
2. If $f\colon U \to V,\ g\colon V \to W$ are $\mathbb{R}$-linear maps, then $(g \circ f)_{\mathbb{C}} = g_{\mathbb{C}} \circ f_{\mathbb{C}}$.

In particular, if f is a $\mathbb{R}$-isomorphism then $f_{\mathbb{C}}$ is a $\mathbb{C}$-isomorphism and $(f_{\mathbb{C}})^{-1} = (f^{-1})_{\mathbb{C}}$.

Proof. 1. Let $\iota\colon V \to V_{\mathbb{C}}$ be the standard embedding. Since $Id_{V_{\mathbb{C}}}$ is a $\mathbb{C}$-linear map such that

$$Id_{V_{\mathbb{C}}} \circ \iota = \iota \circ Id_V$$

by Lemma 6 (uniqueness): $(Id_V)_{\mathbb{C}} = Id_{V_{\mathbb{C}}}$.

2. Denote by $\iota_U,\ \iota_V,\ \iota_W$ the standard embeddings of $U,\ V,\ W$ into their complexification. Since $g_{\mathbb{C}} \circ f_{\mathbb{C}}$ is a $\mathbb{C}$-linear map such that

$$(g_{\mathbb{C}} \circ f_{\mathbb{C}}) \circ \iota_U = g_{\mathbb{C}} \circ (\iota_V \circ f) = \iota_W \circ (g \circ f)$$

by Lemma 6 (uniqueness): $(g \circ f)_{\mathbb{C}} = g_{\mathbb{C}} \circ f_{\mathbb{C}}$.

The last claim follows from combining 1. , 2. and Lemma 6 (uniqueness). □

Lemma 7 (Complexification of the dual). Let V be a real vector space. Consider the complex vector space $\mathrm{Hom}_{\mathbb{R}}(V,\ \mathbb{C})$, where addition and scalar multiplication are defined point-wise. There are $\mathbb{C}$-linear isomorphisms

$$\begin{aligned}\psi\colon \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, \mathbb{C}) &\to \mathrm{Hom}_{\mathbb{R}}(V,\ \mathbb{C})\\ F &\longmapsto F(\,\cdot\,, 0)\end{aligned}$$

$$\begin{aligned}\Psi\colon \mathrm{Hom}_{\mathbb{R}}(V,\ \mathbb{C}) &\to (\mathrm{Hom}_{\mathbb{R}}(V,\ \mathbb{R}))_{\mathbb{C}}\\ f &\longmapsto (\mathrm{Re}(f), \mathrm{Im}(f))\end{aligned}$$

In particular, $(V_{\mathbb{C}})^* \cong (V^*)_{\mathbb{C}}$.

Proof. We first deal with ψ. $\forall\, F, G \in \mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}},\ \mathbb{C})$, $\forall\, \lambda, \mu \in \mathbb{C}$ one has $\forall\, v \in V$

$$\begin{aligned}\psi(\lambda F + \mu G)(v) &= (\lambda F + \mu G)(v, 0) =\\ &= \lambda F(v, 0) + \mu G(v, 0) = (\lambda\psi(F) + \mu\psi(G))(v)\end{aligned}$$

which implies that ψ is $\mathbb{C}$-linear. To prove its injectivity pick $F \in \mathrm{Ker}(\psi)$, and by Remark 3 and $\mathbb{C}$-linearity of F one sees $\forall v, w \in V$

$$F(v, w) = F(v, 0) + \sqrt{-1}F(w, 0) = \psi(F)(v) + \sqrt{-1}\psi(F)(w) = 0$$

that is, $F = 0$. Finally, if $f \in \mathrm{Hom}_{\mathbb{R}}(V,\ \mathbb{C})$, consider the map

$$F\colon V_{\mathbb{C}} \to \mathbb{C} : (v, w) \mapsto f(v) + \sqrt{-1}f(w)$$

Notice that F is $\mathbb{R}$-linear, since $\forall\, u, v, w, z \in V$, $\forall\, a, b \in \mathbb{R}$, the $\mathbb{R}$-linearity of f gives

$$\begin{aligned}F(a(u, v) + b(w, z)) &= F(au + bw, av + bz) =\\ &= f(au + bw) + \sqrt{-1}f(av + bz) =\\ &= af(u) + bf(w) + \sqrt{-1}(af(v) + bf(z)) =\\ &= a(f(u) + \sqrt{-1}f(v)) + b(f(w) + \sqrt{-1}f(z)) =\\ &= aF(u, v) + bF(w, z)\end{aligned}$$

and in particular it is $\mathbb{C}$-linear because $\forall\, v, w \in V$

$$\begin{aligned} F(\sqrt{-1}(v,w)) = F(-w,v) = & \\ &= f(-w) + \sqrt{-1}f(v) = -f(w) + \sqrt{-1}f(v) = \\ &= \sqrt{-1}(f(v) + \sqrt{-1}f(w)) = \sqrt{-1}F(v,w) \end{aligned}$$

By construction: $\psi(F) = f$, so ψ is surjective.

Secondly, we deal with Ψ. If $f, g \in \mathrm{Hom}_{\mathbb{R}}(V,\ \mathbb{C})$ and $a, b \in \mathbb{R}$, we compute

$$\begin{aligned} \cdot\ \Psi(af+bg) &= (\mathrm{Re}(af+bg), \mathrm{Im}(af+bg)) = \\ &= (a\mathrm{Re}(f) + b\mathrm{Re}(g), a\mathrm{Im}(f) + b\mathrm{Im}(g)) = \\ &= a(\mathrm{Re}(f), \mathrm{Im}(f)) + b(\mathrm{Re}(g), \mathrm{Im}(g)) = \\ &= a\Psi(f) + b\Psi(g) \\ \cdot\ \Psi(\sqrt{-1}f) &= ((\mathrm{Re}(\sqrt{-1}f), \mathrm{Im}(\sqrt{-1}f)) = \\ &= (-\mathrm{Im}(f), \mathrm{Re}(f)) = \\ &= \sqrt{-1}(\mathrm{Re}(f), \mathrm{Im}(f)) = \sqrt{-1}\Psi(f) \end{aligned}$$

which implies that Ψ is $\mathbb{C}$-linear. Injectivity follows readily from the definition of Ψ. Moreover, from the previous point and using Lemma 5

$$\begin{aligned} \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{R}}(V,\ \mathbb{C})) &= \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}}(V_{\mathbb{C}},\ \mathbb{C})) = \\ &= \dim_{\mathbb{C}}(V_{\mathbb{C}}) = \dim_{\mathbb{R}}(V) = \\ &= \dim_{\mathbb{R}}(\mathrm{Hom}_{\mathbb{R}}(V,\ \mathbb{R})) = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{R}}(V,\ \mathbb{R}))_{\mathbb{C}} \end{aligned}$$

Hence Ψ is also surjective. We conclude the proof by observing that $\Psi \circ \psi$ is an isomorphism between $(V_{\mathbb{C}})^*$ and $(V^*)_{\mathbb{C}}$. $\square$

With the next result, we generalize Lemma 7 to a broader class of objects. For the sake of notation, we identify in the natural way

$$V \cong \iota(V)\,, \quad (V_{\mathbb{C}})^m \cong (V^m)_{\mathbb{C}}$$

Proposition 1 (Complexification of forms). Let V be a real vector space. For all $m \in \mathbb{N}^{\geq 2}$, there exists a $\mathbb{C}$-linear isomorphism

$$\text{Mult}_{\mathbb{C}}((V_{\mathbb{C}})^m,\ \mathbb{C}) \cong (\text{Mult}_{\mathbb{R}}(V^m,\ \mathbb{R}))_{\mathbb{C}}$$

defined by

$$F \overset{\Phi}{\longmapsto} (\text{Re}(F|_{V^m}),\ \text{Im}(F|_{V^m}))$$

In particular, Φ restricts to a $\mathbb{C}$-linear isomorphism

$$\textstyle\bigwedge_{\mathbb{C}}^m(V_{\mathbb{C}}) \cong (\bigwedge_{\mathbb{R}}^m(V))_{\mathbb{C}}$$

Proof. Analogous computations such as those of Lemma 7 show that Φ is $\mathbb{C}$-linear. To prove that Φ is injective, we first note that for any $F \in \text{Mult}_{\mathbb{C}}((V_{\mathbb{C}})^m,\ \mathbb{C})$ one has $\forall\ (v_1,\ \ldots,\ v_m) \in (V_{\mathbb{C}})^m$

$$F(v_1,\ \ldots,\ v_m) = \sum_{j=0}^{m} (\sqrt{-1})^j A(\text{Re}(\overline{v})_{m-j},\ \text{Im}(\overline{v})_j)$$

where the coefficients $A(\text{Re}(\overline{v})_{m-j},\ \text{Im}(\overline{v})_j)$ are the sums of the evaluations of F on the vectors of V^m made of $m-j$ real parts and j imaginary parts, each extracted from one of the vectors $v_1,\ \ldots,\ v_m$. Hence, since if $F \in \text{Ker}(\Phi)$

$$F|_{V^m} = 0$$

we see that $F = 0$ and Φ is injective. Surjectivity follows from Lemma 5, observing

$$\begin{aligned}\dim_{\mathbb{C}}(\text{Mult}_{\mathbb{C}}((V_{\mathbb{C}})^m,\ \mathbb{C})) &= (\dim_{\mathbb{C}}(V_{\mathbb{C}}))^m = \\ &= (\dim_{\mathbb{R}}(V))^m = \\ &= \dim_{\mathbb{R}}(\text{Mult}_{\mathbb{R}}(V^m,\ \mathbb{R})) = \\ &= \dim_{\mathbb{C}}((\text{Mult}_{\mathbb{R}}(V^m,\ \mathbb{R}))_{\mathbb{C}})\end{aligned}$$

Moreover, for any $F \in \text{Mult}_{\mathbb{C}}((V_{\mathbb{C}})^m,\ \mathbb{C})$ it holds

$$F \text{ is alternating} \iff \text{Re}(F),\ \text{Im}(F) \text{ are alternating}$$

so one has $\Phi(\bigwedge_{\mathbb{C}}^m(V_{\mathbb{C}})) \subseteq (\bigwedge_{\mathbb{R}}^m(V))_{\mathbb{C}}$. The equality follows by Lemma 5 again:

$$\begin{aligned}\dim_{\mathbb{C}}\left(\bigwedge^m{}_{\mathbb{C}}(V_{\mathbb{C}})\right) &= \binom{\dim_{\mathbb{C}}(V_{\mathbb{C}})}{m} = \\ &= \binom{\dim_{\mathbb{R}}(V)}{m} = \\ &= \dim_{\mathbb{R}}\left(\bigwedge^m{}_{\mathbb{R}}(V)\right) = \dim_{\mathbb{C}}\left((\bigwedge^m{}_{\mathbb{R}}(V))_{\mathbb{C}}\right)\end{aligned}$$

□

Remark 4. The same construction as that of Proposition 1 allow us to identify a complex-linear tensor over $V_{\mathbb{C}}$ of any type with the complexification of a (unique) tensor over V of the same type.

Let $p,\ q \in \mathbb{N}$. We can define a sort of complexified wedge product

$$\wedge_{\mathbb{C}} :\ (\bigwedge_{\mathbb{R}}^p V)_{\mathbb{C}} \times (\bigwedge_{\mathbb{R}}^q V)_{\mathbb{C}} \to \left(\bigwedge_{\mathbb{R}}^{(p+q)} V\right)_{\mathbb{C}}$$

by "extending the real wedge product $\wedge_{\mathbb{R}}$ by $\mathbb{C}$-bilinearity". That is, for $\omega \in (\bigwedge_{\mathbb{R}}^p V)_{\mathbb{C}}$ and $\eta \in (\bigwedge_{\mathbb{R}}^q V)_{\mathbb{C}}$, we set

$$\begin{aligned}\mathrm{Re}(\omega \wedge_{\mathbb{C}} \eta) &:= \mathrm{Re}(\omega) \wedge_{\mathbb{R}} \mathrm{Re}(\eta) - \mathrm{Im}(\omega) \wedge_{\mathbb{R}} \mathrm{Im}(\eta) \\ \mathrm{Im}(\omega \wedge_{\mathbb{C}} \eta) &:= \mathrm{Re}(\omega) \wedge_{\mathbb{R}} \mathrm{Im}(\eta) + \mathrm{Im}(\omega) \wedge_{\mathbb{R}} \mathrm{Re}(\eta)\end{aligned}$$

Then $\wedge_{\mathbb{C}}$ coincides with $\wedge_{\mathbb{R}}$ on $(\bigwedge_{\mathbb{R}}^p V) \times (\bigwedge_{\mathbb{R}}^q V)$, it is

associative, anticommutative, distributive

because these properties hold for $\wedge_{\mathbb{R}}$, and by the $\mathbb{R}$-homogeneity of $\wedge_{\mathbb{R}}$ and the complex vector space structure introduced in Lemma 4

$\wedge_{\mathbb{C}}$ is $\mathbb{C}$-homogeneous

Moreover, $\wedge_{\mathbb{C}}$ corresponds to the usual wedge product $\wedge$ on the exterior algebra $\bigwedge_{\mathbb{C}}^*(V_{\mathbb{C}})$, under the isomorphism in Proposition 1. Indeed, for $F \in \bigwedge_{\mathbb{C}}^p(V_{\mathbb{C}}),\ G \in \bigwedge_{\mathbb{C}}^q(V_{\mathbb{C}})$, since

$$(F \wedge G)|_{V^{p+q}} = F|_{V^p} \wedge G|_{V^q}$$

we compute

$$\begin{aligned}
\Phi(F \wedge G) &= \big(\mathrm{Re}((F \wedge G)\big|_{V^{p+q}}),\ \mathrm{Im}((F \wedge G)\big|_{V^{p+q}})\big) = \\
&= \big(\mathrm{Re}(F\big|_{V^p} \wedge G\big|_{V^q}),\ \mathrm{Im}(F\big|_{V^p} \wedge G\big|_{V^q})\big) = \\
&= (\mathrm{Re}(F\big|_{V^p}) \wedge_{\mathbb{R}} \mathrm{Re}(G\big|_{V^q}) - \mathrm{Im}(F\big|_{V^p}) \wedge_{\mathbb{R}} \mathrm{Im}(G\big|_{V^q}), \\
&\qquad \mathrm{Re}(F\big|_{V^p}) \wedge_{\mathbb{R}} \mathrm{Im}(G\big|_{V^q}) + \mathrm{Im}(F\big|_{V^p}) \wedge_{\mathbb{R}} \mathrm{Re}(G\big|_{V^q})) = \\
&= \Phi(F) \wedge_{\mathbb{C}} \Phi(G)
\end{aligned}$$

where we used that $\wedge = \wedge_{\mathbb{R}}$ on the real forms on a real vector subspace of $V_{\mathbb{C}}$. With abuse of notation, we denote $\wedge_{\mathbb{C}}$ and $\wedge_{\mathbb{R}}$ simply by $\wedge$.

Remark 5. We can "complexify" the tensor product of tensors and the trace of tensors with the same procedure, and they will correspond to the tensor product on and trace in the tensor algebra of the complexified space.

1.2.2 Vector bundles setting

In this subsection, our aim is to complexify the canonical vector bundles associated with a manifold. The crucial step will be the construction of a suitable vector bundle structure to employ the theory developed in Subsection 1.2.1.

Here, M denotes a smooth manifold and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 5 (Trivializing chart). Let $E \xrightarrow{\pi} M$ be a $\mathbb{K}$-vector bundle. A local chart (U, ϕ) of M is called *trivializing* if there exists a trivialization φ of the bundle defined on $\pi^{-1}(U)$.

We denote a trivializing chart by $(U,\ \phi,\ \varphi)$. An atlas of M is said to *trivialize* a bundle if each of its charts is trivializing.

Remark 6. Starting from an atlas, one can always obtain a trivializing atlas (just choose around each point a chart and a trivializing neighborhood and take the intersection).

Definition 6 (Transition map). Let $E \xrightarrow{\pi} M$ be a $\mathbb{K}$-vector bundle of rank r. If two trivializing charts $(U_\alpha,\ \phi_\alpha,\ \varphi_\alpha),\ (U_\beta,\ \phi_\beta,\ \varphi_\beta)$ overlap (i.e. $U_\alpha \cap U_\beta \neq \emptyset$), the composition

$$\varphi_\alpha \circ \varphi_\beta^{-1} \colon (U_\alpha \cap U_\beta) \times \mathbb{K}^r \to (U_\alpha \cap U_\beta) \times \mathbb{K}^r$$

is smooth. It follows from the definition of trivializations that there exists a smooth map $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to GL(r,\ \mathbb{K})$ such that

$$(\varphi_\alpha \circ \varphi_\beta^{-1})(p,\ v) = (p,\ g_{\alpha\beta}(p)v) \qquad \forall\, p \in U_\alpha \cap U_\beta,\ \forall\, v \in \mathbb{K}^r$$

The map $g_{\alpha\beta}$ is called a *transition map*.

Trivializing charts and transition maps actually determine the vector bundle structure, as we see in the next result.

Proposition 2 (Bundle through trivializations and transition maps). Let E be a set, and $\pi \colon E \to M$ a surjective map. If there exist an atlas $\mathcal{A} = \{(U_\alpha,\ \phi_\alpha)\}$ of M and bijections $\varphi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^r$ such that

- **(a)** $\pi_1 \circ \varphi_\alpha = \pi$, where π_1 is the projection onto the first factor
- **(b)** for all overlapping charts $(U_\alpha,\ \phi_\alpha),\ (U_\beta,\ \phi_\beta)$ there exists a smooth map $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to GL(r,\ \mathbb{K})$ such that

$$(\varphi_\alpha \circ \varphi_\beta^{-1})(p,\ v) = (p,\ g_{\alpha\beta}(p)v) \qquad \forall\, p \in U_\alpha \cap U_\beta,\ \forall\, v \in \mathbb{K}^r$$

then $E \xrightarrow{\pi} M$ admits a unique structure of $\mathbb{K}$-vector bundle of rank r such that the φ_α are trivializations.

Proof. See [AT11], Proposition 3.1.7. □

For any $p \in M$, let π_p be the projection of $\{p\} \times \mathbb{R}^r$ onto $\mathbb{R}^r$.

Proposition 3 (Complexification of a vector bundle). Let $E \stackrel{\pi}{\to} M$ a real vector bundle or rank r. Denote $E_p := \pi^{-1}(p),\ p \in M$. The set

$$E_{\mathbb{C}} := \bigsqcup_{p \in M} (E_p)_{\mathbb{C}}$$

together with the map

$$\pi_{\mathbb{C}} \colon E_{\mathbb{C}} \to M \ :\ \pi_{\mathbb{C}}(x) := p \ \text{ if } \ x \in (E_p)_{\mathbb{C}}$$

admits a unique structure of complex vector bundle of rank r over M such that the property **(P)** is satisfied $\forall\, p \in M$:

(P) "Let $(U,\ \phi,\ \varphi)$ be a trivializing chart around p for E. Consider the map

$$\varphi_{\mathbb{C}} \colon (\pi_{\mathbb{C}})^{-1}(U) \to U \times \mathbb{C}^r$$

defined as follows: for any $q \in U$, if $(v, w) \in (E_q)_{\mathbb{C}}$

$$\varphi_{\mathbb{C}}(v, w) := (q, (\pi_q \circ \varphi)(v) + \sqrt{-1}(\pi_q \circ \varphi)(w))$$

Then $(U,\ \phi,\ \varphi_{\mathbb{C}})$ is a trivializing chart around p for $E_{\mathbb{C}}$ (which is called the *complexification of* $(U,\ \phi,\ \varphi)$) ."

The vector bundle $E_{\mathbb{C}} \stackrel{\pi_{\mathbb{C}}}{\to} M$ is called the *complexification of* $E \stackrel{\pi}{\to} M$.

Proof. Let $(U,\ \phi,\ \varphi)$ be a trivializing chart and let $(U,\ \phi,\ \varphi_{\mathbb{C}})$ be its complexification. Since

$$(\pi_{\mathbb{C}})^{-1}(U) = \bigsqcup_{p \in U} (E_p)_{\mathbb{C}}\ , \quad U \times \mathbb{C}^r = \bigsqcup_{p \in U} (\{p\} \times \mathbb{C}^r)$$

to prove that $\varphi_{\mathbb{C}}$ is bijective it suffices to show that $\forall\, q \in U$

$$\varphi_{\mathbb{C}}\big|_{(E_q)_{\mathbb{C}}} \colon (E_q)_{\mathbb{C}} \to \{q\} \times \mathbb{C}^r$$

is bijective. But this is true because $\varphi_{\mathbb{C}}\big|_{(E_q)_{\mathbb{C}}} = \Theta \circ (\varphi\big|_{E_q})_{\mathbb{C}}$, where

$$(\varphi\big|_{E_q})_{\mathbb{C}} \colon (E_q)_{\mathbb{C}} \to (\{q\} \times \mathbb{R}^r)_{\mathbb{C}} :$$
$$(v, w) \mapsto ((q, (\pi_q \circ \varphi)(v)),\ (q, (\pi_q \circ \varphi)(w)))$$

is bijective by Corollary 1 and

$$\Theta\colon\ (\{q\}\times\mathbb{R}^r)_{\mathbb{C}}\to\{q\}\times\mathbb{C}^r : ((q,a),(q,b))\mapsto(q,a+\sqrt{-1}b)$$

is also bijective. Moreover

- by construction of $\pi_{\mathbb{C}}$ and $\varphi_{\mathbb{C}}$, $\forall\, x\in E_{\mathbb{C}}$

  $$\pi_{\mathbb{C}}(x)=p=(\pi_1\circ\varphi_{\mathbb{C}})(x)$$

  where $x\in(E_p)_{\mathbb{C}}$

- if $(V,\ \psi,\ \eta)$ is a trivializing chart which overlaps with $(U,\ \phi,\ \varphi)$ and g is a transition map between the two charts, $\forall\, p\in U\cap V$ and $\forall\, v\in\mathbb{C}^r$

  $$\begin{aligned}(\varphi_{\mathbb{C}}\circ\eta_{\mathbb{C}}^{-1})(p,\ v)&=\varphi_{\mathbb{C}}((\eta^{-1}\circ\pi_p^{-1})(\mathrm{Re}(v)),\ (\eta^{-1}\circ\pi_p^{-1})(\mathrm{Im}(v)))=\\&=(p,\ (\pi_p\circ\varphi\circ\eta^{-1}\circ\pi_p^{-1})(\mathrm{Re}(v))+\\&\quad+\sqrt{-1}(\pi_p\circ\varphi\circ\eta^{-1}\circ\pi_p^{-1})(\mathrm{Im}(v)))=\\&=(p,\ g(p)\mathrm{Re}(v)+\sqrt{-1}g(p)\mathrm{Im}(v))\end{aligned}$$

  that is, denoting with $\iota\colon GL(r,\ \mathbb{R})\to GL(r,\ \mathbb{C})$ the inclusion, the smooth map $g_{\mathbb{C}}:=\iota\circ g\colon U\cap V\to GL(r,\ \mathbb{C})$ satisfies

  $$(\varphi_{\mathbb{C}}\circ\eta_{\mathbb{C}}^{-1})(p,\ v)=(p,\ g_{\mathbb{C}}(p)v)$$

The claim follows immediately from Proposition 2, picking the atlas of M made of all its trivializing charts (within the differentiable structure on M). □

Remark 7. Let $(U,\ \phi,\ \varphi)$ be a trivializing chart. Up to proper identification with the isomorphism Θ introduced in Proposition 3, we see that $\varphi_{\mathbb{C}}$ acts on the fibers $(E_p)_{\mathbb{C}}$ as the complexification of φ introduced in Lemma 6.

We can finally fulfill the goal established in this subsection.

Example 1 (Complexified bundles for a manifold). The following complex vector bundles arise from Proposition 3 applied to the canonical vector bundles associated to M:

1. the *complexified tangent bundle* $T_{\mathbb{C}}M := (TM)_{\mathbb{C}}$
2. the *complexified cotangent bundle* $T^*_{\mathbb{C}}M := (T^*M)_{\mathbb{C}}$
3. the *complexified k-forms bundle* $\bigwedge^k_{\mathbb{C}} M := (\bigwedge^k M)_{\mathbb{C}}$
4. the *complexified (k,l)-tensors bundle* $T^{(k,l)}_{\mathbb{C}}M := (T^{(k,l)}M)_{\mathbb{C}}$

The sections of $T_{\mathbb{C}}M$, $T^*_{\mathbb{C}}M$, $\bigwedge^k_{\mathbb{C}} M$, $T^{(k,l)}_{\mathbb{C}}M$ are called *complex vector fields, complex 1-forms* (or *complex covector fields*), *complex k-forms, complex (k,l)-tensors*. We denote the complex vector spaces of smooth such sections by

$$\chi_{\mathbb{C}}(M)\ ,\ \ \Omega^1_{\mathbb{C}}(M)\ ,\ \ \Omega^k_{\mathbb{C}}(M)\ ,\ \ \mathcal{T}^{(k,l)}_{\mathbb{C}}(M)$$

By Remark 3, any of such sections Z admits a unique decomposition

$$Z = \mathrm{Re}(Z) + \sqrt{-1}\,\mathrm{Im}(Z)$$

where $\mathrm{Re}(Z),\ \mathrm{Im}(Z)$ are sections of the real vector bundle associated to the considered one. In particular

$$Z \text{ is smooth} \iff \mathrm{Re}(Z),\ \mathrm{Im}(Z) \text{ are smooth}$$

In this discussion, we will focus exclusively on smooth sections and refer to them simply as sections.

1.3 Almost complex structures

We introduced the canonical complex structure on the Euclidean space, which acts as the multiplication by $\sqrt{-1}$ through the identification $\mathbb{C}^k \equiv \mathbb{R}^{2k}$. This concept can be naturally extended to any finite-dimensional real vector space of even dimension (see [Huy05]).

Nevertheless, we work directly in the setting of smooth manifolds.

Definition 7 (Almost complex structure). An *almost complex structure* on a smooth manifold M is an endomorphism $J \colon TM \to TM$ such that $J^2 = -Id$, i.e. a smooth map such that $\forall\, p \in M$

$$J_p \colon T_pM \to T_pM \text{ is } \mathbb{R}\text{-linear and satisfies } J_p^2 = -Id$$

The couple $(M,\ J)$ is called an *almost complex manifold*.

Being a vector bundle endomorphism, an almost complex structure can be naturally paired with the $(1,\ 1)$-tensor

$$\tilde{J} \colon \Omega^1 M \times \chi(M) \to C^\infty(M) : \ (\omega,\ Y) \mapsto \omega(JY)$$

such that $\tilde{J}(\omega,\ JY) = -\omega(Y)$. With abuse of notation, we denote $\tilde{J}$ by J.

Remark 8. The dimension of an almost complex manifold has to be even: any endomorphism of an odd-dimensional real vector space has a real eigenvalue (by the intermediate value theorem), so it could not square to -1.

As the name suggests, a complex manifold is in particular an almost complex manifold in a natural way.

Example 2 (Natural complex structure). Let M be a complex manifold and choose holomorphic coordinates $z^i = x^i + \sqrt{-1}\, y^i$ on an open U. Setting

$$J_U(\tfrac{\partial}{\partial x^i}) := \tfrac{\partial}{\partial y^i},\ J_U(\tfrac{\partial}{\partial y^i}) := -\tfrac{\partial}{\partial x^i}$$

gives rise to an almost complex structure J_U on U. It extends to a well-defined almost complex structure J on M. Indeed, on two overlapping charts $(U,\ \phi)$, $(V,\ \psi)$, for all $p \in U \cap V$: Lemma 1 results into

$$\begin{aligned} j_n \circ \phi_{*p} \circ (\psi_{*p})^{-1} &= j_n \circ (\phi \circ \psi^{-1})_{*\psi(p)} = \\ &= (\phi \circ \psi^{-1})_{*\psi(p)} \circ j_n = \phi_{*p} \circ (\psi_{*p})^{-1} \circ j_n \end{aligned}$$

while by definition

$$\phi_{*p} \circ J_U(p) \circ (\phi_{*p})^{-1} = j_n = \psi_{*p} \circ J_V(p) \circ (\psi_{*p})^{-1}$$

Hence $\phi_{*p} \circ J_U(p) \circ (\psi_{*p})^{-1} = \phi_{*p} \circ J_V(p) \circ (\psi_{*p})^{-1}$, and $J_U(p) = J_V(p)$. J is called the *natural complex structure* on M.

When dealing with a complex manifold, we always assume that it is equipped with its natural complex structure, unless otherwise stated.

It is natural to ask whether the presence of an almost complex structure on a smooth manifold can grant the existence of a holomorphic atlas.

Definition 8 (Integrability). An almost complex structure on a smooth manifold is called *integrable,* or simply *complex structure,* if it arises from holomorphic charts.

A deep result of Newlander and Nirenberg (see [NN57]) states that the integrability condition can be expressed in analytic terms as follows.

Theorem 2. An almost complex structure J on a smooth manifold M is integrable if and only if its *Nijenhuis tensor*

$$N^J(X,\ Y) := [X,\ Y] + J[X,\ JY] + J[JX,\ Y] - [JX,\ JY]\,,\ \ X, Y \in \chi(M)$$

vanishes.

1.4 The complex exterior algebra

In this section, we dig into the richness of the complex environment, discussing how the structure of complex manifold influences the linear-algebraic structures associated with the underlying space. M will denote a complex manifold of complex dimension n.

Pick holomorphic coordinates $z^i = x^i + \sqrt{-1}\, y^i$. A local frame for TM is given by

$$\{\tfrac{\partial}{\partial x^1},\ \tfrac{\partial}{\partial y^1}, \dots,\ \tfrac{\partial}{\partial x^n},\ \tfrac{\partial}{\partial y^n}\}$$

By Lemma 5, the latter is also a local complex frame for $T_{\mathbb{C}}M$. Hence, a new local complex frame for $T_{\mathbb{C}}M$ is

$$\{\tfrac{\partial}{\partial z^1}, \dots,\ \tfrac{\partial}{\partial z^n},\ \tfrac{\partial}{\partial \overline{z}^1}, \dots,\ \tfrac{\partial}{\partial \overline{z}^n}\}$$

where

$$\tfrac{\partial}{\partial z^i} := \tfrac{1}{2}\big(\tfrac{\partial}{\partial x^i} - \sqrt{-1}\,\tfrac{\partial}{\partial y^i}\big)\ ,\quad \tfrac{\partial}{\partial \overline{z}^i} := \tfrac{1}{2}\big(\tfrac{\partial}{\partial x^i} + \sqrt{-1}\,\tfrac{\partial}{\partial y^i}\big)$$

These complex vector fields are called the *holomorphic coordinate vector fields* and *anti-holomorphic coordinate vector fields*, respectively. Furthermore, a local complex frame for $T^*_{\mathbb{C}}M$ is given by

$$\{dz^1, \dots,\ dz^n, d\overline{z}^1, \dots,\ d\overline{z}^n\}$$

where

$$dz^i = dx^i + \sqrt{-1}\, dy^i\ ,\quad d\overline{z}^i = dx^i - \sqrt{-1}\, dy^i$$

since these are the 1-forms dual to $\tfrac{\partial}{\partial z^i},\ \tfrac{\partial}{\partial \overline{z}^i}$ respectively. These complex 1-forms are called the *holomorphic coordinate covector fields* and the *anti-holomorphic coordinate covector fields*, respectively.

1.4.1 Splitting of complex vector fields

The complex structure J can be fiberwise complexified using Lemma 6, obtaining an endomorphism $J_{\mathbb{C}}$ of $T_{\mathbb{C}}M$ that satisfies (see Corollary 1)

$$(J_{\mathbb{C}})^2 = (J^2)_{\mathbb{C}} = (-Id_{TM})_{\mathbb{C}} = -Id_{T_{\mathbb{C}}M}$$

With abuse of notation, we denote $J_{\mathbb{C}}$ by J. Let $p \in M$. If $T_p^{1,0}M,\ T_p^{0,1}M$ are the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces for J_p , $\forall\, i = 1,\ \ldots,\ n$

$$\frac{\partial}{\partial z^i}\Big|_p \in T_p^{1,0}M\ ,\quad \frac{\partial}{\partial \bar{z}^i}\Big|_p \in T_p^{0,1}M$$

It follows by the dimensional equation that locally

$$T^{1,0}M = \text{span}_{\mathbb{C}}\left(\tfrac{\partial}{\partial z^i}\right)\ ,\quad T^{0,1}M = \text{span}_{\mathbb{C}}\left(\tfrac{\partial}{\partial \bar{z}^i}\right)$$

and one has the pointwise splitting

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$$

$T^{1,0}M,\ T^{0,1}M$ are called the *holomorphic tangent bundle* and the *anti-holomorphic tangent bundle* respectively. The splitting of $T_{\mathbb{C}}M$ induces the splitting of complex vector fields

$$\chi_{\mathbb{C}}(M) = \chi^{1,0}(M) \oplus \chi^{0,1}(M)$$

where $\chi^{1,0}(M),\ \chi^{0,1}(M)$ are the complex vector spaces of sections of $T^{1,0}M$, $T^{0,1}M$ respectively. They are called the spaces of *holomorphic vector fields* and of *anti-holomorphic vector fields*.

1.4.2 Splitting of complex covector fields

Take $J^* \colon T^*_{\mathbb{C}}M \to T^*_{\mathbb{C}}M$ the dual endomorphism of J, defined in $q \in M$ by

$$(J^*)_q(\omega_q)(X_q) = \omega_q(J_q X_q)$$

With abuse of notation, we denote J^* by J. Let $p \in M$ and let $\bigwedge_p^{1,0} M,\ \bigwedge_p^{0,1} M$ be the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces for J_p . By duality, $\forall\, i = 1,\ \ldots,\ n$

$$dz^i\big|_p \in \textstyle\bigwedge_p^{1,0} M\ , \quad d\overline{z}^i\big|_p \in \textstyle\bigwedge_p^{0,1} M$$

Thus, as in the previous case, one has the pointwise splitting

$$T^*_{\mathbb{C}}M = \textstyle\bigwedge^{1,0} M \oplus \textstyle\bigwedge^{0,1} M$$

$\bigwedge^{1,0} M,\ \bigwedge^{0,1} M$ are called the *holomorphic cotangent bundle* and the *anti-holomorphic cotangent bundle* respectively. The splitting of $T^*_{\mathbb{C}}M$ induces the splitting of complex covector fields

$$\Omega^1_{\mathbb{C}}(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$$

where $\Omega^{1,0}(M),\ \Omega^{0,1}(M)$ are the complex vector spaces of sections of $\bigwedge^{1,0} M$, $\bigwedge^{0,1} M$ respectively. We call them the spaces of *holomorphic covector fields* and of *anti-holomorphic covector fields*.

1.4.3 Splitting of complex forms

Fix $k \geq 2$. Let $x \in M$. A $\mathbb{C}$-basis of $(\bigwedge^k_{\mathbb{C}} M)_x$ is

$$\left\{ \begin{matrix} dz^{i_1}\big|_x \wedge \cdots \wedge dz^{i_p}\big|_x \wedge d\overline{z}^{j_1}\big|_x \wedge \cdots \wedge d\overline{z}^{j_q}\big|_x :\ p+q=k, \\ 1 \leq i_1 < \cdots < i_p \leq n,\ 1 \leq j_1 < \cdots < j_q \leq n \end{matrix} \right\}$$

We define for $(p,\ q) \neq (0,0)$

$$\textstyle\bigwedge_x^{p,q} M := \operatorname{span}_{\mathbb{C}} \left(\begin{matrix} \omega_{i_1} \wedge \cdots \wedge \omega_{i_p} \wedge \eta_{j_1} \wedge \cdots \wedge \eta_{j_q} : \\ \omega_{i_r} \in \bigwedge\limits^{1,0} M\ \forall\, r,\ \eta_{j_s} \in \bigwedge\limits^{0,1} M\ \forall\, s \end{matrix} \right)$$

which has the $\mathbb{C}$-basis

$$\left\{ \begin{matrix} dz^{i_1}\big|_x \wedge \cdots \wedge dz^{i_p}\big|_x \wedge d\overline{z}^{j_1}\big|_x \wedge \cdots \wedge d\overline{z}^{j_q}\big|_x : \\ 1 \leq i_1 < \cdots < i_p \leq n,\ 1 \leq j_1 < \cdots < j_q \leq n \end{matrix} \right\}$$

Hence, we have the pointwise splitting

$$\textstyle\bigwedge^k_{\mathbb{C}} M = \bigoplus\limits_{p+q=k} \textstyle\bigwedge^{p,q} M$$

We call $\bigwedge^{p,q} M$ the (p, q)*-forms bundle*, and $(p,\ q)$ is called *bigrading* (or *type*). The splitting of $\bigwedge^k_{\mathbb{C}} M$ induces the splitting of complex forms

$$\Omega^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

where $\Omega^{p,q}(M)$ is the complex vector space of sections of $\bigwedge^{p,q} M$, called the space of (p, q)*-forms*.

1.4.4 Splitting of the exterior derivative

Using Lemma 6, we can define a complexified exterior derivative

$$d_{\mathbb{C}} \colon \Omega^k_{\mathbb{C}} M \to \Omega^{k+1}_{\mathbb{C}} M\ , \quad k = 0,\ \ldots,\ 2n-1$$

by "extending the real exterior derivative d by $\mathbb{C}$-linearity". Explicitly, $d_{\mathbb{C}}$ is given on $\omega \in \Omega^k_{\mathbb{C}} M$ by

$$d_{\mathbb{C}}\omega = d\mathrm{Re}(\omega) + \sqrt{-1} d\mathrm{Im}(\omega)$$

Then $d_{\mathbb{C}}$ coincides with d on real forms and it is $\mathbb{C}$-linear, by Lemma 6. It satisfies $(d_{\mathbb{C}})^2 = 0$ because $d^2 = 0$; furthermore, if $\omega \in \Omega^k_{\mathbb{C}} M,\ \eta \in \Omega^l_{\mathbb{C}} M$

$$\begin{aligned} d_{\mathbb{C}}(\omega \wedge \eta) &= d(\mathrm{Re}(\omega) \wedge \mathrm{Re}(\eta) - \mathrm{Im}(\omega) \wedge \mathrm{Im}(\eta)) + \\ &\quad + \sqrt{-1} d(\mathrm{Re}(\omega) \wedge \mathrm{Im}(\eta) + \mathrm{Im}(\omega) \wedge \mathrm{Re}(\eta)) = \\ &= d\mathrm{Re}(\omega) \wedge \mathrm{Re}(\eta) + (-1)^k\, \mathrm{Re}(\omega) \wedge d\mathrm{Re}(\eta) + \\ &\quad - d\mathrm{Im}(\omega) \wedge \mathrm{Im}(\eta) - (-1)^k\, \mathrm{Im}(\omega) \wedge d\mathrm{Im}(\eta) + \\ &\quad + \sqrt{-1}(d\mathrm{Re}(\omega) \wedge \mathrm{Im}(\eta) + (-1)^k\, \mathrm{Re}(\omega) \wedge d\mathrm{Im}(\eta) + \\ &\quad + d\mathrm{Im}(\omega) \wedge \mathrm{Re}(\eta) + (-1)^k\, \mathrm{Im}(\omega) \wedge d\mathrm{Re}(\eta)) = \\ &= d_{\mathbb{C}}\omega \wedge \eta + (-1)^k\, \omega \wedge d_{\mathbb{C}}\eta \end{aligned}$$

which means that $d_{\mathbb{C}}$ satisfies the Leibniz rule. With abuse of notation, we denote $d_{\mathbb{C}}$ simply by d.

Let $p,\ q \in \mathbb{N}$ and consider the projections on the summand

$$\Pi^{p+1,q} \colon \Omega_{\mathbb{C}}^{p+q+1} M \to \Omega^{p+1,q} M$$
$$\Pi^{p,q+1} \colon \Omega_{\mathbb{C}}^{p+q+1} M \to \Omega^{p,q+1} M$$

The splitting of complex forms induces on each bigrading the splitting of the exterior derivative

$$d = \partial + \overline{\partial}$$

where for all $p,\ q \in \mathbb{N}$, we define the *Dolbeault operators*

$$\partial := \Pi^{p+1,q} \circ d \colon \Omega^{p,q} M \to \Omega^{p+1,q} M$$
$$\overline{\partial} := \Pi^{p,q+1} \circ d \colon \Omega^{p,q} M \to \Omega^{p,q+1} M$$

Due to the properties of d and the splitting of complex forms, it follows readily that both $\partial,\ \overline{\partial}$ are $\mathbb{C}$-linear and satisfy the Leibniz rule. Furthermore

$$0 = d^2 = \partial^2 + \overline{\partial}^2 + \partial\overline{\partial} + \overline{\partial}\partial$$

Thus, since $\partial^2,\ \overline{\partial}^2,\ \partial\overline{\partial} + \overline{\partial}\partial$ take values in different bigradings, we have

$$\partial^2 = 0\ ,\ \overline{\partial}^2 = 0\ ,\ \partial\overline{\partial} + \overline{\partial}\partial = 0$$

Remark 9. By different bigrading: $d = 0 \iff \partial = \overline{\partial} = 0$.

1.4.5 Splitting in local coordinates

For our purposes, it is useful to compute objects such as the exterior derivative in local coordinates. When dealing with a complex manifold, an effective choice often turns out to be to pick holomorphic coordinates in order to exploit the complex structure and its properties, such as the splittings studied so far.

In this subsection, we discuss some examples of these computations, for which we pick holomorphic coordinates $z^i = x^i + \sqrt{-1}\, y^i$.

Example 3. For $f\colon M \to \mathbb{C}$ smooth, we have

$$
\begin{aligned}
df &= d\mathrm{Re}(f) + \sqrt{-1}d\mathrm{Im}(f) = \\
&= \sum_i \frac{\partial \mathrm{Re}(f)}{\partial x^i}\, dx^i + \frac{\partial \mathrm{Re}(f)}{\partial y^i}\, dy^i + \\
&\quad + \sqrt{-1}\sum_i \frac{\partial \mathrm{Im}(f)}{\partial x^i}\, dx^i + \frac{\partial \mathrm{Im}(f)}{\partial y^i}\, dy^i = \\
&= \sum_i \Big(\frac{\partial \mathrm{Re}(f)}{\partial z^i} + \frac{\partial \mathrm{Re}(f)}{\partial \overline{z}^i}\Big)\, \frac{(dz^i + d\overline{z}^i)}{2} + \\
&\qquad + \sqrt{-1}\Big(\frac{\partial \mathrm{Re}(f)}{\partial z^i} - \frac{\partial \mathrm{Re}(f)}{\partial \overline{z}^i}\Big)\, \frac{-\sqrt{-1}(dz^i - d\overline{z}^i)}{2}\Big) + \\
&\quad + \sqrt{-1}\sum_i \Big(\frac{\partial \mathrm{Im}(f)}{\partial z^i} + \frac{\partial \mathrm{Im}(f)}{\partial \overline{z}^i}\Big)\, \frac{(dz^i + d\overline{z}^i)}{2} + \\
&\qquad + \sqrt{-1}\Big(\frac{\partial \mathrm{Im}(f)}{\partial z^i} - \frac{\partial \mathrm{Im}(f)}{\partial \overline{z}^i}\Big)\, \frac{-\sqrt{-1}(dz^i - d\overline{z}^i)}{2}\Big) = \\
&= \sum_i \frac{\partial f}{\partial z^i}\, dz^i + \sum_i \frac{\partial f}{\partial \overline{z}^i}\, d\overline{z}^i
\end{aligned}
$$

Moreover, by definition of $\partial,\ \overline{\partial}$

$$
\partial f = \sum_j \tfrac{\partial f}{\partial z^j}\, dz^j \ , \ \ \overline{\partial} f = \sum_j \tfrac{\partial f}{\partial \overline{z}^j}\, d\overline{z}^j
$$

Example 4. Let $\alpha \in \Omega^{p,q}M$. To ensure notational clarity, in the following we use the multi-index convention

$$
\begin{aligned}
&\frac{\partial}{\partial z^I} := \left(\frac{\partial}{\partial z^{i_1}}, \dots, \frac{\partial}{\partial z^{i_p}}\right) \ , \quad \frac{\partial}{\partial \overline{z}^J} := \left(\frac{\partial}{\partial \overline{z}^{j_1}}, \dots, \frac{\partial}{\partial \overline{z}^{j_q}}\right) \\
&dz^I := dz^{i_1} \wedge \dots \wedge dz^{i_p} \ , \quad d\overline{z}^J := d\overline{z}^{j_1} \wedge \dots \wedge d\overline{z}^{j_q}
\end{aligned}
$$

for any $I,\ J$ strictly-ordered multi-indices over $\{1, \dots, n\}$, of length $p,\ q$ respectively. As observed, α admits a local decomposition

$$
\alpha = \sum_{|J|=p, |K|=q} \alpha_{J\overline{K}}\, dz^J \wedge d\overline{z}^K
$$

where $\alpha_{J\overline{K}} \colon M \to \mathbb{C}$ are the smooth functions $\alpha(\frac{\partial}{\partial z^I}, \frac{\partial}{\partial \overline{z}^J})$. Since

$$d(dz^i) = d(dx^i) + \sqrt{-1}d(dy^i) = 0$$
$$d(d\overline{z}^i) = d(dx^i) - \sqrt{-1}d(dy^i) = 0$$

one gets applying the Leibniz rule

$$\begin{aligned} d\alpha &= \sum_{|J|=p,\ |K|=q} d\alpha_{J\overline{K}} \wedge dz^J \wedge d\overline{z}^K = \\ &= \sum_{|J|=p,\ |K|=q} (\sum_i \frac{\partial \alpha_{J\overline{K}}}{\partial z^i}\, dz^i + \frac{\partial \alpha_{J\overline{K}}}{\partial \overline{z}^i}\, d\overline{z}^i) \wedge dz^J \wedge d\overline{z}^K = \\ &= \sum_{|J|=p,\ |K|=q,\ i} \frac{\partial \alpha_{J\overline{K}}}{\partial z^i}\, dz^i \wedge dz^J \wedge d\overline{z}^K + \\ &\quad + \sum_{|J|=p,\ |K|=q,\ i} \frac{\partial \alpha_{J\overline{K}}}{\partial \overline{z}^i}\, d\overline{z}^i \wedge dz^J \wedge d\overline{z}^K \end{aligned}$$

and again by definition of $\partial,\ \overline{\partial}$

$$\partial\alpha = \sum_{|J|=p,\ |K|=q,\ i} \frac{\partial \alpha_{J\overline{K}}}{\partial z^i}\, dz^i \wedge dz^J \wedge d\overline{z}^K$$
$$\overline{\partial}\alpha = \sum_{|J|=p,\ |K|=q,\ i} \frac{\partial \alpha_{J\overline{K}}}{\partial \overline{z}^i}\, d\overline{z}^i \wedge dz^J \wedge d\overline{z}^K$$

Throughout this work, the following special kind of $(1,1)$-forms will play a crucial role.

Example 5. Let $f \colon M \to \mathbb{R}$ be smooth and consider the $(1,1)$-form $\sqrt{-1}\partial\overline{\partial}f$. By different bigrading, the conditions $d(dz^i) = d(d\overline{z}^i) = 0$ imply

$$\partial(dz^i) = \overline{\partial}(dz^i) = \partial(d\overline{z}^i) = \overline{\partial}(d\overline{z}^i) = 0$$

Hence, we compute

$$\sqrt{-1}\partial\overline{\partial}f = \sqrt{-1}\partial\left(\sum_i \frac{\partial f}{\partial \overline{z}^i}\, d\overline{z}^i\right) = \sqrt{-1}\sum_{i,\ j} \frac{\partial^2 f}{\partial z^j \partial \overline{z}^i}\, dz^j \wedge d\overline{z}^i$$

An important detail regarding the kind of form exhibited in Example 5 is that it is actually a real form. The computation done may not highlight this fact, so we introduce a new concept in order to prove it.

For the remainder of this subsection, let $X \in \{T_{\mathbb{C}}M,\, T^*_{\mathbb{C}}M,\, \bigwedge^k_{\mathbb{C}} M,\, T^{(a,b)}_{\mathbb{C}} M\}$.

Definition 9 (Conjugation). The *conjugation of* X is the $\mathbb{C}$-antilinear automorphism $\mathcal{C}$ of X defined for $p \in M$ by

$$\mathcal{C}_p(V_p) := \mathrm{Re}(V_p) - \sqrt{-1}\mathrm{Im}(V_p)$$

$\mathcal{C}$ induces naturally a $\mathbb{C}$-antilinear automorphism on the space of (smooth) sections of X, also called *conjugation* and denoted by $\mathcal{C}$, as follows:

$$\mathcal{C}(\omega) := \mathrm{Re}(\omega) - \sqrt{-1}\mathrm{Im}(\omega)$$

To simplify the notation, we denote $\mathcal{C}(\omega)$ by $\overline{\omega}$. The following properties arise readily from Definition 9.

Lemma 8 (Properties of conjugation). Let $\omega,\ \eta$ be sections of X.

1. $\overline{\omega} = \omega \iff \mathrm{Im}(\omega) = 0\,, \quad \overline{\omega} = -\omega \iff \mathrm{Re}(\omega) = 0$

If $\omega,\ \eta$ are complex forms on M

2. $\overline{\omega \wedge \eta} = \overline{\omega} \wedge \overline{\eta}$
3. $\overline{\partial\overline{\omega}} = \overline{\partial}\omega$

Proof. The first claim holds because

$$2\sqrt{-1}\mathrm{Im}(\omega) = \omega - \overline{\omega}\,, \quad 2\mathrm{Re}(\omega) = \omega + \overline{\omega}$$

For the second claim, we compute

$$\begin{aligned}
\overline{\omega \wedge \eta} &= \overline{\mathrm{Re}(\omega) \wedge \mathrm{Re}(\eta) - \mathrm{Im}(\omega) \wedge \mathrm{Im}(\eta) +} \\
&\quad \overline{+\sqrt{-1}(\mathrm{Re}(\omega) \wedge \mathrm{Im}(\eta) + \mathrm{Im}(\omega) \wedge \mathrm{Re}(\eta))} = \\
&= \mathrm{Re}(\omega) \wedge \mathrm{Re}(\eta) - \mathrm{Im}(\omega) \wedge \mathrm{Im}(\eta) + \\
&\quad - \sqrt{-1}(\mathrm{Re}(\omega) \wedge \mathrm{Im}(\eta) + \mathrm{Im}(\omega) \wedge \mathrm{Re}(\eta)) = \\
&= \mathrm{Re}(\omega) \wedge \mathrm{Re}(\eta) - (-\mathrm{Im}(\omega)) \wedge (-\mathrm{Im}(\eta)) + \\
&\quad + \sqrt{-1}(\mathrm{Re}(\omega) \wedge (-\mathrm{Im}(\eta)) + (-\mathrm{Im}(\omega)) \wedge \mathrm{Re}(\eta)) = \\
&= \overline{\omega} \wedge \overline{\eta}
\end{aligned}$$

For the third claim, pick holomorphic coordinates $z^1, \dots, z^n$. We first consider the case where $\omega = f\colon M \to \mathbb{C}$ is a smooth function. By Example 3 and the second claim

$$\overline{\partial \overline{f}} = \overline{\sum_j \tfrac{\partial \overline{f}}{\partial z^j}\, dz^j} = \sum_j \overline{\tfrac{\partial \overline{f}}{\partial z^j}}\, \overline{dz^j}$$

Notice that for all i: by definition

$$\overline{dz^i} = d\overline{z}^i$$

and we compute

$$\begin{aligned}
\overline{\frac{\partial \overline{f}}{\partial z^i}} &= \overline{\frac{1}{2}(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i})(\mathrm{Re}(f) - \sqrt{-1}\mathrm{Im}(f))} = \\
&= \frac{1}{2}\overline{\left(\frac{\partial \mathrm{Re}(f)}{\partial x^i} - \frac{\partial \mathrm{Im}(f)}{\partial y^i} - \sqrt{-1}(\frac{\partial \mathrm{Im}(f)}{\partial x^i} + \frac{\partial \mathrm{Re}(f)}{\partial y^i})\right)} = \\
&= \frac{1}{2}\left(\frac{\partial \mathrm{Re}(f)}{\partial x^i} - \frac{\partial \mathrm{Im}(f)}{\partial y^i} + \sqrt{-1}(\frac{\partial \mathrm{Im}(f)}{\partial x^i} + \frac{\partial \mathrm{Re}(f)}{\partial y^i})\right) = \\
&= \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i})(\mathrm{Re}(f) + \sqrt{-1}\,\mathrm{Im}(f)) = \frac{\partial f}{\partial \overline{z}^i}
\end{aligned}$$

Thus, by Example 3 we conclude

$$\overline{\partial \overline{f}} = \overline{\partial} f$$

Now consider ω a (p,q)-form on M. Notice that by the second claim

$$\overline{\omega} = \overline{\sum_{|J|=p,\ |K|=q} \omega_{J\overline{K}}\, dz^J \wedge d\overline{z}^K} = \sum_{|J|=p,\ |K|=q} \overline{\omega_{J\overline{K}}}\, d\overline{z}^J \wedge dz^K$$

so that $\overline{\omega}$ is a (q,p)-form. By Example 4, the second claim and the the previous case one has

$$\begin{aligned}
\overline{\partial\overline{\omega}} &= \overline{\sum_{|J|=p,\ |K|=q} \partial\overline{\omega_{J\overline{K}}} \wedge d\overline{z}^J \wedge dz^K} = \\
&= \sum_{|J|=p,\ |K|=q} \overline{\partial\overline{\omega_{J\overline{K}}}} \wedge dz^J \wedge d\overline{z}^K = \\
&= \sum_{|J|=p,\ |K|=q} \overline{\partial}\omega_{J\overline{K}} \wedge dz^J \wedge d\overline{z}^K = \overline{\partial}\omega
\end{aligned}$$

Finally, for any complex form ω, the third claim holds by the splitting in bigraded forms and linearity of $\partial,\ \overline{\partial},\ \mathcal{C}$. □

Using Lemma 8, we can show easily that for $f\colon M \to \mathbb{R}$ smooth: $\sqrt{-1}\partial\overline{\partial} f$ is a real form. Indeed, since f is real, the third point of Lemma 8 yields

$$\begin{aligned}
\overline{\sqrt{-1}\partial\overline{\partial} f} &= \overline{\sqrt{-1}}\ \overline{\partial\overline{\partial} f} = \\
&= -\sqrt{-1}\ \overline{\partial\overline{\overline{\partial} \overline{f}}} = \\
&= -\sqrt{-1}\ \overline{\partial}\partial\overline{f} = \sqrt{-1}\partial\overline{\partial} f
\end{aligned}$$

which by the first point of Lemma 8 means $\mathrm{Im}(\sqrt{-1}\partial\overline{\partial} f) = 0$.

Chapter 2

Kähler manifolds

Here, M always denotes a complex manifold of complex dimension n and J its natural complex structure.

2.1 Hermitian metrics

In this section, we begin our journey into the intersection between complex geometry and Riemannian geometry. Our goal is to analyze the interplay between the two objects that represent these branches, that is, a complex structure and a Riemannian metric.

Definition 10 (Hermitian metric). A Riemannian metric g on M is called *Hermitian* if for all X, Y real vector fields

$$g(JX,\ JY) = g(X,\ Y)$$

The triple $(M,\ J,\ g)$ is called a *Hermitian manifold*.

The following object, which arises naturally from a Hermitian metric, will be the main focus of this section.

Definition 11 (Fundamental form). Let g be a Hermitian metric on M. Setting

$$\omega(X,\ Y) := g(JX,\ Y)\ , \quad X,\ Y \text{ real vector fields}$$

gives a real 2-form on M. Indeed, ω is $C^\infty(M)$-bilinear because of the $C^\infty(M)$-bilinearity of g and the $C^\infty(M)$-linearity of J. Moreover, for all $X,\ Y$ real vector fields one has

$$\omega(Y,\ X) = g(JY,\ X) = -g(JY,\ J^2X) = -g(JX,\ Y) = -\omega(X,\ Y)$$

so that ω is skew-symmetric. ω is called the *fundamental form of* g.

We readily see from Definition 11 that the fundamental form ω satisfies the three following properties:

- ω is *positive definite*, in the sense that for any $X \neq 0$ real vector field

  $$\omega(X,\ JX) > 0$$

  This holds because $\omega(X,\ JX) = g(JX, JX)$ and g is positive definite.

- ω is *non-degenerate*, in the sense that $\forall\, p \in M$: if for $X_p \in T_pM$

  $$\omega_p(X_p, Y_p) = 0 \quad \forall\, Y_p \in T_pM$$

  then $X_p = 0$. This holds because for any $p \in M$, if $X_p \in T_pM$ satisfies the condition, then

  $$g_p(X_p,\ X_p) = g_p(J_pX_p,\ J_pX_p) = \omega(X_p,\ J_pX_p) = 0$$

  so $X_p = 0$ because g is everywhere non-degenerate.

- ω preserves J, since for all $X,\ Y$ real vector fields

  $$\omega(JX,\ JY) = g(J^2X,\ JY) = g(JX,\ Y) = \omega(X,\ Y)$$

Remark 10. If ω is a real 2-form on M that satisfies the above three properties, then setting

$$g(X,\ Y) := \omega(X,\ JY)\ , \quad X,\ Y \text{ real vector fields}$$

defines a Hermitian metric on M with fundamental form ω.

Consider a Hermitian metric g with fundamental form ω. Using Proposition 1 and Remark 4, we can "extend g and ω by $\mathbb{C}$-bilinearity". Explicitly, for $X,\ Y$ complex vector fields, the complexifications $g_{\mathbb{C}},\ \omega_{\mathbb{C}}$ are given by

$$\begin{aligned} g_{\mathbb{C}}(X,\ Y) &= g(\mathrm{Re}(X),\ \mathrm{Re}(Y)) - g(\mathrm{Im}(X),\ \mathrm{Im}(Y)) + \\ &\quad + \sqrt{-1}\,(g(\mathrm{Re}(X),\ \mathrm{Im}(Y)) + g(\mathrm{Im}(X),\ \mathrm{Re}(Y))) \\ \omega_{\mathbb{C}}(X,\ Y) &= \omega(\mathrm{Re}(X),\ \mathrm{Re}(Y)) - \omega(\mathrm{Im}(X),\ \mathrm{Im}(Y)) + \\ &\quad + \sqrt{-1}\,(\omega(\mathrm{Re}(X),\ \mathrm{Im}(Y)) + \omega(\mathrm{Im}(X),\ \mathrm{Re}(Y))) \end{aligned}$$

Then Proposition 1, Remark 4 and the properties of $g,\ \omega$ result in the following:

- $g_{\mathbb{C}},\ \omega_{\mathbb{C}}$ coincide with $g,\ \omega$ on real vector fields;
- $g_{\mathbb{C}},\ \omega_{\mathbb{C}}$ are $C^\infty(M,\ \mathbb{C})$-bilinear and preserve J;
- $g_{\mathbb{C}}$ is symmetric and $\omega_{\mathbb{C}}$ is skew-symmetric;
- $\omega_{\mathbb{C}}(X,\ Y) = g_{\mathbb{C}}(JX,\ Y)$ for all $X,\ Y$ complex vector fields.

With abuse of notation, we denote $g_{\mathbb{C}}$ by g and $\omega_{\mathbb{C}}$ by ω.

Lemma 9. g satisfies the following relations:

1. $\forall\, Z \in \chi_{\mathbb{C}}(M) \setminus \{0\}\ : \quad g(Z,\ \overline{Z})$ is a real positive function
2. $\forall\, Z,\ W \in \chi_{\mathbb{C}}(M)\ : \quad g(\overline{Z},\ \overline{W}) = \overline{g(Z,\ W)}$
3. $\forall\, X,\ Y \in \chi^{1,0}(M),\ \forall\, U,\ V \in \chi^{0,1}(M)\ : \quad g(X,\ Y) = g(U,\ V) = 0$

Proof. For the first claim, just notice that for any complex vector field Z

$$\begin{aligned} g(Z,\, \overline{Z}) &:= g(\mathrm{Re}(Z),\, \mathrm{Re}(Z)) - g(\mathrm{Im}(Z),\, -\mathrm{Im}(Z)) + \\ &\quad + \sqrt{-1}\,(g(\mathrm{Re}(Z),\, -\mathrm{Im}(Z)) + g(\mathrm{Im}(Z),\, \mathrm{Re}(Z))) = \\ &= g(\mathrm{Re}(Z),\, \mathrm{Re}(Z)) + g(\mathrm{Im}(Z),\, \mathrm{Im}(Z)) \end{aligned}$$

which is positive whenever $Z \neq 0$, because g is positive definite on real vector fields. For the second claim, compute

$$\begin{aligned} g(\overline{Z},\, \overline{W}) &:= g(\mathrm{Re}(Z),\, \mathrm{Re}(W)) - g(-\mathrm{Im}(Z),\, -\mathrm{Im}(W)) + \\ &\quad + \sqrt{-1}\,(g(\mathrm{Re}(Z),\, -\mathrm{Im}(W)) + g(-\mathrm{Im}(Z),\, \mathrm{Re}(W))) = \\ &= g(\mathrm{Re}(Z),\, \mathrm{Re}(W)) - g(\mathrm{Im}(Z),\, \mathrm{Im}(W)) + \\ &\quad - \sqrt{-1}\,(g(\mathrm{Re}(Z),\, \mathrm{Im}(W)) + g(\mathrm{Im}(Z),\, \mathrm{Re}(W))) = \\ &= \overline{g(Z,\, W)} \end{aligned}$$

For the third claim, if $X,\ Y$ are holomorphic vector fields then

$$g(X,\, Y) = g(JX,\, JY) = g(\sqrt{-1}X,\, \sqrt{-1}Y) = -g(X,\, Y)$$

Thus, $g(X,\, Y) = 0$. This relation also holds for anti-holomorphic vector fields, by the second claim. □

Pick holomorphic coordinates $z^1, \ldots, z^n$. By Lemma 9 (3.), if we set

$$g_{j\overline{k}} := g(\tfrac{\partial}{\partial z^j},\, \tfrac{\partial}{\partial \overline{z}^k})$$

then $g_{j\overline{k}} = g_{\overline{k}j}$ by symmetry and g admits the local decomposition

$$g = \sum_{j,k} g_{j\overline{k}}\,(dz^j \otimes d\overline{z}^k + d\overline{z}^k \otimes dz^j)$$

Moreover, g can be pointwisely represented by the symmetric matrix of order $2n$

$$(g) := \left(\begin{array}{c|c} 0 & (g_{j\overline{k}}) \\ \hline (g_{j\overline{k}})^T & 0 \end{array}\right)$$

Let us further analyze the matrix $(g_{j\bar{k}})$, which has order n. By the symmetry of g and Lemma 9 (2.), one also has for all $j,\ k$

$$\begin{aligned}\overline{g_{j\bar{k}}} &= \overline{g(\frac{\partial}{\partial z^j},\ \frac{\partial}{\partial \bar{z}^k})} = \\ &= g(\overline{\frac{\partial}{\partial z^j}},\ \overline{\frac{\partial}{\partial \bar{z}^k}}) = \\ &= g(\frac{\partial}{\partial z^k},\ \frac{\partial}{\partial \bar{z}^j}) = g_{k\bar{j}}\end{aligned}$$

which means that $(g_{j\bar{k}})$ is a Hermitian matrix. For a complex vector field X, let

$$[X]_{\mathcal{Z}} := \begin{bmatrix} X^{1,0} \\ X^{0,1} \end{bmatrix}$$

be the $2n$-column vector of its components with respect to the local frame

$$\mathcal{Z} := \left(\tfrac{\partial}{\partial z^i},\ \tfrac{\partial}{\partial \bar{z}^j}\right)_{i,j=1,\dots,n}$$

where $X^{1,0},\ X^{0,1}$ are the n-column vectors of the components of the $(1,0)$ and $(0,1)$ parts of X. For all $X,\ Y$ complex vector fields, we have at each point

$$\begin{aligned} g(X,\ Y) &= [X]_{\mathcal{Z}}^T\,(g)\,[Y]_{\mathcal{Z}} = \\ &= \left(X^{1,0}\right)^T (g_{j\bar{k}})\, Y^{0,1} + \left(X^{0,1}\right)^T (g_{j\bar{k}})^T\, Y^{1,0}\end{aligned}$$

In particular, since for any anti-holomorphic vector field X

$$[X]_{\mathcal{Z}} := \begin{bmatrix} 0 \\ X^{0,1} \end{bmatrix} \quad \text{and} \quad \left[\overline{X}\right]_{\mathcal{Z}} := \begin{bmatrix} \overline{X^{0,1}} \\ 0 \end{bmatrix}$$

it holds by Lemma 9 (1.)

$$\left(\overline{X^{0,1}}\right)^T (g_{j\bar{k}})\, X^{0,1} = g(\overline{X},\ X) > 0$$

which implies that $(g_{j\bar{k}})$ is positive definite at each point.

The matrix representation of g clearly yields

$$\big(\det(g_{j\overline{k}})\big)^2 = \det(g_{\mathbb{C}}) = \det(g_{\mathbb{R}})$$

where the latter holds because $g_{\mathbb{C}}$ coincides with $g_{\mathbb{R}}$ on real vector fields. Consequently, we can write

$$\big(\det(g_{j\overline{k}})\big)^2 = \det(g)$$

with no risk of misunderstanding.

Due to the connection between g and ω, the matrix $(g_{j\overline{k}})$ is useful to compute the fundamental form. Indeed, by Lemma 9 (3.) see that for all $i,\ j$

$$\omega(\frac{\partial}{\partial z^i},\ \frac{\partial}{\partial z^j}) = \sqrt{-1}g(\frac{\partial}{\partial z^i},\ \frac{\partial}{\partial z^j}) = 0$$
$$\omega(\frac{\partial}{\partial \overline{z}^i},\ \frac{\partial}{\partial \overline{z}^j}) = -\sqrt{-1}g(\frac{\partial}{\partial \overline{z}^i},\ \frac{\partial}{\partial \overline{z}^j}) = 0$$

so ω admits the local decomposition

$$\omega = \sum_{j,k} \omega(\tfrac{\partial}{\partial z^j},\ \tfrac{\partial}{\partial \overline{z}^k})\, dz^j \wedge d\overline{z}^k = \sqrt{-1} \sum_{j,k} g_{j\overline{k}}\, dz^j \wedge d\overline{z}^k$$

We notice that ω is a $(1,\ 1)$-form, and being $(g_{j\overline{k}})$ Hermitian gives

$$\overline{\omega} = -\sqrt{-1} \sum_{j,k} \overline{g_{j\overline{k}}}\, d\overline{z}^j \wedge dz^k = \sqrt{-1} \sum_{j,k} g_{k\overline{j}}\, dz^k \wedge d\overline{z}^j = \omega$$

so by Lemma 8: ω is a real form.

Remark 11. Sometimes, it is useful to consider a normalized local expression for the fundamental form ω:

$$\omega = \tfrac{\sqrt{-1}}{2} \sum_{j,k} g_{j\overline{k}}\, dz^j \wedge d\overline{z}^k$$

2.2 Kähler metrics

This section is devoted to present the main type of object we deal with in this work. Its importance lies in the richness of its geometry: a first example can be found in Proposition 4 below.

Definition 12 (Kähler metric). A Hermitian metric g on M is called *Kähler* if its fundamental form is d-closed, i.e.

$$d\omega = 0$$

In this case, ω is called the *Kähler form* of g and the couple $(M,\ g)$ (equivalently the couple $(M,\ \omega)$, by Remark 10) is called a *Kähler manifold*.

We refer to the condition $d\omega = 0$ as the *Kähler condition*. Although it is a global condition, during computations it is useful to apply its local equivalent, which is our next result.

Lemma 10 (Kähler metrics in local coordinates). Pick holomorphic coordinates $z^1, \dots, z^n$. Then

$$d\omega = 0 \iff \tfrac{\partial g_{j\bar{k}}}{\partial z^i} = \tfrac{\partial g_{i\bar{k}}}{\partial z^j} \quad \forall\, i,\ j,\ k \iff \tfrac{\partial g_{j\bar{k}}}{\partial \bar{z}^i} = \tfrac{\partial g_{j\bar{i}}}{\partial \bar{z}^k} \quad \forall\, i,\ j,\ k$$

Proof. Since ω is real, by Lemma 8

$$\bar{\partial}\omega = \overline{\partial\overline{\omega}} = \overline{\partial\omega}$$

which implies together with Remark 9

$$d\omega = 0 \iff \partial\omega = 0 \iff \bar{\partial}\omega = 0$$

From Example 4 we have

$$\cdot\ \partial\omega = \sqrt{-1}\sum_{i,j,k}\frac{\partial g_{j\bar{k}}}{\partial z^i}\,dz^i\wedge dz^j\wedge d\overline{z}^k =$$

$$= \sqrt{-1}\sum_{i<j,\ k}\Big(\frac{\partial g_{j\bar{k}}}{\partial z^i}-\frac{\partial g_{i\bar{k}}}{\partial z^j}\Big)\,dz^i\wedge dz^j\wedge d\overline{z}^k$$

$$\cdot\ \overline{\partial}\omega = \sqrt{-1}\sum_{i,j,k}\frac{\partial g_{j\bar{k}}}{\partial \overline{z}^i}\,d\overline{z}^i\wedge dz^j\wedge d\overline{z}^k =$$

$$= -\sqrt{-1}\sum_{j,\ i<k}\Big(\frac{\partial g_{j\bar{k}}}{\partial \overline{z}^i}-\frac{\partial g_{j\bar{i}}}{\partial \overline{z}^k}\Big)\,dz^j\wedge d\overline{z}^i\wedge d\overline{z}^k$$

Hence, the ∂-closure and $\overline{\partial}$-closure of ω translate respectively into

$$\cdot\ \frac{\partial g_{j\bar{k}}}{\partial z^i}-\frac{\partial g_{i\bar{k}}}{\partial z^j}=0 \quad \forall\, i<j,\ k$$

$$\cdot\ \frac{\partial g_{j\bar{k}}}{\partial \overline{z}^i}-\frac{\partial g_{j\bar{i}}}{\partial \overline{z}^k}=0 \quad \forall\, j,\ i<k$$

The claim holds for all $i,\ j,\ k$ because of the symmetry of the conditions $\frac{\partial g_{j\bar{k}}}{\partial z^i} = \frac{\partial g_{i\bar{k}}}{\partial z^j}$ with respect to $i,\ j$ and $\frac{\partial g_{j\bar{k}}}{\partial \overline{z}^i} = \frac{\partial g_{j\bar{i}}}{\partial \overline{z}^k}$ with respect to $i,\ k$. □

The following is one of the key concepts introduced by Erich Kähler in his seminal paper [Käh32], where he demonstrated their importance in simplifying the local geometry of Kähler manifolds.

Proposition 4 (Normal coordinates). Let g be a Kähler metric on M. Around any point $p\in M$, there are holomorphic coordinates $z^1,\dots,z^n$ such that $\forall\ j,\ k$

$$g_{j\bar{k}}(p)=\delta_{jk}\ , \quad \forall\, i:\ \frac{\partial g_{j\bar{k}}}{\partial z^i}(p)=\frac{\partial g_{j\bar{k}}}{\partial \overline{z}^i}(p)=0$$

where $\delta_{jk}=0$ if $j\neq k$ and $\delta_{jk}=1$ if $j=k$ (called *normal coordinates* at p).

Proof. First, note that it suffices to show that there exist holomorphic coordinates $z^1,\dots,z^n$ centered at p such that the local expression of the fundamental form ω with respect to these is

$$\omega = \sqrt{-1} \sum_{j,k} \left(\delta_{jk} + O(|z|^2) \right) dz^j \wedge d\overline{z}^k$$

where $O(|z|^2)$ denotes terms which are at least quadratic in $z^i,\ \overline{z}^i$. Indeed, this condition is equivalent by uniqueness of components to

$$g_{j\overline{k}} = \delta_{jk} + O(|z|^2)$$

Assume the latter is satisfied: then for all i

$$\frac{\partial g_{j\overline{k}}}{\partial z^i} = O(|z|)\ ,\ \frac{\partial g_{j\overline{k}}}{\partial \overline{z}^i} = O(|z|)$$

where $O(|z|)$ denotes terms which are at least linear in $z^i,\ \overline{z}^i$. The claim then follows from the chart being centered at p. We proceed by steps.

(i) Pick any holomorphic coordinates system $v^1, \dots, v^n$ centered at p and denote by $g^v_{j\overline{k}}$ the components of the metric with respect to these coordinates. Since $(g^v_{j\overline{k}}(p))$ is a Hermitian matrix of dimension n, by the Spectral Theorem there exists $U \in U(n)$ such that

$$U^* \left(g^v_{j\overline{k}}(p)\right) U = \text{diag}(\lambda_1, \dots, \lambda_n)$$

where $\lambda_l \in \mathbb{R}$ are the eigenvalues of $(g^v_{j\overline{k}}(p))$, and $\lambda_l > 0$ for all l because $(g^v_{j\overline{k}}(p))$ is positive definite. Hence, there exists $C \in GL_n(\mathbb{C})$ such that

$$C^* \left(g^v_{j\overline{k}}(p)\right) C = I_n$$

where I_n is the identity matrix. Therefore one computes

$$\left(\begin{array}{c|c} 0 & \overline{C} \\ \hline C & 0 \end{array}\right)^T (g^v(p)) \left(\begin{array}{c|c} 0 & \overline{C} \\ \hline C & 0 \end{array}\right) =$$

$$= \left(\begin{array}{c|c} 0 & C^T \\ \hline C^* & 0 \end{array}\right) (g^v(p)) \left(\begin{array}{c|c} 0 & \overline{C} \\ \hline C & 0 \end{array}\right) = \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right)$$

(ii) We can define new holomorphic coordinates $w^1, \dots, w^n$ by pointwise applying a $\mathbb{C}$-linear isomorphism to the frame

$$\mathcal{V} := \left\{ \frac{\partial}{\partial v^1},\ \dots,\ \frac{\partial}{\partial v^n},\ \frac{\partial}{\partial \overline{v}^1},\ \dots,\ \frac{\partial}{\partial \overline{v}^n} \right\}$$

because $\mathbb{C}$-linear maps on the complex space are holomorphic, linear maps can be naturally identified with their differential at any point, and we can retrieve the real coordinates from the complex ones by

$$a^r := \tfrac{w^r+\overline{w}^r}{2}\ ,\ b^r := \tfrac{\overline{w}^r-w^r}{2\sqrt{-1}}$$

Being C invertible, such $\mathbb{C}$-linear isomorphism can be given by

$$\left[\frac{\partial}{\partial w^j}\right]_{\mathcal{V}} = \left(\begin{array}{c|c} 0 & \overline{C} \\ \hline C & 0 \end{array}\right) \left[\frac{\partial}{\partial v^j}\right]_{\mathcal{V}}$$

$$\left[\frac{\partial}{\partial \overline{w}^k}\right]_{\mathcal{V}} = \left(\begin{array}{c|c} 0 & \overline{C} \\ \hline C & 0 \end{array}\right) \left[\frac{\partial}{\partial \overline{v}^k}\right]_{\mathcal{V}}$$

The new chart is still centered at p, by linearity. Denote by $g^w_{j\bar{k}}$ the components of the metric with respect to $\mathcal{W}$. We compute

$$\begin{aligned} g^w_{j\bar{k}}(p) &= g_p(\frac{\partial}{\partial w^j}\bigg|_p, \frac{\partial}{\partial \overline{w}^k}\bigg|_p) = \\ &= \left[\frac{\partial}{\partial w^j}\bigg|_p\right]^T_{\mathcal{V}} (g^v(p)) \left[\frac{\partial}{\partial \overline{w}^k}\bigg|_p\right]_{\mathcal{V}} = \\ &= \left[\frac{\partial}{\partial v^j}\bigg|_p\right]^T_{\mathcal{V}} \left(\begin{array}{c|c} 0 & \overline{C} \\ \hline C & 0 \end{array}\right)^T (g^v(p)) \left(\begin{array}{c|c} 0 & \overline{C} \\ \hline C & 0 \end{array}\right) \left[\frac{\partial}{\partial \overline{v}^k}\bigg|_p\right]_{\mathcal{V}} = \\ &= \left[\frac{\partial}{\partial v^j}\bigg|_p\right]^T_{\mathcal{V}} \left(\begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array}\right) \left[\frac{\partial}{\partial \overline{v}^k}\bigg|_p\right]_{\mathcal{V}} = \\ &= \left[\frac{\partial}{\partial v^j}\bigg|_p\right]^T_{\mathcal{V}} \left[\frac{\partial}{\partial v^k}\bigg|_p\right]_{\mathcal{V}} = \delta_{jk} \end{aligned}$$

Therefore, Taylor's approximation yields

$$g^w_{j\bar{k}} = \delta_{jk} + \sum_i a_{jki}\, w^i + \sum_i a'_{jki}\, \overline{w}^i + O(|w|^2)$$

where $a_{jki},\ a'_{jki} \in \mathbb{C}$ for all $i,\ j,\ k$. In particular, the chart being centered at p implies that for all l

$$\frac{\partial g^w_{j\bar{k}}}{\partial w^l}(p) = a_{jkl}\ ,\ \frac{\partial g^w_{j\bar{k}}}{\partial \overline{w}^l}(p) = a'_{jkl}$$

and Lemma 10 reads $a_{jkl} = a_{lkj},\ a'_{jkl} = a'_{jlk}$. Furthermore, by Lemma 9

$$\overline{\frac{\partial g^w_{k\bar{j}}}{\partial w^i}} = \frac{\partial \overline{g^w_{j\bar{k}}}}{\partial w^i} = \frac{\partial g^w_{j\bar{k}}}{\partial \overline{w}^i}$$

so $\overline{a_{kji}} = a'_{jki}$. Also notice that $\overline{a_{jki}} = \overline{a_{ikj}} = a'_{kij}$.

(iii) Finally, define a holomorphic map $\phi = (z^1, \dots, z^n)$ by

$$z^k := w^k + \tfrac{1}{2} \sum_{j,i} a_{jki}\, w^j\, w^i$$

Notice $\phi(p) = 0$. For all $k,\ l$ one finds

$$\frac{\partial z^k}{\partial w^l} = \delta_{kl} + \sum_{j,i} a_{jki}\, w^j\, \delta_{il} \qquad \Longrightarrow \qquad \frac{\partial z^k}{\partial w^l}(p) =\ \delta_{kl}$$

so $\det(\frac{\partial z^k}{\partial w^l}(p)) \neq 0$. The chart is centered at p, so ϕ defines a holomorphic chart by the Holomorphic Inverse Function Theorem (up to choosing a smaller neighborhood of p). Differentiation and conjugation give

$$dz^k = dw^k + \sum_{j,i} a_{jki}\, w^j\, dw^i\ ,\ d\overline{z}^k = d\overline{w}^k + \sum_{j,i} a'_{kij}\, \overline{w}^j\, d\overline{w}^i$$

Consequently, it holds up to term of order at least two

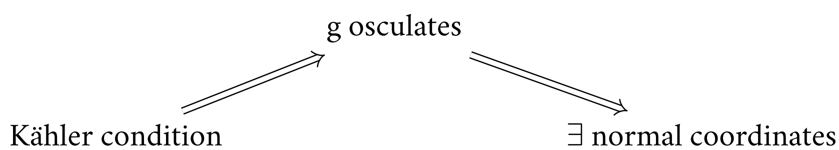
$$\begin{aligned}
&\sqrt{-1} \sum_k dz^k \wedge d\overline{z}^k = \\
&\qquad = \sqrt{-1} \sum_k (dw^k \wedge d\overline{w}^k + \sum_{j,i} (a_{jki}\, w^j\, dw^i) \wedge d\overline{w}^k + \\
&\qquad\quad + dw^k \wedge \sum_{j,i} a'_{kij}\, \overline{w}^j\, d\overline{w}^i) = \\
&\qquad = \sqrt{-1}\, (\sum_{j,k} \delta_{jk}\, dw^j \wedge d\overline{w}^k + \sum_{j,k} (\sum_i a_{jki}\, w^i)\, dw^j \wedge d\overline{w}^k + \\
&\qquad\quad + \sum_{j,k} (\sum_i a'_{jki}\, \overline{w}^i)\, dw^j \wedge d\overline{w}^k) = \\
&\qquad = \sqrt{-1} \sum_{j,k} g^w_{j\bar{k}}\, dw^j \wedge d\overline{w}^k = \omega
\end{aligned}$$

that is, $g^z_{j\bar{k}} = \delta_{jk} + O(|z|^2)$. □

Remark 12. The possibility of finding around a point holomorphic coordinates centered at that point such that with respect to these

$$g_{j\bar{k}} = \delta_{jk} + O(|z|^2)$$

is referred to as *"the metric g osculates in the origin to order two to the standard metric"*. In Proposition 4 we showed that



This diagram can be closed so that the properties are equivalent, i.e. the existence of normal coordinates implies the metric is Kähler. Indeed, let $p \in M$ and pick normal coordinates $z^1, \ldots, z^n$ at p. Then

$$\frac{\partial g_{j\bar{k}}}{\partial z^i}(p) = 0 = \frac{\partial g_{i\bar{k}}}{\partial z^j}(p)$$

and we can apply Lemma 10.

In addition to their geometric meaning, normal coordinates are a powerful tool to simplify computations. We will use them extensively throughout this work for this purpose.

An immediate application can be found in the next property. By Lemma 2, M admits a natural orientation: choose the holomorphic structure as the atlas that gives positive orientation.

If g is a Kähler metric on M, we can describe the volume form of g with respect to the natural orientation of M in terms of the Kähler form. More precisely:

Proposition 5. Let $(M,\ g,\ \omega)$ be a connected Kähler manifold. Then $\frac{\omega^n}{n!}$ is the volume form of g with respect to the natural orientation of M, where ω^n is the wedge product of ω with itself n-times.

Proof. Pick normal coordinates $z^i = x^i + \sqrt{-1}y^i$ at $p \in M$. Then $\det(g(p)) = 1$ and the volume form of g at p is given by

$$\mathrm{vol}_g(p) = \bigwedge_i dx^i\big|_p \wedge dy^i\big|_p$$

Notice that for all i

$$dx^i \wedge dy^i = \tfrac{\sqrt{-1}}{2}\, dz^i \wedge d\overline{z}^i$$

which implies

$$\mathrm{vol}_g(p) = (\tfrac{\sqrt{-1}}{2})^n \bigwedge_i dz^i\big|_p \wedge d\overline{z}^i\big|_p$$

On the other hand (see Remark 11), in normal coordinates

$$\omega_p = \tfrac{\sqrt{-1}}{2} \sum_i dz^i\big|_p \wedge d\overline{z}^i\big|_p$$

Hence, by the Multinomial Theorem

$$\begin{aligned}\omega_p^n &= (\frac{\sqrt{-1}}{2})^n \sum_{k_1+\cdots+k_n=n} \frac{n!}{k_1!\dots k_n!} \bigwedge_i (dz^i\big|_p \wedge d\overline{z}^i\big|_p)^{k_i} = \\ &= (\frac{\sqrt{-1}}{2})^n\, n! \bigwedge_i dz^i\big|_p \wedge d\overline{z}^i\big|_p = n!\, \mathrm{vol}_g(p)\end{aligned}$$

where we used that $(dz^i\big|_p \wedge d\overline{z}^i\big|_p)^{k_i} = 0$ whenever $k_i \geq 2$. $\square$

Remark 13. We could apply the Multinomial Theorem in Proposition 5 because $\wedge$ is commutative on 2-forms. We can also use it to compute (in normal coordinates at p) for $1 \leq a \leq n$

$$\begin{aligned}\omega^a &= (\frac{\sqrt{-1}}{2})^a \sum_{k_1+\cdots+k_n=a} \frac{a!}{k_1!\dots k_n!} \bigwedge_{i=1}^{m} (dz^i\big|_p \wedge d\overline{z}^i\big|_p)^{k_i} = \\ &= (\frac{\sqrt{-1}}{2})^a\, a! \sum_{r_1<\cdots<r_a} \bigwedge_{i=1}^{a} dz^{r_i}\big|_p \wedge d\overline{z}^{r_i}\big|_p\end{aligned}$$

2.3 Intrinsic geometry of Kähler manifolds

Throughout this section, we explore the more geometric side of Hermitian and Kähler manifolds, focusing on developing the complex version of the tools that describe the intrinsic geometry of a Riemannian manifold in the real case.

We need the following preliminary construction. We can "extend the Lie bracket $[\cdot,\ \cdot]$ by $\mathbb{C}$-bilinearity": that is, for $X,\ Y$ complex vector fields, we set

$$\begin{aligned}[X,\ Y]_{\mathbb{C}} := [\text{Re}(X),\ \text{Re}(Y)] - [\text{Im}(X),\ \text{Im}(Y)] +\\ + \sqrt{-1}\,([\text{Re}(X),\ \text{Im}(Y)] + [\text{Im}(X),\ \text{Re}(Y)])\end{aligned}$$

Then $[\cdot,\ \cdot]_{\mathbb{C}}$ coincides with $[\cdot,\ \cdot]$ on real vector fields, it is $\mathbb{C}$-bilinear and skew-symmetric. For $U,\ V,\ W$ complex vector fields, we compute explicitly

- $\begin{aligned}\text{Re}([U,\ [V,\ W]_{\mathbb{C}}]_{\mathbb{C}}) = [\text{Re}(U),\ [\text{Re}(V),\ \text{Re}(W)]] - [\text{Re}(U),\ [\text{Im}(V),\ \text{Im}(W)]] +\\ - [\text{Im}(U),\ [\text{Re}(V),\ \text{Im}(W)]] - [\text{Im}(U),\ [\text{Im}(V),\ \text{Re}(W)]]\end{aligned}$
- $\begin{aligned}\text{Im}([U,\ [V,\ W]_{\mathbb{C}}]_{\mathbb{C}}) = [\text{Im}(U),\ [\text{Re}(V),\ \text{Re}(W)]] - [\text{Im}(U),\ [\text{Im}(V),\ \text{Im}(W)]] +\\ + [\text{Re}(U),\ [\text{Re}(V),\ \text{Im}(W)]] + [\text{Re}(U),\ [\text{Im}(V),\ \text{Re}(W)]]\end{aligned}$

from which we deduce that $[\cdot,\ \cdot]_{\mathbb{C}}$ satisfies the Jacobi identity. Furthermore, let $f\colon M \to \mathbb{R}$ be smooth. We compute for $X,\ Y$ complex vector fields

- $\begin{aligned}\text{Re}([X,\ fY]_{\mathbb{C}}) &= [\text{Re}(X),\ f\text{Re}(Y)] - [\text{Im}(X),\ f\text{Im}(Y)] =\\ &= \text{Re}(X)(f)\text{Re}(Y) + f[\text{Re}(X),\ \text{Re}(Y)] +\\ &\quad - \text{Im}(X)(f)\text{Im}(Y) - f[\text{Im}(X),\ \text{Im}(Y)] =\\ &= \text{Re}(X(f)Y + f[X,\ Y]_{\mathbb{C}})\end{aligned}$
- $\begin{aligned}\text{Im}([X,\ fY]_{\mathbb{C}}) &= [\text{Re}(X),\ f\text{Im}(Y)] + [\text{Im}(X),\ f\text{Re}(Y)] =\\ &= \text{Re}(X)(f)\text{Im}(Y) + f[\text{Re}(X),\ \text{Im}(Y)] +\\ &\quad + \text{Im}(X)(f)\text{Re}(Y) + f[\text{Im}(X),\ \text{Re}(Y)] =\\ &= \text{Im}(X(f)Y + f[X,\ Y]_{\mathbb{C}})\end{aligned}$

which means $[X,\ fY]_{\mathbb{C}} = X(f)Y + f[X,\ Y]_{\mathbb{C}}$. Then, for $g\colon M \to \mathbb{C}$ smooth

$$\begin{aligned}[X,\ gY]_{\mathbb{C}} = [X,\ \mathrm{Re}(g)Y]_{\mathbb{C}} + \sqrt{-1}[X,\ \mathrm{Im}(g)Y]_{\mathbb{C}} =& \\ = X(\mathrm{Re}(g))Y + \mathrm{Re}(g)[X,\ Y]_{\mathbb{C}} +& \\ + \sqrt{-1}\left(X(\mathrm{Im}(g))Y + \mathrm{Im}(g)[X,\ Y]_{\mathbb{C}}\right) =& \\ = X(g)Y + g[X,\ Y]_{\mathbb{C}}&\end{aligned}$$

Moreover, it readily follows from the definition that

$$\overline{[X,\ Y]_{\mathbb{C}}} = [\overline{X},\ \overline{Y}]_{\mathbb{C}}$$

With abuse of notation, we denote $[\cdot,\ \cdot]_{\mathbb{C}}$ simply by $[\cdot,\ \cdot]$.

Lemma 11 (Schwarz). Pick $z^1, \dots, z^n$ holomorphic coordinates. For all $i,\ j$

- $[\frac{\partial}{\partial z^i},\ \frac{\partial}{\partial z^j}] = [\frac{\partial}{\partial \overline{z}^i},\ \frac{\partial}{\partial z^j}] = [\frac{\partial}{\partial \overline{z}^i},\ \frac{\partial}{\partial \overline{z}^j}] = 0$
- $[\frac{\partial}{\partial z^i},\ J\frac{\partial}{\partial z^j}] = [\frac{\partial}{\partial \overline{z}^i},\ J\frac{\partial}{\partial z^j}] = [\frac{\partial}{\partial \overline{z}^i},\ J\frac{\partial}{\partial \overline{z}^j}] = [J\frac{\partial}{\partial \overline{z}^i},\ \frac{\partial}{\partial z^j}] = 0$
- $[J\frac{\partial}{\partial z^i},\ J\frac{\partial}{\partial z^j}] = [J\frac{\partial}{\partial \overline{z}^i},\ J\frac{\partial}{\partial z^j}] = [J\frac{\partial}{\partial \overline{z}^i},\ J\frac{\partial}{\partial \overline{z}^j}] = 0$

Proof. By Schwarz's lemma

$$\begin{aligned}4[\frac{\partial}{\partial z^k},\ \frac{\partial}{\partial z^j}] = [\frac{\partial}{\partial x^k},\ \frac{\partial}{\partial x^j}] - [\frac{\partial}{\partial y^k},\ \frac{\partial}{\partial y^j}] +& \\ + \sqrt{-1}\left([\frac{\partial}{\partial x^k},\ \frac{\partial}{\partial y^j}] + [\frac{\partial}{\partial y^k},\ \frac{\partial}{\partial x^j}]\right) = 0&\end{aligned}$$

The others relations of the first point are similar; the claim follows by $\mathbb{C}$-bilinearity of $[\cdot,\ \cdot]$ and the fact that holomorphic and anti-holomorphic coordinate vector fields are eigenvectors for J. □

The next result is deduced because the complexified exterior derivative and Lie bracket satisfy the same formal properties of their real counterparts.

Corollary 2. Let ω be a complex k-form. Then $d\omega$ is given on complex vector fields $X_1, \ldots, X_{k+1}$ by the formula

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) = \\ &= \sum_{j=1}^{k+1} (-1)^{j+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) + \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}) \end{aligned}$$

2.3.1 The Levi-Civita connection and the Kähler condition

From now on, let g be a Hermitian metric on M with fundamental form ω. The Levi-Civita connection ∇ of g can be "extended to the complex tensor algebra": that is, for X complex vector field and T complex tensor, we set

$$\begin{aligned} \nabla^{\mathbb{C}}_X T := &\nabla_{\text{Re}(X)}\text{Re}(T) - \nabla_{\text{Im}(X)}\text{Im}(T) + \\ &+ \sqrt{-1}\left(\nabla_{\text{Re}(X)}\text{Im}(T) + \nabla_{\text{Im}(X)}\text{Re}(T)\right) \end{aligned}$$

Then $\nabla^{\mathbb{C}}$ coincides with ∇ on real vector fields and tensors. Since ∇ is a connection, we have the following properties for $\nabla^{\mathbb{C}}$:

- $\nabla^{\mathbb{C}}$ is $C^\infty(M, \mathbb{C})$-linear on the first entry and $\mathbb{C}$-linear on the second entry.
- **(Leibniz rule)** If $f\colon M \to \mathbb{R}$ is smooth, we compute for X complex vector field and T complex tensor

$$\begin{aligned} \nabla^{\mathbb{C}}_X fT = &\nabla_{\text{Re}(X)} f\text{Re}(T) - \nabla_{\text{Im}(X)} f\text{Im}(T) + \\ &+ \sqrt{-1}\left(\nabla_{\text{Re}(X)} f\text{Im}(T) + \nabla_{\text{Im}(X)} f\text{Re}(T)\right) = \\ = &\text{Re}(X)(f)\text{Re}(T) + f\nabla_{\text{Re}(X)}\text{Re}(T) + \\ &- \text{Im}(X)(f)\text{Im}(T) - f\nabla_{\text{Im}(X)}\text{Im}(T) + \\ &+ \sqrt{-1}(\text{Re}(X)(f)\text{Im}(T) + f\nabla_{\text{Re}(X)}\text{Im}(T) + \\ &+ \text{Im}(X)(f)\text{Re}(T) + f\nabla_{\text{Im}(X)}\text{Re}(T)) = X(f)T + f\nabla^{\mathbb{C}}_X T \end{aligned}$$

Then, for $g\colon M \to \mathbb{C}$ smooth we have

$$
\begin{aligned}
\nabla^{\mathbb{C}}_X gT &= \nabla^{\mathbb{C}}_X \mathrm{Re}(g)T + \sqrt{-1}\nabla^{\mathbb{C}}_X \mathrm{Im}(g)T = \\
&= X(\mathrm{Re}(g))T + \mathrm{Re}(g)\nabla^{\mathbb{C}}_X T + \\
&\quad + \sqrt{-1}\left(X(\mathrm{Im}(g))T + \mathrm{Im}(g)\nabla^{\mathbb{C}}_X T\right) = \\
&= X(g)T + g\nabla^{\mathbb{C}}_X T
\end{aligned}
$$

- For $g\colon M \to \mathbb{C}$ smooth

$$
\begin{aligned}
\nabla^{\mathbb{C}}_X g &= \nabla_{\mathrm{Re}(X)}\mathrm{Re}(g) - \nabla_{\mathrm{Im}(X)}\mathrm{Im}(g) + \\
&\quad + \sqrt{-1}\left(\nabla_{\mathrm{Re}(X)}\mathrm{Im}(g) + \nabla_{\mathrm{Im}(X)}\mathrm{Re}(g)\right) \\
&= \mathrm{Re}(X)(\mathrm{Re}(g)) - \mathrm{Im}(X)(\mathrm{Im}(g)) + \\
&\quad + \sqrt{-1}\left(\mathrm{Re}(X)(\mathrm{Im}(g) + \mathrm{Im}(X)(\mathrm{Re}(g))\right) = X(g)
\end{aligned}
$$

- **(Leibniz rule)** For X real vector field and T, S complex tensors

$$
\begin{aligned}
* \ \mathrm{Re}(\nabla^{\mathbb{C}}_X T \otimes S) &= \nabla_X \mathrm{Re}(T \otimes S) = \\
&= (\nabla_X \mathrm{Re}(T)) \otimes \mathrm{Re}(S) + \mathrm{Re}(T) \otimes (\nabla_X \mathrm{Re}(S)) + \\
&\quad - (\nabla_X \mathrm{Im}(T)) \otimes \mathrm{Im}(S) - \mathrm{Im}(T) \otimes (\nabla_X \mathrm{Im}(S)) = \\
&= \mathrm{Re}((\nabla^{\mathbb{C}}_X T) \otimes S + T \otimes (\nabla^{\mathbb{C}}_X S)) \\
* \ \mathrm{Im}(\nabla^{\mathbb{C}}_X T \otimes S) &= \nabla_X \mathrm{Im}(T \otimes S) = \\
&= (\nabla_X \mathrm{Re}(T)) \otimes \mathrm{Im}(S) + \mathrm{Re}(T) \otimes (\nabla_X \mathrm{Im}(S)) + \\
&\quad + (\nabla_X \mathrm{Im}(T)) \otimes \mathrm{Re}(S) + \mathrm{Im}(T) \otimes (\nabla_X \mathrm{Re}(S)) = \\
&= \mathrm{Im}((\nabla^{\mathbb{C}}_X T) \otimes S + T \otimes (\nabla^{\mathbb{C}}_X S))
\end{aligned}
$$

which means that

$$
\nabla^{\mathbb{C}}_X T \otimes S = (\nabla^{\mathbb{C}}_X T) \otimes S + T \otimes (\nabla^{\mathbb{C}}_X S)
$$

Then, for Y complex vector field

$$\begin{aligned}\nabla^{\mathbb{C}}_Y T\otimes S &= \nabla^{\mathbb{C}}_{\mathrm{Re}(Y)} T\otimes S + \sqrt{-1}\nabla^{\mathbb{C}}_{\mathrm{Im}(Y)} T\otimes S = \\ &= (\nabla^{\mathbb{C}}_{\mathrm{Re}(Y)} T)\otimes S + T\otimes(\nabla^{\mathbb{C}}_{\mathrm{Re}(Y)} S) + \\ &\quad + \sqrt{-1}((\nabla^{\mathbb{C}}_{\mathrm{Im}(Y)} T)\otimes S + T\otimes(\nabla^{\mathbb{C}}_{\mathrm{Im}(Y)} S)) = \\ &= (\nabla^{\mathbb{C}}_Y T)\otimes S + T\otimes(\nabla^{\mathbb{C}}_Y S)\end{aligned}$$

- Let tr be a contraction $(k+1,\ l+1)\to(k,\ l)$. Then, for X complex vector field and T a complex $(k+1,\ l+1)$-tensor

$$\begin{aligned}\mathrm{tr}(\nabla^{\mathbb{C}}_X T) &= \mathrm{tr}(\nabla_{\mathrm{Re}(X)}\mathrm{Re}(T) - \nabla_{\mathrm{Im}(X)}\mathrm{Im}(T)) + \\ &\quad + \sqrt{-1}\mathrm{tr}(\big(\nabla_{\mathrm{Re}(X)}\mathrm{Im}(T) + \nabla_{\mathrm{Im}(X)}\mathrm{Re}(T)\big)) = \\ &= \nabla_{\mathrm{Re}(X)}\mathrm{tr}(\mathrm{Re}(T)) - \nabla_{\mathrm{Im}(X)}\mathrm{tr}(\mathrm{Im}(T)) + \\ &\quad + \sqrt{-1}\left(\nabla_{\mathrm{Re}(X)}\mathrm{tr}(\mathrm{Im}(T)) + \nabla_{\mathrm{Im}(X)}\mathrm{tr}(\mathrm{Re}(T))\right)) = \\ &= \nabla^{\mathbb{C}}_X\mathrm{tr}(T)\end{aligned}$$

From the properties described so far, it follows as in the real case that

- **(P)** for ω complex 1-form and $X,\ Y$ complex vector fields

$$\nabla^{\mathbb{C}}_X\omega(Y) = (\nabla^{\mathbb{C}}_X\omega)(Y) + \omega(\nabla^{\mathbb{C}}_X Y)$$

- **(E)** for a complex $(k,\ l)$-tensor T, if X is a complex vector field, then for any l complex vector fields Y_i and k complex 1-forms ω_j

$$\begin{aligned}&(\nabla^{\mathbb{C}}_X T)(\omega_1,\dots,\omega_k,Y_1,\dots,Y_l) = \\ &\quad = X(T(\omega_1,\dots,\omega_k,Y_1,\dots,Y_l)) + \\ &\quad\quad - \sum_{j=1}^{k} T(\omega_1,\dots,\nabla^{\mathbb{C}}_X\omega_j,\dots,\omega_k,Y_1,\dots,Y_l) + \\ &\quad\quad - \sum_{i=1}^{l} T(\omega_1,\dots,\omega_k,Y_1,\dots,\nabla^{\mathbb{C}}_X Y_i,\dots,Y_l)\end{aligned}$$

Furthermore, since ∇ is, in particular, the Levi-Civita connection, we have the following relations.

- **($\nabla^{\mathbb{C}}$ is symmetric)** For $X,\ Y$ complex vector fields

$$\begin{aligned}\nabla^{\mathbb{C}}_X Y - \nabla^{\mathbb{C}}_Y X &= \nabla_{\mathrm{Re}(X)}\mathrm{Re}(Y) - \nabla_{\mathrm{Im}(X)}\mathrm{Im}(Y) + \\ &\quad - \nabla_{\mathrm{Re}(Y)}\mathrm{Re}(X) + \nabla_{\mathrm{Im}(Y)}\mathrm{Im}(X) + \\ &\quad + \sqrt{-1}\left(\nabla_{\mathrm{Re}(X)}\mathrm{Im}(Y) + \nabla_{\mathrm{Im}(X)}\mathrm{Re}(Y) + \right. \\ &\quad \left. - \nabla_{\mathrm{Re}(Y)}\mathrm{Im}(X) + \nabla_{\mathrm{Im}(Y)}\mathrm{Re}(X)\right) = \\ &= [\mathrm{Re}(X),\ \mathrm{Re}(Y)] - [\mathrm{Im}(X),\ \mathrm{Im}(Y)] + \\ &\quad + \sqrt{-1}\left([\mathrm{Re}(X),\ \mathrm{Im}(Y)] + [\mathrm{Im}(X),\ \mathrm{Re}(Y)]\right) = \\ &= [X,\ Y]\end{aligned}$$

- **($\nabla^{\mathbb{C}}$ is metric)** For X real vector field and $Y,\ Z$ complex vector fields

$$\begin{aligned}* \ \mathrm{Re}(\nabla^{\mathbb{C}}_X g(Y,\ Z)) &= \nabla_X g(\mathrm{Re}(Y),\ \mathrm{Re}(Z)) - \nabla_X g(\mathrm{Im}(Y),\ \mathrm{Im}(Z)) = \\ &= g(\nabla_X \mathrm{Re}(Y),\ \mathrm{Re}(Z)) + g(\mathrm{Re}(Y),\ \nabla_X \mathrm{Re}(Z)) + \\ &\quad - g(\nabla_X \mathrm{Im}(Y),\ \mathrm{Im}(Z)) - g(\mathrm{Im}(Y),\ \nabla_X \mathrm{Im}(Z)) = \\ &= \mathrm{Re}(g(\nabla^{\mathbb{C}}_X Y,\ Z) + g(Y,\ \nabla^{\mathbb{C}}_X Z)) \\ * \ \mathrm{Im}(\nabla^{\mathbb{C}}_X g(Y,\ Z)) &= \nabla_X g(\mathrm{Re}(Y),\ \mathrm{Im}(Z)) + \nabla_X g(\mathrm{Im}(Y),\ \mathrm{Re}(Z)) = \\ &= g(\nabla_X \mathrm{Re}(Y),\ \mathrm{Im}(Z)) + g(\mathrm{Re}(Y),\ \nabla_X \mathrm{Im}(Z)) + \\ &\quad + g(\nabla_X \mathrm{Im}(Y),\ \mathrm{Re}(Z)) + g(\mathrm{Im}(Y),\ \nabla_X \mathrm{Re}(Z)) = \\ &= \mathrm{Im}(g(\nabla^{\mathbb{C}}_X Y,\ Z) + g(Y,\ \nabla^{\mathbb{C}}_X Z))\end{aligned}$$

  which means $\nabla^{\mathbb{C}}_X g(Y,\ Z) = g(\nabla^{\mathbb{C}}_X Y,\ Z) + g(Y,\ \nabla^{\mathbb{C}}_X Z)$. But then for W complex vector field

$$\begin{aligned}\nabla^{\mathbb{C}}_W g(Y,\ Z) &= \nabla^{\mathbb{C}}_{\mathrm{Re}(W)} g(Y,\ Z) + \sqrt{-1}\nabla^{\mathbb{C}}_{\mathrm{Im}(W)} g(Y,\ Z) = \\ &= g(\nabla^{\mathbb{C}}_{\mathrm{Re}(W)} Y,\ Z) + g(Y,\ \nabla^{\mathbb{C}}_{\mathrm{Re}(W)} Z) + \\ &\quad + \sqrt{-1}(g(\nabla^{\mathbb{C}}_{\mathrm{Im}(W)} Y,\ Z) + g(Y,\ \nabla^{\mathbb{C}}_{\mathrm{Im}(W)} Z)) = \\ &= g(\nabla^{\mathbb{C}}_W Y,\ Z)\end{aligned}$$

From these relations, it follows as in the real case the *Koszul formula*

$$g(\nabla^{\mathbb{C}}_X Y,\ Z) = \frac{1}{2}(X(g(Y,\ Z)) + Y(g(Z,\ X)) - Z(g(X,\ Y)) + \\ - g(X,\ [Y,\ Z]) + g(Y,\ [Z,\ X]) + g(Z,\ [X,\ Y]))$$

where $X,\ Y,\ Z$ are complex vector fields. Finally, it is clear from the definition that for X complex vector field and T complex tensor

$$\overline{\nabla^{\mathbb{C}}_X T} = \nabla^{\mathbb{C}}_{\overline{X}} \overline{T}$$

With abuse of notation, we denote $\nabla^{\mathbb{C}}$ with ∇. When dealing with a Hermitian manifold, we always consider it equipped with its Levi-Civita connection.

We are now ready to analyze the strict interplay among the many structures that we have introduced.

Proposition 6. For all $X,\ Y,\ Z$ complex vector fields, the following identities hold:

1) $d\omega(X, Y, Z) = g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y)$

2) $2g((\nabla_X J)Y, Z) = d\omega(X, Y, Z) - d\omega(X, JY, JZ)$

Proof. The claim regards tensorial identities, so it is sufficient to prove it for a local frame, thanks to the local behavior of $d,\ \nabla$. Moreover, by Lemma 11, it is sufficient to prove the equations for $X,\ Y,\ Z$ complex vector fields such that

$$X,\ Y,\ Z,\ JY,\ JZ \ \text{ all commute}$$

Then, by Corollary 2 one has

a) $d\omega(X, Y, Z) = X(\omega(Y, Z)) + Y(\omega(Z, X)) + Z(\omega(X, Y))$

b) $d\omega(X, JY, JZ) = X(\omega(JY,\ JZ)) + JY(\omega(JZ,\ X)) + JZ(\omega(X,\ JY))$

On the other hand, for all $U,\ V$ complex vector fields it holds

$$\nabla_U JV = (\nabla_U J)V + J\nabla_U V \qquad (J)$$

Indeed, for all η complex 1-forms and $U,\ V$ complex vector fields, by **(E)** and **(P)**

$$\begin{aligned}\eta((\nabla_U J)V) &= (\nabla_U J)(\eta,\ V) = \\ &= \nabla_U J(\eta,\ V) - J(\eta,\ \nabla_U V) - J(\nabla_U \eta,\ V) = \\ &= \nabla_U \eta(JV) - (\nabla_U \eta)JV - \eta(J\nabla_U V) = \\ &= \eta(\nabla_U JV) - \eta(J\nabla_U V) = \eta(\nabla_U JV - J\nabla_U V)\end{aligned}$$

so $(\nabla_U J)V = \nabla_U JV - J\nabla_U V$. Since ω is skew-symmetric and ∇ is metric and symmetric, one computes

$$\begin{aligned}X(\omega(Y,Z)) &+ Y(\omega(Z,X)) + Z(\omega(X,Y)) = \\ &= \nabla_X g(JY,\ Z) + \nabla_Y g(JZ,\ X) + \nabla_Z g(JX,\ Y) = \\ &= g(\nabla_X JY,\ Z) + \omega(Y,\ \nabla_X Z) + \\ &\quad +\ g(\nabla_Y JZ,\ X) + \omega(Z,\ \nabla_Y X) + \\ &\quad +\ g(\nabla_Z JX,\ Y) + \omega(X,\ \nabla_Z Y) = \\ &= g((\nabla_X J)Y,\ Z) + \omega(\nabla_X Y,\ Z) + \omega(Y,\ \nabla_X Z) + \\ &\quad +\ g((\nabla_Y J)Z,\ X) + \omega(\nabla_Y Z,\ X) + \omega(Z,\ \nabla_Y X) + \\ &\quad +\ g((\nabla_Z J)X,\ Y) + \omega(\nabla_Z X,\ Y) + \omega(X,\ \nabla_Z Y)\ = \\ &= g((\nabla_X J)Y,\ Z) + \omega(\nabla_X Y,\ Z) + \omega(Y,\ \nabla_Z X) + \\ &\quad +\ g((\nabla_Y J)Z,\ X) + \omega(\nabla_Y Z,\ X) + \omega(Z,\ \nabla_X Y) + \\ &\quad +\ g((\nabla_Z J)X,\ Y) + \omega(\nabla_Z X,\ Y) + \omega(X,\ \nabla_Y Z)\ = \\ &= g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y)\end{aligned}$$

which proves **1)** because of **a)**. As for **2)**, observe that since g preserves J

$$\begin{aligned}g((\nabla_X J)Y, Z) &= g(\nabla_X JY, Z) - g(J(\nabla_X Y), Z) = \\ &= g(\nabla_X JY, Z) + g((\nabla_X Y), JZ)\end{aligned}$$

By the Koszul formula we then get

$$\begin{aligned}2g(\nabla_X JY, Z) &= X(g(JY, Z)) + JY(g(Z, X)) - Z(g(X, JY))\ = \\ &= X\omega(Y,\ Z) - JY\omega(JZ,\ X) + Z\omega(X,\ Y)\end{aligned}$$

and also

$$2g(\nabla_X Y, JZ) = X(g(Y, JZ)) + Y(g(JZ, X)) - JZ(g(X, Y)) =$$
$$= -X\omega(JY,\ JZ) + Y\omega(Z,\ X) - JZ\omega(X,\ JY)$$

Using **b)**, we consequently have

$$2g((\nabla_X J)Y, Z) = X\omega(Y,\ Z) - JY\omega(JZ,\ X) + Z\omega(X,\ Y) +$$
$$- X\omega(JY,\ JZ) + Y\omega(Z,\ X) - JZ\omega(X,\ JY) =$$
$$= d\omega(X, Y, Z) - d\omega(X, JY, JZ)$$

□

Corollary 3. g is Kähler if and only if *J is parallel with respect to* ∇, i.e. $\nabla J = 0$. In this case, it holds for any complex vector fields $X,\ Y$

$$\nabla_X JY = J\nabla_X Y$$

Proof. " $\Longrightarrow$ " By Proposition 6 (**2)**), one has for all $X,\ Y,\ Z$ real vector fields

$$g((\nabla_X J)Y, Z) = 0$$

and since the real g is positive definite, it follows for all $X,\ Y$ real vector fields

$$\nabla J(X,\ Y) = (\nabla_X J)Y = 0$$

Due to $\mathbb{C}$-linearity, we conclude that $\nabla J = 0$. In this case, from (J) of Proposition 6 we have for all $U,\ V$ complex vector fields

$$\nabla_U JV = J\nabla_U V$$

" $\Longleftarrow$ " It is a direct application of **1)** of Proposition 6. □

2.3.2 Curvature

Here, we define the objects related to the curvature of a Kähler manifold, using the Levi-Civita connection of its metric. For our purposes, we focus on a local description of such objects.

For the remainder of this section, assume g is Kähler. We adopt the following notational convention: denote

$$\nabla_{\frac{\partial}{\partial z^i}} \equiv \nabla_i \ , \ \nabla_{\frac{\partial}{\partial \bar{z}^i}} \equiv \nabla_{\bar{i}} \ , \ \frac{\partial}{\partial z^i} \equiv \partial_i \ , \ \frac{\partial}{\partial \bar{z}^i} \equiv \partial_{\bar{i}}$$

and let $(g^{i\bar{j}})$ denote the transpose-inverse to $(g_{i\bar{j}})$, i.e.

$$\sum_k g^{i\bar{k}} g_{j\bar{k}} = \delta_{ij} \ , \quad \sum_k g^{k\bar{i}} g_{k\bar{j}} = \delta_{ij}$$

Definition 13 (Christoffel's symbols). From Corollary 3, one has

$$J\nabla_j \partial_k = \nabla_j J \partial_k = \sqrt{-1}\, \nabla_j \partial_k$$

Hence, $\nabla_j \partial_k$ is a holomorphic vector field. The *holomorphic Christoffel's symbols* Γ^i_{jk} of the Levi-Civita connection on M are defined by

$$\nabla_j \partial_k = \sum_i \Gamma^i_{jk}\, \partial_i$$

Similarly, since $\nabla_{\bar{j}} \partial_{\bar{k}}$ is an anti-holomorphic vector field, the *anti-holomorphic Christoffel's symbols* $\Gamma^{\bar{i}}_{\bar{j}\bar{k}}$ are defined by

$$\nabla_{\bar{j}} \partial_{\bar{k}} = \sum_i \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\, \partial_{\bar{i}}$$

Notice that the mixed terms $\nabla_{\bar{a}} \partial_b, \ \nabla_a \partial_{\bar{b}}$ have not been considered. They are not relevant: as in Definition 13, for all $j, \ k$

- $\nabla_{\bar{j}} \partial_k$ is a holomorphic vector field
- $\nabla_k \partial_{\bar{j}}$ is an anti-holomorphic vector field

but Lemma 11 and the symmetry of ∇ yield $\nabla_{\bar{j}} \partial_k = \nabla_k \partial_{\bar{j}}$, so these equal 0 because of the splitting of complex vector fields.

Remark 14. Since ∇ preserves conjugation, we see that

$$\Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \overline{\Gamma^{i}_{jk}}$$

In particular, ∇ is locally completely determined by Γ^i_{jk}. Furthermore, by Lemma 11 and the symmetry of ∇ we have $\nabla_j \partial_k = \nabla_k \partial_j$, which translates to

$$\Gamma^i_{jk} = \Gamma^i_{kj} \text{ and } \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \Gamma^{\bar{i}}_{\bar{k}\bar{j}}$$

It is clear from the definition that the Christoffel symbols are strictly related to covariant derivation. The next examples highlight this fact.

Example 6. Fix $i,\ j,\ k$. Differentiate $dz^k(\partial_j) = \delta_{kj}$ with respect to ∇_i to obtain

$$0 = \nabla_i(dz^k \partial_j) = (\nabla_i dz^k)\partial_j + dz^k(\nabla_i \partial_j) = (\nabla_i dz^k)\partial_j + \Gamma^k_{ij}$$

which means

$$(\nabla_i dz^k)\partial_j = -\Gamma^k_{ij}$$

On the other hand, since $\nabla_i \partial_{\bar{j}} = 0$ and $\nabla_i dz^k(\partial_{\bar{j}}) = 0$, we deduce from above

$$(\nabla_i dz^k)\partial_{\bar{j}} = 0$$

Thus, $\nabla_i dz^k$ is a $(1,0)$-form and it admits the local decomposition

$$\nabla_i dz^k = -\sum_j \Gamma^k_{ij}\, dz^j$$

By conjugation, $\nabla_{\bar{i}} d\bar{z}^k$ is a $(0,1)$-form and it admits the local decomposition

$$\nabla_{\bar{i}} d\bar{z}^k = -\sum_j \Gamma^{\bar{k}}_{\bar{i}\bar{j}}\, d\bar{z}^j$$

The mixed terms $\nabla_{\bar{i}} dz^k$ are 0: for all j

- $\nabla_{\bar{i}} \partial_j = 0 \quad \overset{above}{\Longrightarrow} \quad (\nabla_{\bar{i}} dz^k)\partial_j = 0$
- $dz^k(\nabla_{\bar{i}} \partial_{\bar{j}}) = 0 \quad \overset{above}{\Longrightarrow} \quad (\nabla_{\bar{i}} dz^k)\partial_{\bar{j}} = 0$

By conjugation, $\nabla_i d\bar{z}^k = 0$.

Example 7. Let $\sum_{i,j} a_{i\bar{j}}\, dz^i \otimes d\bar{z}^j$ be a tensor. By Example 6 and the Leibniz rule, we compute for any p

- $$\begin{aligned}\nabla_p\Big(\sum_{i,j} a_{i\bar{j}}\, dz^i \otimes d\bar{z}^j\Big) &= \\ &= \sum_{i,j} (\partial_p\, a_{i\bar{j}})\, dz^i \otimes d\bar{z}^j + a_{i\bar{j}}(\nabla_p\, dz^i) \otimes d\bar{z}^j + a_{i\bar{j}}\, dz^i \otimes (\nabla_p\, d\bar{z}^j) \ = \\ &= \sum_{i,j} (\partial_{\overline{p}}\, a_{i\bar{j}})\, dz^i \otimes d\bar{z}^j - a_{i\bar{j}} \left(\sum_l \Gamma^i_{pl}\, dz^l\right) \otimes d\bar{z}^j = \\ &= \sum_{i,j} (\partial_{\overline{p}}\, a_{i\bar{j}} - \sum_l \Gamma^l_{pi}\, a_{l\bar{j}})\, dz^i \otimes d\bar{z}^j\end{aligned}$$
- $$\begin{aligned}\nabla_{\overline{p}}\Big(\sum_{i,j} a_{i\bar{j}}\, dz^i \otimes d\bar{z}^j\Big) &= \\ &= \sum_{i,j} (\partial_{\overline{p}}\, a_{i\bar{j}})\, dz^i \otimes d\bar{z}^j + a_{i\bar{j}}(\nabla_{\overline{p}}\, dz^i) \otimes d\bar{z}^j + a_{i\bar{j}}\, dz^i \otimes (\nabla_{\overline{p}}\, d\bar{z}^j) \ = \\ &= \sum_{i,j} (\partial_{\overline{p}}\, a_{i\bar{j}})\, dz^i \otimes d\bar{z}^j - a_{i\bar{j}}\, dz^i \otimes \left(\sum_l \Gamma^{\bar{j}}_{\overline{p}\bar{l}}\, d\bar{z}^l\right) = \\ &= \sum_{i,j} (\partial_{\overline{p}}\, a_{i\bar{j}} - \sum_l \Gamma^{\bar{l}}_{\overline{p}\bar{j}}\, a_{i\bar{l}})\, dz^i \otimes d\bar{z}^j\end{aligned}$$

Lemma 12. In terms of the coefficients $g_{j\bar{k}}$, the Christoffel symbols are given by

$$\Gamma^i_{jk} = \sum_l g^{i\bar{l}} \partial_j g_{k\bar{l}}$$

Proof. Since ∇ is metric, for all p one has $\nabla_p g = 0$. In holomorphic coordinates, this reads by Example 7

$$0 = \ \partial_p\, g_{j\bar{k}} - \sum_a \Gamma^a_{pj}\, g_{a\bar{k}}$$

Consequently

$$\sum_l g^{i\bar{l}} \partial_j g_{k\bar{l}} = \ \sum_{l,a} \Gamma^a_{jk}\, g_{a\bar{l}}\, g^{i\bar{l}} \ = \sum_{l,a} \Gamma^a_{jk}\, \delta_{ia} = \Gamma^i_{jk}$$

□

In Riemannian geometry, one interpretation of the curvature tensor is the measure of commutativity between covariant derivatives. In Kähler geometry, we adapt this concept to the complex structure on the manifold.

Definition 14 (Curvature tensor). Using Corollary 3, we see that

$$(\nabla_k \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_k)\partial_i \text{ is a holomorphic vector field}$$

The $(1,3)$-*curvature tensor* is defined in holomorphic coordinates by

$$(\nabla_k \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_k)\partial_i = \sum_j R_i{}^j{}_{k\bar{l}}\, \partial_j$$

To better understand the meaning of the curvature tensor, let us find its relation to the metric. Since we compute

$$\begin{aligned}(\nabla_k \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_k)\partial_i &= \nabla_k \nabla_{\bar{l}}\, \partial_i - \nabla_{\bar{l}} \nabla_k \partial_i = \\ &= -\nabla_{\bar{l}} \left(\sum_j \Gamma^j_{ki}\, \partial_j \right) = \\ &= -\sum_j \big((\nabla_{\bar{l}} \Gamma^j_{ki}) \partial_j + \Gamma^j_{ki} \nabla_{\bar{l}}\, \partial_j \big) = -\sum_j \partial_{\bar{l}}\, \Gamma^j_{ki}\, \partial_j\end{aligned}$$

by uniqueness of components, we can express the $(1,3)$-curvature tensor in terms of the Christoffel symbols as

$$R_i{}^j{}_{k\bar{l}} = -\partial_{\bar{l}}\, \Gamma^j_{ki}$$

Furthermore, by Lemma 12 and the Leibniz rule

$$\begin{aligned}R_i{}^j{}_{k\bar{l}} &= -\sum_q \partial_{\bar{l}}\, (g^{j\bar{q}} \partial_k\, g_{i\bar{q}}) = \\ &= -\sum_q \big((\partial_{\bar{l}}\, g^{j\bar{q}})(\partial_k\, g_{i\bar{q}}) + g^{j\bar{q}} \partial_{\bar{l}}\, \partial_k\, g_{i\bar{q}} \big)\end{aligned}$$

Again, by the Leibniz rule

$$0 = \partial_{\bar{l}}\, (\sum_k g^{k\bar{i}} g_{k\bar{j}}) = \sum_k\, (\partial_{\bar{l}}\, g^{k\bar{i}}) g_{k\bar{j}} + g^{k\bar{i}} (\partial_{\bar{l}}\, g_{k\bar{j}})$$

which leads to

$$\partial_{\bar{l}}\, g^{j\bar{q}} = \sum_k (\partial_{\bar{l}}\, g^{k\bar{q}})\delta_{jk} =$$
$$= \sum_{p,k} g^{j\bar{p}}(\partial_{\bar{l}}\, g^{k\bar{q}})g_{k\bar{p}} = -\sum_{p,k} g^{j\bar{p}}(\partial_{\bar{l}}\, g_{k\bar{p}})g^{k\bar{q}}$$

Applying this to the previous expression, we can express the $(1,3)$-curvature tensor in terms of the coefficients $g_{j\bar{k}}$ as

$$R_i{}^j{}_{k\bar{l}} = -\sum_q g^{j\bar{q}}\partial_{\bar{l}}\,\partial_k\, g_{i\bar{q}} + \sum_{p,r,q} g^{r\bar{q}}g^{j\bar{p}}(\partial_k\, g_{i\bar{q}})(\partial_{\bar{l}}\, g_{r\bar{p}}) \qquad (R1)$$

Now consider the $(0,4)$-*curvature tensor*, defined "lowering the second index of the $(1,3)$-curvature tensor through the metric": that is, in holomorphic coordinates

$$R_{i\bar{j}k\bar{l}} := \sum_t g_{t\bar{j}}\, R_i{}^t{}_{k\bar{l}}$$

Then $(R1)$ becomes

$$R_{i\bar{j}k\bar{l}} = \partial_{\bar{l}}\,\partial_k\, g_{i\bar{j}} + \sum_{r,q} g^{r\bar{q}}(\partial_k\, g_{i\bar{q}})(\partial_{\bar{l}}\, g_{r\bar{j}}) \qquad (R2)$$

Picking normal coordinates around $p \in M$ results into

$$R_i{}^j{}_{k\bar{l}}(p) = R_{i\bar{j}k\bar{l}}(p) = -(\partial_k\,\partial_{\bar{l}}\, g_{i\bar{j}})(p)$$

Hence, we deduce the following.

Remark 15 (Geometric meaning of the curvature tensor). The curvature tensor measures the obstruction, for the chosen holomorphic coordinates, to be normal coordinates (around a specific point).

By Remark 12, this indicates the deviation of the metric from the Euclidean metric, up to second order, in the selected chart.

Lemma 13 (Symmetries of the curvature tensor). The $(0,4)$-curvature tensor enjoys the following symmetries:

$$R_{i\bar{j}k\bar{l}} = R_{i\bar{l}k\bar{j}} = R_{k\bar{j}i\bar{l}} = R_{k\bar{l}i\bar{j}}$$

Proof. For any $i,\ j,\ k,\ l$, by Lemma 10, Lemma 11 and ($R2$)

$$\begin{aligned}
\cdot\ R_{i\bar{j}k\bar{l}} &= \partial_{\bar{l}}\,\partial_k\, g_{i\bar{j}} + \sum_{r,q} g^{r\bar{q}}(\partial_k\, g_{i\bar{q}})(\partial_{\bar{l}}\, g_{r\bar{j}}) = \\
&= \partial_{\bar{l}}\,\partial_i\, g_{k\bar{j}} + \sum_{r,q} g^{r\bar{q}}(\partial_i\, g_{k\bar{q}})(\partial_{\bar{l}}\, g_{r\bar{j}}) = R_{k\bar{j}i\bar{l}} \\
\cdot\ R_{i\bar{j}k\bar{l}} &= \partial_{\bar{l}}\,\partial_k\, g_{i\bar{j}} + \sum_{r,q} g^{r\bar{q}}(\partial_k\, g_{i\bar{q}})(\partial_{\bar{l}}\, g_{r\bar{j}}) = \\
&= \partial_{\bar{j}}\,\partial_i\, g_{k\bar{l}} + \sum_{r,q} g^{r\bar{q}}(\partial_i\, g_{k\bar{q}})(\partial_{\bar{j}}\, g_{r\bar{l}}) = R_{k\bar{l}i\bar{j}}
\end{aligned}$$

□

The following objects play a crucial role in the study of Kähler manifolds, as they encode not only geometric but also significant topological information about the underlying space.

Definition 15 (Ricci curvature). The *Ricci curvature* is defined to be the contraction over the third and fourth indices of the $(0,4)$-curvature tensor with the metric tensor. That is, in holomorphic coordinates

$$R_{i\bar{j}} = \sum_{k,l} g^{k\bar{l}}\, R_{i\bar{j}k\bar{l}}$$

A characteristic fact regarding the Ricci curvature of a Kähler manifold is that, unlike in the Riemannian case, its components can be expressed in a nice way as functions depending only on the coefficient $g_{j\bar{k}}$.

Lemma 14. In holomorphic coordinates it holds

$$R_{i\bar{j}} = -\partial_i\, \partial_{\bar{j}} \log \det(g_{p\bar{q}})$$

Proof. For an invertible Hermitian matrix $A = A(z)$, Jacobi's formula states

$$\partial_i \log \det(A) = \operatorname{tr}(A^{-1}\, \partial_i A)$$

Consequently, by Lemma 11, Lemma 12, Lemma 13

$$\begin{aligned} -\partial_i\, \partial_{\bar{j}} \log \det(g_{p\bar{q}}) &= -\partial_{\bar{j}} \sum_{p,q} (g^{p\bar{q}} \partial_i\, g_{p\bar{q}}) = \\ &= -\sum_p \partial_{\bar{j}}\, \Gamma^p_{ip} = \\ &= \sum_p R_p{}^p{}_{i\bar{j}} = \\ &= \sum_{p,l} g^{p\bar{l}}\, R_{p\bar{l}i\bar{j}} = \\ &= \sum_{p,l} g^{p\bar{l}}\, R_{i\bar{j}p\bar{l}} = R_{i\bar{j}} \end{aligned}$$

□

Definition 16 (Ricci form). The *Ricci form* $\operatorname{Ric}(\omega)$ is the $(1,\ 1)$-form defined in holomorphic coordinates by

$$\operatorname{Ric}(\omega) := \sqrt{-1}\, R_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$$

Notice that by Lemma 14 and Example 5

$$\operatorname{Ric}(\omega) = -\sqrt{-1}\, \partial\bar{\partial} \log \det(g_{j\bar{k}})$$

Then $\operatorname{Ric}(\omega)$ is a closed form. Moreover, it is real since $(g_{j\bar{k}})$ is a positive definite Hermitian matrix, so it defines a cohomology class in the De Rham cohomology of M. Furthermore, if $\omega_h,\ \omega_g$ are two Kähler forms on M

$$\operatorname{Ric}(\omega_h) - \operatorname{Ric}(\omega_g) = -\sqrt{-1}\, \partial\bar{\partial} \log \tfrac{\det(h_{j\bar{k}})}{\det(g_{j\bar{k}})}$$

Since the determinant of the metric transforms under a change of holomorphic coordinates by the formula

$$\det(g'_{j\bar{k}}) = \det(J^{-1})^2 \det(g_{j\bar{k}})$$

where J is the complex Jacobian matrix of the change of coordinates, $\frac{\det(h_{j\bar{k}})}{\det(g_{j\bar{k}})}$ is a globally defined function. This means that

$$\mathrm{Ric}(\omega_h) - \mathrm{Ric}(\omega_g) \text{ is an exact form}$$

and the two Ricci forms define the same cohomology class.

Definition 17 (First Chern class). The De Rham cohomology class

$$c_1(M) := \tfrac{1}{2\pi} \, [\mathrm{Ric}(\omega)] \in H^2(M, \, \mathbb{R})$$

uniquely determined by any Ricci form on M, is called the *first Chern class*.

Chapter 3

Calabi's conjecture

This chapter is devoted to present the solution proposed by S. T. Yau ([Yau77], [Yau78]) of the Calabi conjecture [Cal54].

Throughout this chapter, M always denotes a compact, connected Kähler manifold of complex dimension $m \geq 2$ with Kähler metric g and Kähler form ω.

3.1 Preliminary tools

In this section, we develop the necessary tools to address the resolution of Calabi's conjecture.

> **Lemma 15** ($\partial\overline{\partial}$ **lemma).** If ϕ and η are two real $(1,\ 1)$-forms on M in the same cohomology class, then there is a smooth map $h\colon\ M \to \mathbb{R}$ such that
>
> $$\eta = \phi + \sqrt{-1}\partial\overline{\partial}h$$

Proof. Since $\phi,\ \eta$ are cohomologous 2-forms, there exists a real 1-form α such that

$$\eta = \phi + d\alpha$$

In particular, since $\phi,\ \eta$ are $(1,\ 1)$-forms the same holds for $d\alpha$. If we split $\alpha = \alpha^{1,0} + \alpha^{0,1}$ into its $(1,\ 0)$ and $(0,\ 1)$ parts, by different bigrading

$$\partial \alpha^{1,0} = \overline{\partial} \alpha^{0,1} = 0$$

and consequently

$$\eta = \phi + \partial \alpha^{0,1} + \overline{\partial} \alpha^{1,0}$$

The function

$$\partial^* \alpha^{1,0} \ := -\sum_{j,k} g^{j\overline{k}} \, \nabla_{\overline{k}} \, \alpha_j$$

has zero integral on M, so (see [Szé14], Thm. 2.12 p. 33) there is a smooth map $f\colon M \to \mathbb{C}$ such that

$$\partial^* \alpha^{1,0} \ = \Delta f = -\partial^* \partial f$$

which implies $\partial^*(\alpha^{1,0} + \partial f) = 0$. Moreover, since $\partial(\alpha^{1,0} + \partial f) = 0$, then $\alpha^{1,0} + \partial f$ is a ∂-harmonic form; but g is Kähler, so the form is also $\overline{\partial}$-harmonic and in particular it is $\overline{\partial}$-closed. This means

$$\overline{\partial} \alpha^{1,0} = -\overline{\partial} \partial f$$

Notice that $\alpha^{0,1} = \overline{\alpha^{1,0}}$ because α is real. Hence

$$\eta - \phi = -\overline{\partial}\partial f - \partial \overline{\partial f} = \partial\overline{\partial}(f - \overline{f}) = \sqrt{-1}\, \partial\overline{\partial}(2\mathrm{Im}(f))$$

□

Theorem 3 (**Maximum principle**)**.** The only subharmonic maps on M are the constant maps.

Proof. Recall that for f, g smooth maps on M

$$\Delta(fg) = \Delta(f)g + f\Delta(g) + 2g(\nabla f,\ \nabla g)$$

By Stokes' theorem

$$\begin{aligned} 0 &= \int_M \Delta(fg)\, dV_g = \\ &= \int_M \Delta(f) g\, dV_g \ + \int_M f\Delta(g)\, dV_g \ + \int_M 2g(\nabla f,\ \nabla g)\, dV_g \end{aligned}$$

Now let $f \in C^\infty(M)$ be subharmonic. First, assume $f \geq 0$ on M. Applying the above relation with $g = \frac{1}{2}f$ gives

$$0 = \int_M \Delta(f) f \, dV_g \, + 2 \int_M (||\nabla f||_g)^2 \, dV_g \, \geq \int_M (||\nabla f||_g)^2 \, dV_g$$

which implies $||\nabla f||_g = 0$ on M, and f is constant being M connected. In the general case, being M compact

$$\inf_M (f) \in \mathbb{R}$$

The map $f - \inf_M (f)$ is then subharmonic and non-negative on M, hence constant. This implies that f is constant. $\square$

3.2 The Monge-Ampère equation

We have seen that necessary conditions for a $(1,1)$-form to be the Ricci form of some Kähler metric are to be real, closed and represent the first Chern class in cohomology. We now reduce Calabi's conjecture to a complex partial differential equation through these conditions (and an ulterior hypothesis).

Let η be a closed real $(1,1)$-form on M such that $[\eta] = 2\pi c_1(M)$. If $\eta = \text{Ric}(\omega')$ for some Kähler metric g' on M, then

$$\eta = -\sqrt{-1}\partial\overline{\partial}\log\det(g'_{j\overline{k}})$$

and being $\eta, \text{Ric}(\omega)$ cohomologous, by Lemma 15 there is $F \in C^\infty(M)$ such that

$$\sqrt{-1}\partial\overline{\partial}F = \text{Ric}(\omega) - \eta = \sqrt{-1}\partial\overline{\partial}\log\frac{\det(g'_{j\overline{k}})}{\det(g_{j\overline{k}})}$$

It follows that

$$\partial\overline{\partial}\left(F - \log\frac{\det(g'_{j\overline{k}})}{\det(g_{j\overline{k}})}\right) = 0$$

that is, $\log\frac{\det(g'_{j\overline{k}})}{\det(g_{j\overline{k}})} - F$ is harmonic on M. By Theorem 3, for a constant $c \in \mathbb{R}$

$$\log\frac{\det(g'_{j\overline{k}})}{\det(g_{j\overline{k}})} = F + c$$

or equivalently

$$\det(g'_{j\overline{k}}) = Ce^F\det(g_{j\overline{k}})$$

Notice $C > 0$. If we further assume that $\omega,\ \omega'$ are cohomologous, then by Lemma 15 there is $\varphi \in C^\infty(M)$ such that

$$\sqrt{-1}\partial\overline{\partial}\varphi = \omega' - \omega$$

In holomorphic coordinates, this becomes (recall Example 5)

$$\sqrt{-1}\sum_{i,j}\partial_i\,\partial_{\overline{j}}\,\varphi\,dz^i \wedge d\overline{z}^j = \sqrt{-1}\sum_{i,j}(g'_{i\overline{j}} - g_{i\overline{j}})\,dz^i \wedge d\overline{z}^j$$

Thus, by uniqueness of components, we can write the previous equation as

$$\det(g_{j\bar{k}} + \partial_j \, \partial_{\bar{k}} \, \varphi) = Ce^F \det(g_{j\bar{k}}) \tag{M-A}$$

Equation (M-A) is called the *Monge-Ampère* equation. We just proved that if the existence part of Calabi's conjecture is verified, then (M-A) has a solution φ that is smooth.

Conversely, let η be a closed real $(1,1)$-form on M such that $[\eta] = 2\pi c_1(M)$. Set (M-A), where $F \in C^\infty(M)$ is given by

$$\sqrt{-1}\partial\overline{\partial}F = \mathrm{Ric}(\omega) - \eta$$

If for some constant $C > 0$ we can solve (M-A) for a smooth φ such that the tensor given in local coordinates by

$$\sum_{j,k} (g_{j\bar{k}} + \partial_j \, \partial_{\bar{k}} \, \varphi) \, (dz^j \otimes d\overline{z}^k + d\overline{z}^k \otimes dz^j)$$

defines a Kähler metric g' on M, then

- $\omega' - \omega = \sqrt{-1}\partial\overline{\partial}\varphi$
- proceeding backwards in the above discussion, we have

$$\begin{aligned}\mathrm{Ric}(\omega) - \eta = \sqrt{-1}\partial\overline{\partial} \log \frac{\det(g'_{j\bar{k}})}{\det(g_{j\bar{k}})} = \\ &= -\sqrt{-1}\partial\overline{\partial} \log\det\,(g_{j\bar{k}}) + \sqrt{-1}\partial\overline{\partial}\log\det\,(g'_{j\bar{k}}) = \\ &= \mathrm{Ric}(\omega) - \mathrm{Ric}(\omega')\end{aligned}$$

which means that $\omega' \in [\omega]$ and $\eta = \mathrm{Ric}(\omega')$, and the existence part of Calabi's conjecture is solved.

Also note that by construction, the uniqueness part of Calabi's conjecture corresponds to the uniqueness of the smooth solution of (M-A) such that the above expression defines a Kähler metric.

Remark 16. In fact, there is only one possibility for the constant C in (M-A). Indeed, if $\omega' - \omega = d\phi$, then

$$(\omega')^2 = \omega^2 + d\phi \wedge \omega + \omega \wedge d\phi + (d\phi)^2 = \omega^2 + d\psi$$

that is, $(\omega')^2 - \omega^2$ are cohomologous. Inductively, we get by Proposition 5 that

$$dV_{g'} = \tfrac{(\omega')^m}{m!}, \; dV_g = \tfrac{\omega^m}{m!} \text{ are cohomologous}$$

Therefore, we compute by Stokes' theorem

$$\begin{aligned}
\int_M Ce^F dV_g &= \sum_i \int_{(U_i)} \alpha_i \, Ce^F \sqrt{\det(g)} \, dx^1 \wedge \cdots \wedge dy^m = \\
&= \sum_i \int_{(U_i)} \alpha_i \, Ce^F \det(g_{j\overline{k}}) \, dx^1 \wedge \cdots \wedge dy^m = \\
&= \sum_i \int_{(U_i)} \alpha_i \det(g'_{j\overline{k}}) \, dx^1 \wedge \cdots \wedge dy^m = \\
&= \sum_i \int_{(U_i)} \alpha_i \sqrt{\det(g')} \, dx^1 \wedge \cdots \wedge dy^m = \\
&= \int_M dV_{g'} = \int_M dV_g = \text{vol}(M)
\end{aligned}$$

where we picked a holomorphic atlas $\{(U_i, \; \psi_i = (x^i, \; y^i))\}_i$ and a partition of unity $\{\alpha_i\}_i$ subordinated to it in order to compute the integral.

Remark 17. Since g is a Kähler metric, the expression

$$\sum_{j,k} (g_{j\overline{k}} + \partial_j \, \partial_{\overline{k}} \, \varphi) \, (dz^j \otimes d\overline{z}^k + d\overline{z}^k \otimes dz^j)$$

already defines a symmetric tensor g' compatible with J and whose fundamental form satisfies the Kähler condition. Hence, g' is actually a Kähler metric if and only if g' is positive-definite.

For the remainder of this work, we say in this case that φ is *g-positive*.

3.2.1 The strategy

We plan to solve (M-A), using the continuity method. To do this, we need *a priori estimates* of φ. For this purpose, we make a few arrangements. We can translate F by a constant so that

$$C = 1$$

and the relation $\sqrt{-1}\partial\overline{\partial}F = \mathrm{Ric}(\omega) - \eta$ still holds, because only the second-order derivatives of F are involved. Similarly, we can translate φ by a constant because the relation $\omega' - \omega = \sqrt{-1}\partial\overline{\partial}\varphi$ involves only the second-order derivatives of φ. In particular, we can assume

$$\textstyle\int_M \varphi \, dV_g = 0$$

and then translate again φ during our discussion, when necessary.

To simplify the notation, we will omit the evaluation on the point during the upcoming computations. Furthermore, we will make most of our calculations using the following.

> **Lemma 16 (Special coordinates).** Around $p \in M$, we can choose normal coordinates with respect to g such that the matrix $(\partial_j \, \partial_{\overline{k}} \, \varphi)\,(p)$ is diagonal.

Proof. Start with normal coordinates $w^1, \dots, w^m$ with respect to g around p. Denote the coefficients of $g,\ g'$ and the partial derivatives of φ at p with respect to these coordinates by

$$g^w_{j\overline{k}}\,, \quad (g')^w_{j\overline{k}}\,, \quad \partial_j \, \partial_{\overline{k}} \, \varphi^w$$

Since $(\partial_j \, \partial_{\overline{k}} \, \varphi^w) = \left((g')^w_{j\overline{k}}\right) - \left(g^w_{j\overline{k}}\right)$ is Hermitian, by the Spectral Theorem exists $C \in U(m)$ such that

$$C^* \, (\partial_j \, \partial_{\overline{k}} \, \varphi^w) \, C = \mathrm{diag}(\lambda_1, \dots, \lambda_m)$$

where $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ are the eigenvalues of $(\partial_j \, \partial_{\overline{k}} \, \varphi^w)$. Then

$$C^* \left(g^w_{j\overline{k}}\right) C = I_m \,, \quad C^* \left((g')^w_{j\overline{k}}\right) C = I_m + \mathrm{diag}(\lambda_1, \dots, \lambda_m)$$

As described in Proposition 4, take the holomorphic coordinates $z^1, \ldots, z^m$ obtained by the linear change of coordinates

$$\left[\frac{\partial}{\partial z^j}\right]_{\mathcal{W}} = \left(\begin{array}{c|c} 0 & \overline{C} \\ \hline C & 0 \end{array}\right) \left[\frac{\partial}{\partial w^j}\right]_{\mathcal{W}}$$

$$\left[\frac{\partial}{\partial \overline{z}^k}\right]_{\mathcal{W}} = \left(\begin{array}{c|c} 0 & \overline{C} \\ \hline C & 0 \end{array}\right) \left[\frac{\partial}{\partial \overline{w}^k}\right]_{\mathcal{W}}$$

Denote the coefficients of $g,\ g'$ and the partial derivatives of φ at p with respect to the new coordinates by

$$g^z_{j\overline{k}}\,, \quad (g')^z_{j\overline{k}}\,, \quad \partial_j\, \partial_{\overline{k}}\, \varphi^z$$

Then, since $w^1, \ldots, w^m$ are normal coordinates, we compute at p

- for all $j,\ k$

$$\begin{aligned} g^z_{j\overline{k}} &= g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \overline{z}^k}) = \\ &= \left[\frac{\partial}{\partial w^j}\right]^T_{\mathcal{W}} \left(\begin{array}{c|c} 0 & \overline{C} \\ \hline C & 0 \end{array}\right)^T (g^w) \left(\begin{array}{c|c} 0 & \overline{C} \\ \hline C & 0 \end{array}\right) \left[\frac{\partial}{\partial \overline{w}^k}\right]_{\mathcal{W}} = \\ &= \left[\frac{\partial}{\partial w^j}\right]^T_{\mathcal{W}} \left(\begin{array}{c|c} 0 & C^T \left(g^w_{j\overline{k}}\right)^T \overline{C} \\ \hline C^* \left(g^w_{j\overline{k}}\right) C & 0 \end{array}\right) \left[\frac{\partial}{\partial \overline{w}^k}\right]_{\mathcal{W}} = \delta_{jk} \end{aligned}$$

- for all $i,\ j,\ k$

$$\begin{aligned} \partial_i\, g^z_{j\overline{k}} = &\left(\partial_i \left[\frac{\partial}{\partial z^j}\right]^T_{\mathcal{W}}\right) (g^w) \left[\frac{\partial}{\partial \overline{z}^k}\right]_{\mathcal{W}} + \\ &+ \left[\frac{\partial}{\partial z^j}\right]^T_{\mathcal{W}} (\partial_i\, (g^w)\,) \left[\frac{\partial}{\partial \overline{z}^k}\right]_{\mathcal{W}} + \\ &+ \left[\frac{\partial}{\partial z^j}\right]^T_{\mathcal{W}} (g^w) \left(\partial_i \left[\frac{\partial}{\partial \overline{z}^k}\right]_{\mathcal{W}}\right) = 0 \end{aligned}$$

  where we used that the components of the new vector fields with respect to the previous ones are constant. By conjugation: $\partial_{\overline{i}}\, g^z_{j\overline{k}} = 0$ for all $i,\ j,\ k$.

This means that $z^1, \ldots, z^m$ are normal coordinates at p. Furthermore, denoting diag$(\lambda_1, \ldots, \lambda_m)$ by D, we compute at p

$$(g')^z_{j\bar{k}} = \left[\frac{\partial}{\partial z^j}\right]^T_{\mathcal{W}} (((g')^w)\,) \left[\frac{\partial}{\partial \bar{z}^k}\right]_{\mathcal{W}} =$$
$$= \left[\frac{\partial}{\partial w^j}\right]^T_{\mathcal{W}} \left(\begin{array}{c|c} 0 & I_m + D \\ \hline I_m + D & 0 \end{array}\right) \left[\frac{\partial}{\partial \overline{w}^k}\right]_{\mathcal{W}} = \delta_{jk} + \delta_{jk}\lambda_k$$

This means that $((g')^z_{j\bar{k}})$ is diagonal. Hence, the same holds for $(\partial_j\, \partial_{\bar{k}}\, \varphi^z)$. □

Here we present some useful relations regarding the normal coordinates given by Lemma 16, which we call *special coordinates*.

Let $p \in M$ and pick special coordinates around p. Then $(g'_{j\bar{k}})$ is diagonal and

$$(g'_{j\bar{j}}) = 1 + \partial_j\, \partial_{\bar{j}}\, \varphi \ \text{ for all } j$$

In particular, the $1 + \partial_j\, \partial_{\bar{j}}\, \varphi$'s are the eigenvalues of $(g'_{j\bar{k}})$, which is Hermitian and positive definite. Hence

$$1 + \partial_j\, \partial_{\bar{j}}\, \varphi \in \mathbb{R}^{>0} \ \text{ for all } j$$

We then also have

$$0 < \sum_j (1 + \partial_j\, \partial_{\bar{j}}\, \varphi) = m + \Delta\varphi$$

where Δ is the Laplacian with respect to g, and for all $i,\ j$

$$(g')^{i\bar{j}} = \tfrac{\delta_{ij}}{1 + \partial_i\, \partial_{\bar{i}}\, \varphi}$$

3.2.2 Second Order Estimates

For this step, we follow [Yau78] and look for an upper estimate of $\Delta\varphi$. Pick holomorphic coordinates. Rewrite (M-A) as

$$\log \det \left(g_{i\bar{j}} + \partial_i\, \partial_{\bar{j}}\, \varphi\right) - \log \det \left(g_{i\bar{j}}\right) = F$$

We differentiate it with respect to ∂_k by Jacobi's formula, obtaining

$$\partial_k\, F = \sum_{i,j} (g')^{i\bar{j}} \left(\partial_k\, g_{i\bar{j}} + \partial_i\, \partial_{\bar{j}}\, \partial_k\, \varphi\right) - \sum_{i,j} g^{i\bar{j}} \left(\partial_k\, g_{i\bar{j}}\right) \tag{$E1$}$$

Recall that we computed

$$\partial_{\bar{l}}\, (g')^{i\bar{j}} = -\sum_{t,n} (g')^{i\bar{n}} (\partial_{\bar{l}}\, (g')_{t\bar{n}}) (g')^{t\bar{j}}$$

Therefore, differentiating ($E1$) with respect to $\partial_{\bar{l}}$ gives

$$\begin{aligned}
\partial_k\, \partial_{\bar{l}}\, F = & \\
= -\sum_{i,j,t,n} & (g')^{t\bar{j}}\, (g')^{i\bar{n}} \left(\partial_{\bar{l}}\, g_{t\bar{n}} + \partial_t\, \partial_{\bar{n}}\, \partial_{\bar{l}}\, \varphi\right) \left(\partial_k\, g_{i\bar{j}} + \partial_i\, \partial_{\bar{j}}\, \partial_k\, \varphi\right) + \\
& + \sum_{i,j} (g')^{i\bar{j}} \left(\partial_k\, \partial_{\bar{l}}\, g_{i\bar{j}} + \partial_i\, \partial_{\bar{j}}\, \partial_k\, \partial_{\bar{l}}\, \varphi\right) + \\
& + \sum_{i,j,t,n} g^{t\bar{j}}\, g^{i\bar{n}} (\partial_{\bar{l}}\, g_{t\bar{n}})(\partial_k\, g_{i\bar{j}}) - \sum_{i,j} g^{i\bar{j}} \left(\partial_k\, \partial_{\bar{l}}\, g_{i\bar{j}}\right)
\end{aligned} \tag{$E2$}$$

If Δ' is the Laplacian with respect to g', then by the Leibniz rule

$$\begin{aligned}
\Delta'(\Delta\varphi) &= \sum_{k,l} (g')^{k\bar{l}}\, \partial_k\, \partial_{\bar{l}} \left(\sum_{i,j} g^{i\bar{j}}\, \partial_i\, \partial_{\bar{j}}\, \varphi\right) = \\
&= \sum_{i,j,k,l} (g')^{k\bar{l}}\, g^{i\bar{j}}\, \partial_i\, \partial_{\bar{j}}\, \partial_k\, \partial_{\bar{l}}\, \varphi + \\
&\quad + \sum_{i,j,k,l} (g')^{k\bar{l}}\, (\partial_k\, \partial_{\bar{l}}\, g^{i\bar{j}})(\partial_i\, \partial_{\bar{j}}\, \varphi) + \\
&\quad + \sum_{i,j,k,l} (g')^{k\bar{l}}\, (\partial_k\, g^{i\bar{j}})(\partial_i\, \partial_{\bar{j}}\, \partial_{\bar{l}}\, \varphi) + \\
&\quad + \sum_{i,j,k,l} (g')^{k\bar{l}}\, (\partial_{\bar{l}}\, g^{i\bar{j}})(\partial_i\, \partial_{\bar{j}}\, \partial_k\, \varphi)
\end{aligned} \tag{$E3$}$$

Now fix $p \in M$. Since the Laplacian is independent of the choice of coordinates, we can pick special coordinates around p. Consequently, the following hold:

- $g_{i\bar{j}} = g^{i\bar{j}} = \delta_{ij}$
- $\partial_k \, g_{i\bar{j}} = \partial_{\bar{l}} \, g_{i\bar{j}} = 0$
- $\partial_k \, g^{i\bar{j}} = -\sum_{t,n} g^{i\bar{n}}(\partial_k \, g_{t\bar{n}}) g^{t\bar{j}} = 0 \,, \quad \partial_{\bar{l}} \, g^{i\bar{j}} = -\sum_{t,n} g^{i\bar{n}}(\partial_{\bar{l}} \, g_{t\bar{n}}) g^{t\bar{j}} = 0$
- $R_{i\bar{j}k\bar{l}} = -\partial_k \, \partial_{\bar{l}} \, g_{i\bar{j}}$
- $\partial_i \, \partial_{\bar{j}} \, \varphi = \delta_{ij} \, \partial_i \, \partial_{\bar{i}} \, \varphi$

and we compute

$$
\begin{aligned}
\partial_k \, \partial_{\bar{l}} \, g^{i\bar{j}} &= -\partial_k \left(\sum_{t,n} g^{i\bar{n}}(\partial_{\bar{l}} \, g_{t\bar{n}}) g^{t\bar{j}} \right) = \\
&= -\sum_{t,n} (\partial_k \, g^{i\bar{n}})(\partial_{\bar{l}} \, g_{t\bar{n}}) \, g^{t\bar{j}} \; - \sum_{t,n} g^{i\bar{n}} \, (\partial_k \, \partial_{\bar{l}} \, g_{t\bar{n}}) \, g^{t\bar{j}} + \\
&\quad - \sum_{t,n} g^{i\bar{n}} \, (\partial_{\bar{l}} \, g_{t\bar{n}}) \, (\partial_k \, g^{t\bar{j}}) = -\partial_k \, \partial_{\bar{l}} \, g_{j\bar{i}} = R_{j\bar{i}k\bar{l}}
\end{aligned}
$$

We apply these relations to $(E2)$, obtaining

$$
\begin{aligned}
\Delta F &= \sum_k \partial_k \, \partial_{\bar{k}} \, F = \\
&= -\sum_{i,j,t,n,k} \frac{\delta_{tj}}{1 + \partial_t \, \partial_{\bar{t}} \, \varphi} \, \frac{\delta_{in}}{1 + \partial_i \, \partial_{\bar{i}} \, \varphi} \, (\partial_t \, \partial_{\bar{n}} \, \partial_{\bar{k}} \, \varphi) \left(\partial_i \, \partial_{\bar{j}} \, \partial_k \, \varphi \right) + \\
&\quad + \sum_{k,i,j} \frac{\delta_{ij}}{1 + \partial_i \, \partial_{\bar{i}} \, \varphi} \left(-R_{i\bar{j}k\bar{k}} + \partial_i \, \partial_{\bar{j}} \, \partial_k \, \partial_{\bar{k}} \, \varphi \right) + \sum_{i,j,k} \delta_{ij} \left(R_{i\bar{j}k\bar{k}} \right) = \qquad (E4) \\
&= -\sum_{i,j,k} \frac{1}{1 + \partial_j \, \partial_{\bar{j}} \, \varphi} \, \frac{1}{1 + \partial_i \, \partial_{\bar{i}} \, \varphi} \, (\partial_j \, \partial_{\bar{i}} \, \partial_{\bar{k}} \, \varphi) \left(\partial_i \, \partial_{\bar{j}} \, \partial_k \, \varphi \right) + \\
&\quad + \sum_{i,k} \frac{1}{1 + \partial_i \, \partial_{\bar{i}} \, \varphi} \, (-R_{i\bar{i}k\bar{k}} + \partial_i \, \partial_{\bar{i}} \, \partial_k \, \partial_{\bar{k}} \, \varphi) + \sum_{i,k} R_{i\bar{i}k\bar{k}}
\end{aligned}
$$

and to $(E3)$, obtaining

$$\begin{aligned}\Delta'(\Delta\varphi) = & \sum_{i,j,k,l} \frac{\delta_{kl}}{1+\partial_k\,\partial_{\bar{l}}\,\varphi}\,\delta_{ij}\,\partial_i\,\partial_{\bar{j}}\,\partial_k\,\partial_{\bar{l}}\,\varphi + \\ & + \sum_{i,j,k,l} \frac{\delta_{kl}}{1+\partial_k\,\partial_{\bar{l}}\,\varphi}\,(R_{j\bar{i}k\bar{l}})(\delta_{ij}\,\partial_i\,\partial_{\bar{i}}\,\varphi) = \\ = & \sum_{i,k} \frac{1}{1+\partial_k\,\partial_{\bar{k}}\,\varphi}\,\partial_i\,\partial_{\bar{i}}\,\partial_k\,\partial_{\bar{k}}\,\varphi + \\ & + \sum_{i,k} \frac{1}{1+\partial_k\,\partial_{\bar{k}}\,\varphi}\,(R_{i\bar{i}k\bar{k}})(\partial_i\,\partial_{\bar{i}}\,\varphi)\end{aligned} \tag{$E5$}$$

Then, we combine $(E4)$ and $(E5)$ to get

$$\begin{aligned}&\Delta'(\Delta\varphi) - \Delta F = \\ &= \sum_{i,k} \frac{\partial_i\,\partial_{\bar{i}}\,\partial_k\,\partial_{\bar{k}}\,\varphi}{1+\partial_k\,\partial_{\bar{k}}\,\varphi} + \sum_{i,k} \frac{(R_{i\bar{i}k\bar{k}})(\partial_i\,\partial_{\bar{i}}\,\varphi)}{1+\partial_k\,\partial_{\bar{k}}\,\varphi} + \\ &+ \sum_{i,j,k} \frac{(\partial_j\,\partial_{\bar{i}}\,\partial_{\bar{k}}\,\varphi)\big(\partial_i\,\partial_{\bar{j}}\,\partial_k\,\varphi\big)}{(1+\partial_j\,\partial_{\bar{j}}\,\varphi)(1+\partial_i\,\partial_{\bar{i}}\,\varphi)} + \\ &- \sum_{i,k} \frac{(-R_{i\bar{i}k\bar{k}} + \partial_i\,\partial_{\bar{i}}\,\partial_k\,\partial_{\bar{k}}\,\varphi)}{1+\partial_i\,\partial_{\bar{i}}\,\varphi} - \sum_{i,k} R_{i\bar{i}k\bar{k}} = \\ &= \sum_{i,j,k} \frac{(\partial_j\,\partial_{\bar{i}}\,\partial_{\bar{k}}\,\varphi)\big(\partial_i\,\partial_{\bar{j}}\,\partial_k\,\varphi\big)}{(1+\partial_j\,\partial_{\bar{j}}\,\varphi)(1+\partial_i\,\partial_{\bar{i}}\,\varphi)} + \\ &+ \sum_{i,k} \frac{(R_{i\bar{i}k\bar{k}})(\partial_i\,\partial_{\bar{i}}\,\varphi)}{1+\partial_k\,\partial_{\bar{k}}\,\varphi} + \sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{1+\partial_i\,\partial_{\bar{i}}\,\varphi} - \sum_{i,k} R_{i\bar{i}k\bar{k}}\end{aligned} \tag{$E6$}$$

The left-hand side of $(E6)$ is a real number, so it can be estimated. In particular, by Lemma 8, Lemma 11, φ being real and $g_{i\bar{j}}$ being Hermitian, we see for all $i,\ j,\ k$

- $\overline{\partial_i\,\partial_{\bar{i}}\,\varphi} = \overline{\partial_i\,\overline{\partial_i\,\varphi}} = \partial_{\bar{i}}\,\partial_i\,\varphi = \partial_i\,\partial_{\bar{i}}\,\varphi$
- $\overline{(\partial_j\,\partial_{\bar{i}}\,\partial_{\bar{k}}\,\varphi)\big(\partial_i\,\partial_{\bar{j}}\,\partial_k\,\varphi\big)} = \big(\,\partial_{\bar{j}}\,\partial_i\,\partial_k\,\varphi\big)\,(\,\partial_{\bar{i}}\,\partial_j\,\partial_{\bar{k}}\,\varphi) = (\partial_j\,\partial_{\bar{i}}\,\partial_{\bar{k}}\,\varphi)\,\big(\partial_i\,\partial_{\bar{j}}\,\partial_k\,\varphi\big)$
- $\overline{R_{i\bar{i}k\bar{k}}} = -\overline{\partial_k\,\partial_{\bar{k}}\,g_{i\bar{i}}} = -\partial_{\bar{k}}\,\partial_k\,\overline{g_{i\bar{i}}} = -\partial_k\,\partial_{\bar{k}}\,g_{i\bar{i}} = R_{i\bar{i}k\bar{k}}$

so each summand on the right-hand side of $(E6)$ is a real number, and it can be estimated separately. We compute by Lemma 13

$$
\begin{aligned}
&\sum_{i,k} \frac{(R_{i\bar{i}k\bar{k}})(\partial_i \partial_{\bar{i}} \varphi)}{1+\partial_k \partial_{\bar{k}} \varphi} + \sum_{i,k} \frac{R_{i\bar{i}k\bar{k}}}{1+\partial_i \partial_{\bar{i}} \varphi} - \sum_{i,k} R_{i\bar{i}k\bar{k}} = \\
&= \sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{\partial_i \partial_{\bar{i}} \varphi}{1+\partial_k \partial_{\bar{k}} \varphi} - \sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{\partial_i \partial_{\bar{i}} \varphi}{1+\partial_i \partial_{\bar{i}} \varphi} = \\
&= \sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{\partial_i \partial_{\bar{i}} \varphi(\partial_i \partial_{\bar{i}} \varphi - \partial_k \partial_{\bar{k}} \varphi)}{(1+\partial_i \partial_{\bar{i}} \varphi)(1+\partial_k \partial_{\bar{k}} \varphi)} = \\
&= \frac{1}{2} (\sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{\partial_i \partial_{\bar{i}} \varphi(\partial_i \partial_{\bar{i}} \varphi - \partial_k \partial_{\bar{k}} \varphi)}{(1+\partial_i \partial_{\bar{i}} \varphi)(1+\partial_k \partial_{\bar{k}} \varphi)} + \\
&\quad + \sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{\partial_i \partial_{\bar{i}} \varphi(\partial_i \partial_{\bar{i}} \varphi - \partial_k \partial_{\bar{k}} \varphi)}{(1+\partial_i \partial_{\bar{i}} \varphi)(1+\partial_k \partial_{\bar{k}} \varphi)}) = \\
&= \frac{1}{2} (\sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{\partial_i \partial_{\bar{i}} \varphi(\partial_i \partial_{\bar{i}} \varphi - \partial_k \partial_{\bar{k}} \varphi)}{(1+\partial_i \partial_{\bar{i}} \varphi)(1+\partial_k \partial_{\bar{k}} \varphi)} + \\
&\quad + \sum_{i,k} R_{k\bar{k}i\bar{i}} \frac{\partial_i \partial_{\bar{i}} \varphi(\partial_i \partial_{\bar{i}} \varphi - \partial_k \partial_{\bar{k}} \varphi)}{(1+\partial_i \partial_{\bar{i}} \varphi)(1+\partial_k \partial_{\bar{k}} \varphi)}) = \\
&= \frac{1}{2} (\sum_{i,k} R_{i\bar{i}k\bar{k}} (\frac{\partial_i \partial_{\bar{i}} \varphi(\partial_i \partial_{\bar{i}} \varphi - \partial_k \partial_{\bar{k}} \varphi)}{(1+\partial_i \partial_{\bar{i}} \varphi)(1+\partial_k \partial_{\bar{k}} \varphi)} + \\
&\quad + \frac{\partial_k \partial_{\bar{k}} \varphi(\partial_k \partial_{\bar{k}} \varphi - \partial_i \partial_{\bar{i}} \varphi)}{(1+\partial_i \partial_{\bar{i}} \varphi)(1+\partial_k \partial_{\bar{k}} \varphi)})) = \\
&= \frac{1}{2} \sum_{i,k} R_{i\bar{i}k\bar{k}} \frac{(\partial_k \partial_{\bar{k}} \varphi - \partial_i \partial_{\bar{i}} \varphi)^2}{(1+\partial_i \partial_{\bar{i}} \varphi)(1+\partial_k \partial_{\bar{k}} \varphi)} \geq \\
&\geq \frac{1}{2} \inf_{i \neq k} (R_{i\bar{i}k\bar{k}}) (\sum_{i,k} \frac{(1+\partial_k \partial_{\bar{k}} \varphi)^2}{(1+\partial_i \partial_{\bar{i}} \varphi)(1+\partial_k \partial_{\bar{k}} \varphi)} + \\
&\quad + \sum_{i,k} \frac{(1+\partial_i \partial_{\bar{i}} \varphi)^2}{(1+\partial_i \partial_{\bar{i}} \varphi)(1+\partial_k \partial_{\bar{k}} \varphi)} + \\
&\quad - 2 \sum_{i,k} \frac{(1+\partial_k \partial_{\bar{k}} \varphi)(1+\partial_i \partial_{\bar{i}} \varphi)}{(1+\partial_i \partial_{\bar{i}} \varphi)(1+\partial_k \partial_{\bar{k}} \varphi)}) = \\
&= \inf_{i \neq k} (R_{i\bar{i}k\bar{k}}) (\sum_{i,k} \frac{1+\partial_i \partial_{\bar{i}} \varphi}{1+\partial_k \partial_{\bar{k}} \varphi} - m^2)
\end{aligned}
\qquad (D1)
$$

We insert $(D1)$ into $(E6)$, obtaining

$$\begin{aligned}\Delta'(\Delta\varphi) \geq \Delta F + \sum_{i,j,k} \frac{(\partial_j\, \partial_{\bar{i}}\, \partial_{\bar{k}}\, \varphi)\, \big(\partial_i\, \partial_{\bar{j}}\, \partial_k\, \varphi\big)}{(1+\partial_j\, \partial_{\bar{j}}\, \varphi)(1+\partial_i\, \partial_{\bar{i}}\, \varphi)} + \\ + \inf_{i\neq k}(R_{i\bar{i}k\bar{k}})\, (\sum_{i,k} \frac{1+\partial_i\, \partial_{\bar{i}}\, \varphi}{1+\partial_k\, \partial_{\bar{k}}\, \varphi} - m^2)\end{aligned} \tag{D2}$$

Let $C > 0$ be a constant to be determined later. By the Leibniz rule

$$\begin{aligned}&\Delta'(e^{-C\varphi}\,(m+\Delta\varphi)) = \\ &\quad = \sum_i \frac{1}{1+\partial_i\, \partial_{\bar{i}}\, \varphi}\, \partial_i\, \partial_{\bar{i}}\, (e^{-C\varphi}\,(m+\Delta\varphi)) = \\ &\quad = \sum_i \frac{1}{1+\partial_i\, \partial_{\bar{i}}\, \varphi}\, \partial_i\, \big(-Ce^{-C\varphi}(\partial_{\bar{i}}\, \varphi)(m+\Delta\varphi) + e^{-C\varphi}\, \partial_{\bar{i}}\, (\Delta\varphi)\big) = \\ &\quad = C^2\, e^{-C\varphi}\, \left(\sum_i \frac{(\partial_i\, \varphi)(\partial_{\bar{i}}\, \varphi)}{1+\partial_i\, \partial_{\bar{i}}\, \varphi}\right)(m+\Delta\varphi) + \\ &\qquad - C\, e^{-C\varphi} \sum_i \frac{(\partial_i\, \varphi)(\partial_{\bar{i}}\, (\Delta\varphi)) + (\partial_i\, (\Delta\varphi))(\partial_{\bar{i}}\, \varphi)}{1+\partial_i\, \partial_{\bar{i}}\, \varphi} + \\ &\qquad - C\, e^{-C\varphi}\, (\Delta'\varphi)(m+\Delta\varphi)\; + e^{-C\varphi}\, \Delta'(\Delta\varphi)\end{aligned} \tag{E7}$$

The left-hand side of $(E7)$ is a real number, thus it can be estimated. In particular, by Lemma 8 and being φ real, we see for all i

- $\overline{(\partial_i\, \varphi)(\partial_{\bar{i}}\, \varphi)} = (\partial_{\bar{i}}\, \varphi)(\partial_i\, \varphi)$
- $\overline{(\partial_i\, \varphi)(\partial_{\bar{i}}\, (\Delta\varphi)) + (\partial_i\, (\Delta\varphi))(\partial_{\bar{i}}\, \varphi)} = (\partial_{\bar{i}}\, \varphi)(\partial_i\, (\Delta\varphi)) + (\partial_{\bar{i}}\, (\Delta\varphi))(\partial_i\, \varphi)$

so each summand on the right-hand side of $(E7)$ is a real number and can be estimated separately. By the triangular inequality

$$\begin{aligned}&(\partial_i\, \varphi)(\partial_{\bar{i}}\, (\Delta\varphi)) + (\partial_i\, (\Delta\varphi))(\partial_{\bar{i}}\, \varphi) \leq \\ &\qquad \leq |(\partial_i\, \varphi)(\partial_{\bar{i}}\, (\Delta\varphi)) + (\partial_i\, (\Delta\varphi))(\partial_{\bar{i}}\, \varphi)| = \\ &\qquad = |(\partial_i\, \varphi)(\partial_{\bar{i}}\, (\Delta\varphi)) + (\partial_i\, (\Delta\varphi))(\partial_{\bar{i}}\, \varphi)|_{\mathbb{C}} \leq \\ &\qquad \leq |(\partial_i\, \varphi)(\partial_{\bar{i}}\, (\Delta\varphi))|_{\mathbb{C}} + |(\partial_i\, (\Delta\varphi))(\partial_{\bar{i}}\, \varphi)|_{\mathbb{C}} = 2|(\partial_i\, \varphi)(\partial_{\bar{i}}\, (\Delta\varphi))|_{\mathbb{C}}\end{aligned}$$

where we used $\overline{(\partial_i \varphi)(\partial_{\bar{i}} (\Delta\varphi))} = (\partial_i (\Delta\varphi))(\partial_{\bar{i}} \varphi)$, since φ is real. We estimate using Young's inequality

$$\begin{aligned} C|(\partial_i \varphi)(\partial_{\bar{i}} (\Delta\varphi))|_{\mathbb{C}} = \\ &= C\,|\partial_i \varphi|_{\mathbb{C}}(m+\Delta\varphi)^{\frac{1}{2}}\,|\partial_{\bar{i}} (\Delta\varphi)|_{\mathbb{C}}(m+\Delta\varphi)^{-\frac{1}{2}} \leq \\ &\leq \frac{1}{2}\,\left(C^2\,|\partial_i \varphi|^2_{\mathbb{C}}(m+\Delta\varphi) + |\partial_{\bar{i}} (\Delta\varphi)|^2_{\mathbb{C}}(m+\Delta\varphi)^{-1}\right) = \\ &= \frac{1}{2}\,\left(C^2\,(\partial_i \varphi)(\partial_{\bar{i}} \varphi)(m+\Delta\varphi) + |\partial_{\bar{i}} (\Delta\varphi)|^2_{\mathbb{C}}\,(m+\Delta\varphi)^{-1}\right) \end{aligned}$$

where we compute

$$\begin{aligned} \partial_{\bar{i}} (\Delta\varphi) = \sum_{j,k} \partial_{\bar{i}} \left(g^{j\bar{k}}\,\partial_j\,\partial_{\bar{k}}\,\varphi\right) = \\ &= \sum_{j,k} (\partial_{\bar{i}}\,g^{j\bar{k}})\,\partial_j\,\partial_{\bar{k}}\,\varphi + g^{j\bar{k}}\,(\partial_{\bar{i}}\,\partial_j\,\partial_{\bar{k}}\,\varphi) = \sum_k \partial_{\bar{i}}\,\partial_k\,\partial_{\bar{k}}\,\varphi \end{aligned}$$

Therefore

$$\begin{aligned} -C\,e^{-C\varphi} \sum_i \frac{(\partial_i \varphi)(\partial_{\bar{i}} (\Delta\varphi)) + (\partial_i (\Delta\varphi))(\partial_{\bar{i}} \varphi)}{1+\partial_i\,\partial_{\bar{i}}\,\varphi} \geq \\ &\geq -C^2\,e^{-C\varphi}\,\left(\sum_i \frac{(\partial_i \varphi)(\partial_{\bar{i}} \varphi)}{1+\partial_i\,\partial_{\bar{i}}\,\varphi}\right)(m+\Delta\varphi) + \\ &\quad - e^{-C\varphi}\,\left(\sum_i \frac{|\sum_k \partial_{\bar{i}}\,\partial_k\,\partial_{\bar{k}}\,\varphi|^2_{\mathbb{C}}}{1+\partial_i\,\partial_{\bar{i}}\,\varphi}\right)(m+\Delta\varphi)^{-1} \end{aligned}$$

We insert this inequality and ($D2$), using also Lemma 8, in ($E7$) to obtain

$$\begin{aligned} \Delta'(e^{-C\varphi}\,(m+\Delta\varphi)) \geq -C\,e^{-C\varphi}\,(\Delta'\varphi)(m+\Delta\varphi) + \\ &+ e^{-C\varphi}\,(-(m+\Delta\varphi)^{-1}\sum_i \frac{|\sum_k \partial_{\bar{i}}\,\partial_k\,\partial_{\bar{k}}\,\varphi|^2_{\mathbb{C}}}{1+\partial_i\,\partial_{\bar{i}}\,\varphi} + \\ &+ \Delta F + \sum_{i,j,k} \frac{|\partial_i\,\partial_{\bar{j}}\,\partial_k\,\varphi|^2_{\mathbb{C}}}{(1+\partial_j\,\partial_{\bar{j}}\,\varphi)(1+\partial_i\,\partial_{\bar{i}}\,\varphi)} + \\ &+ \inf_{i\neq k}(R_{i\bar{i}k\bar{k}})\,(\sum_{i,k} \frac{1+\partial_i\,\partial_{\bar{i}}\,\varphi}{1+\partial_k\,\partial_{\bar{k}}\,\varphi} - m^2)) \end{aligned} \tag{D3}$$

Observe that by the Schwarz inequality and being φ real

$$
\begin{aligned}
(m+\Delta\varphi)^{-1}\sum_i \frac{|\sum_k \partial_{\bar{i}}\,\partial_k\,\partial_{\overline{k}}\,\varphi|^2_{\mathbb{C}}}{1+\partial_i\,\partial_{\bar{i}}\,\varphi} = \\
&= (m+\Delta\varphi)^{-1}\sum_i (1+\partial_i\,\partial_{\bar{i}}\,\varphi)^{-1}\times \\
&\quad \times\ |\sum_k \frac{\partial_{\bar{i}}\,\partial_k\,\partial_{\overline{k}}\,\varphi}{(1+\partial_k\,\partial_{\overline{k}}\,\varphi)^{\frac{1}{2}}}(1+\partial_k\,\partial_{\overline{k}}\,\varphi)^{\frac{1}{2}}|^2_{\mathbb{C}}\ \leq \\
&\leq (m+\Delta\varphi)^{-1}\sum_i (1+\partial_i\,\partial_{\bar{i}}\,\varphi)^{-1}\times \\
&\quad \times\ \left(\sum_k |\frac{\partial_{\bar{i}}\,\partial_k\,\partial_{\overline{k}}\,\varphi}{(1+\partial_k\,\partial_{\overline{k}}\,\varphi)^{\frac{1}{2}}}|^2_{\mathbb{C}}\right)\left(\sum_k (1+\partial_k\,\partial_{\overline{k}}\,\varphi)^{\frac{1}{2}}|^2_{\mathbb{C}}\right) = \qquad (D4)\\
&= \sum_i (1+\partial_i\,\partial_{\bar{i}}\,\varphi)^{-1}\ \left(\sum_k \frac{|\partial_{\bar{i}}\,\partial_k\,\partial_{\overline{k}}\,\varphi|^2_{\mathbb{C}}}{1+\partial_k\,\partial_{\overline{k}}\,\varphi}\right) \leq \\
&\leq \sum_i (1+\partial_i\,\partial_{\bar{i}}\,\varphi)^{-1}\ \left(\sum_{j,k} \frac{|\partial_{\bar{i}}\,\partial_j\,\partial_{\overline{k}}\,\varphi|^2_{\mathbb{C}}}{1+\partial_j\,\partial_{\bar{j}}\,\varphi}\right) = \\
&= \sum_{i,j,k} \frac{|\partial_i\,\partial_{\bar{j}}\,\partial_k\,\varphi|^2_{\mathbb{C}}}{(1+\partial_i\,\partial_{\bar{i}}\,\varphi)(1+\partial_j\,\partial_{\bar{j}}\,\varphi)}
\end{aligned}
$$

where the last inequality holds because we added non-negative terms. Moreover

$$
\Delta'\varphi = \sum_i \frac{\partial_i\,\partial_{\bar{i}}\,\varphi}{1+\partial_i\,\partial_{\bar{i}}\,\varphi} = m - \sum_i \frac{1}{1+\partial_i\,\partial_{\bar{i}}\,\varphi}
$$

Applying this and $(D4)$ to $(D3)$, we get

$$
\begin{aligned}
\Delta'(e^{-C\varphi}\,(m+\Delta\varphi)) \geq e^{-C\varphi}\,(\Delta F - m^2\inf_{i\neq k}(R_{i\bar{i}k\bar{k}})) + \\
&+ e^{-C\varphi}\,\inf_{i\neq k}(R_{i\bar{i}k\bar{k}})\ \left(\sum_{i,k}\frac{1+\partial_i\,\partial_{\bar{i}}\,\varphi}{1+\partial_k\,\partial_{\overline{k}}\,\varphi}\right) + \qquad (D5)\\
&- mC\,e^{-C\varphi}\,(m+\Delta\varphi) + C\,e^{-C\varphi}\,(m+\Delta\varphi)\left(\sum_i \frac{1}{1+\partial_i\,\partial_{\bar{i}}\,\varphi}\right)
\end{aligned}
$$

and since

$$\sum_{i,k} \frac{1 + \partial_i \, \partial_{\bar{i}} \, \varphi}{1 + \partial_k \, \partial_{\bar{k}} \, \varphi} = \sum_i \frac{\left(\sum\limits_k 1 + \partial_k \, \partial_{\bar{k}} \, \varphi \right)}{1 + \partial_i \, \partial_{\bar{i}} \, \varphi} = \sum_i \frac{(m + \Delta\varphi)}{1 + \partial_i \, \partial_{\bar{i}} \, \varphi}$$

then $(D5)$ becomes

$$\begin{aligned} \Delta'(e^{-C\varphi} \, (m + \Delta\varphi)) \geq {} & e^{-C\varphi} \, (\Delta F - m^2 \inf_{i \neq k}(R_{i\bar{i}k\bar{k}})) + \\ & - mC \, e^{-C\varphi} \, (m + \Delta\varphi) + \\ & + (C + \inf_{i \neq k}(R_{i\bar{i}k\bar{k}})) \, e^{-C\varphi} \, (m + \Delta\varphi) \left(\sum_i \frac{1}{1 + \partial_i \, \partial_{\bar{i}} \, \varphi} \right) \end{aligned} \qquad (D6)$$

Observe that in special coordinates, $(M\text{-}A)$ reads

$$e^F = \det(\text{diag}(1 + \partial_i \, \partial_{\bar{i}} \, \varphi)) \det(I_m)^{-1} = \prod_i (1 + \partial_i \, \partial_{\bar{i}} \, \varphi)$$

Thus, by the Multinomial Theorem, since $m \geq 1$ we see

$$\begin{aligned} (\sum_i \frac{1}{1 + \partial_i \, \partial_{\bar{i}} \, \varphi})^{m-1} = & \\ & = \sum_{k_1 + \cdots + k_m = \, m-1} \frac{(m-1)!}{k_1! \ldots k_m!} \prod_j \frac{1}{(1 + \partial_j \, \partial_{\bar{j}} \, \varphi)^{k_j}} \geq \\ & \geq (m-1)! \sum_i \prod_{j \neq i} \frac{1}{1 + \partial_j \, \partial_{\bar{j}} \, \varphi} \geq \\ & \geq \sum_i (1 + \partial_i \, \partial_{\bar{i}} \, \varphi) \prod_j \frac{1}{1 + \partial_j \, \partial_{\bar{j}} \, \varphi} = \\ & = \frac{\sum_i 1 + \partial_i \, \partial_{\bar{i}} \, \varphi}{\prod_i 1 + \partial_i \, \partial_{\bar{i}} \, \varphi} = \frac{m + \Delta\varphi}{e^F} \end{aligned} \qquad (D7)$$

Being M compact, we can now choose C such that

$$C + \inf_M(\inf_{i \neq k}(R_{i\bar{i}k\bar{k}})) > 1$$

Then C depends only on M. With this value of C, insert $(D7)$ into $(D6)$ to obtain

$$\begin{aligned}\Delta'(e^{-C\varphi}\,(m+\Delta\varphi)) \geq e^{-C\varphi}\,(\Delta F - m^2\inf_{i\neq k}(R_{i\bar{i}k\bar{k}})) +\\ - mC\,e^{-C\varphi}\,(m+\Delta\varphi)\; + e^{-C\varphi}\,e^{\frac{-F}{m-1}}(m+\Delta\varphi)^{1+\frac{1}{m-1}}\end{aligned} \tag{D8}$$

Recall that $(D8)$ is valid at any point of M. Since M is compact, the continuous map $e^{-C\varphi}\,(m+\Delta\varphi)$ achieves its maximum at a point $q\in M$. Then at q

$$\Delta'(e^{-C\varphi}\,(m+\Delta\varphi)) \leq 0$$

which by $(D8)$ yields

$$\begin{aligned}(m+\Delta\varphi)^{1+\frac{1}{m-1}} &\leq e^{\frac{F}{m-1}}\left(m^2\inf_{i\neq k}(R_{i\bar{i}k\bar{k}})) - \Delta F\right) +\\ &\quad + mCe^{\frac{F}{m-1}}(m+\Delta\varphi) \leq\\ &\leq e^{\frac{\sup_M(F)}{m-1}}\left(m^2\sup_M\left(\inf_{i\neq k}(R_{i\bar{i}k\bar{k}})\right) + \sup_M(-\Delta F)\right) +\\ &\quad + mCe^{\frac{\sup_M(F)}{m-1}}(m+\Delta\varphi)\end{aligned} \tag{D9}$$

Observe that by denoting

$$\begin{aligned}A &:= e^{\frac{\sup_M(F)}{m-1}}\left(m^2\sup_M\left(\inf_{i\neq k}(R_{i\bar{i}k\bar{k}})\right) + \sup_M(-\Delta F)\right)\\ b &:= e^{\frac{\sup_M(F)}{m-1}}\;, \quad y := (m+\Delta\varphi)\end{aligned}$$

then $A,\ b$ are independent of φ and $(D9)$ is an inequality of the type

$$y^{\frac{m}{m-1}} \leq A + by$$

which lead to the cases

$$\cdot\, by \leq A \quad \Longrightarrow \quad y \leq (2A)^{\frac{m-1}{m}} \qquad \cdot\ A \leq by \quad \Longrightarrow \quad y \leq (2b)^{m-1}$$

In particular, y can be estimated only in terms of $A,\ b$. In our case, for a constant C_1 depending only on $\sup_M(-\Delta F),\ \sup_M(\inf_{i\neq k}(R_{i\bar{i}k\bar{k}})),\ \sup_M(F),\ M$

$$m+\Delta\varphi \leq C_1 \tag{D10}$$

Recall that $(D10)$ holds at q. Being q a point of maximum for $e^{-C\varphi}\,(m+\Delta\varphi)$, we have for all $p \in M$

$$e^{-C\varphi(p)}\,(m+(\Delta\varphi)(p)) \le e^{-C\varphi(q)}\,(m+(\Delta\varphi)(q)) \le C_1\, e^{-C\inf\limits_M(\varphi)}$$

which gives

$$(m+\Delta\varphi) \le C_1\, e^{C(\sup\limits_M(\varphi)-\inf\limits_M(\varphi))} \tag{$D11$}$$

Inequality $(D11)$ gives the required estimate for $\Delta\varphi$ if we have an upper estimate for $\sup\limits_M(\varphi)$ and a lower estimate for $\inf\limits_M(\varphi)$, that is, we need to estimate $||\varphi||_{C^0}$.

Remark 18. We have only searched for an upper estimate of $\Delta\varphi$, because we already have a lower estimate given by

$$\Delta\varphi > -m$$

Furthermore, the estimate for $\Delta\varphi$ also grants an estimate for the mixed derivatives of φ due to the inequality

$$||g_{i\bar{j}} + \frac{\partial\varphi}{\partial z^i \partial \overline{z}^j}||_g \le \mathrm{tr}_g\left(g_{i\bar{j}} + \frac{\partial\varphi}{\partial z^i \partial \overline{z}^j}\right) = m + \Delta\varphi$$

We deduce that g' is uniformly equivalent to g, since the coefficients of g' involve only the second-order derivatives of φ.

3.2.3 C^0 Estimate

For the estimate of $\sup_M(\varphi)$, we follow [Yau78]. For the estimate of $\inf_M(\varphi)$, we combine the approaches reported in [Yau78] and [Tia00]. The reason behind this choice will be clarified while developing the result.

Let G be Green's function of Δ on M (see [Aub82] for details). The properties of G that we need are the following:

- G is smooth on $(M \times M) \setminus \text{diag}(M)$
- if $\phi \in C^\infty(M)$, then for all $x \in M$

$$\phi(x) = -\frac{1}{\text{vol}(M)} \int_M \phi \, dV_g - \int_M G(x, \cdot) \, \Delta\phi \, dV_g$$

- G is defined up to a constant. In particular, we can assume $G \geq 0$

For all $p \in M$, since φ has zero integral over M and $m + \Delta\varphi > 0$

$$\varphi(p) = -\int_M G(p, \cdot) \, \Delta\varphi \, dV_g \leq m \int_M G(p, \cdot) \, dV_g \qquad (D12)$$

Since G is smooth on $(M \times M) \setminus \text{diag}(M)$ and the integral is defined up to measure zero sets, the map

$$x \in M \overset{\Theta}{\mapsto} \int_M G(x, \cdot) \, dV_g$$

is smooth on M, which is compact, so $A := \sup_M(\Theta) < \infty$ and A depends only on M. We then get from $(D12)$

$$\sup_M(\varphi) \leq mA \qquad (D13)$$

which is the desired estimate for $\sup_M(\varphi)$.

We now begin to search for an estimate for $\inf_M(\varphi)$. First, notice that since φ has zero integral over M

$$\sup_M(\varphi) \geq 0$$

Indeed, if $\sup_M(\varphi) < 0$ we would have

$$\int_M \varphi \, dV_g \leq \int_M \sup_M(\varphi) \, dV_g < 0$$

We can then estimate $||\varphi||_{L^1}$ using $(D13)$ as follows:

$$\begin{aligned}\int_M |\varphi| \, dV_g &\leq \int_M |\sup_M(\varphi) - \varphi| \, dV_g + \int_M |\sup_M(\varphi)| \, dV_g \leq \\ &\leq (\sup_M(\varphi)) \operatorname{vol}(M) - \int_M \varphi \, dV_g + (\sup_M(\varphi)) \operatorname{vol}(M) \leq \\ &\leq 2m \operatorname{vol}(M) \, A\end{aligned} \tag{D14}$$

Having already estimated $\sup_M(\varphi)$ from above, in the upcoming computations we can assume up to translation that

$$\sup_M(\varphi) \leq -1$$

Notice that here we lose the fact that φ has zero integral over M. We are reduced to finding an upper estimate of $\sup_M(-\varphi)$, where

$$\sup_M(-\varphi) = -\inf_M(\varphi) \geq 1$$

(The change of reference). At this point, in [Yau78] the author proceeds to find the desired estimate by means of the Mean Value Theorem, after stating that there is a constant $C' > 0$ such that

$$\sup_M(|\nabla\varphi|) \leq C' \left(e^{-C \inf_M(\varphi)} + ||\varphi||_{L^1} \right) \tag{X}$$

The author asserts that (X) is a straightforward consequence of $(D11)$ and $(D13)$, after applying the following Schauder Estimate ([Mor64], p. 156 inequality $5.5.23$)

$$||\varphi||_{C^{1,\alpha}} \leq D \left(||\Delta\varphi||_{C^{0,\alpha}} + ||\varphi||_{L^1} \right)$$

where $\alpha > 0$, and $D > 0$ is a constant independent of φ. However, it is unclear how (X) can actually be deduced as described. The description suggests that the Hölder norm is controlled through the C^0 norm, but this is generally not possible. Hence, for the remainder of this subsection, we follow [Tia00].

Let $p \in \mathbb{R}^{\geq 2}$. Starting from ($D14$), we look for estimates for $||\varphi||_{L^p}$. Using Proposition 5, we can rewrite (M-A) as

$$(\omega')^m = e^{CF}\omega^m \tag{E8}$$

Since $\wedge$ commutes on 2-forms, we compute

$$\begin{aligned}(e^{CF}-1)\,\omega^m &= (\omega')^m - \omega^m = \\ &= (\omega'-\omega)\wedge\left(\sum_i (\omega')^{m-i}\wedge\omega^{i-1}\right) = \\ &= -\sqrt{-1}\partial\overline{\partial}(-\varphi)\wedge\left(\sum_i (\omega')^{m-i}\wedge\omega^{i-1}\right)\end{aligned} \tag{E9}$$

The following three relations will be crucial throughout this step. Let $f \in C^\infty(M)$.
i) By Example 3 and Remark 13, we compute in normal coordinates

$$\begin{aligned}\sqrt{-1}m\,\partial f\wedge\overline{\partial}f\wedge\omega^{m-1} &= (\sqrt{-1})^m\, m!\,\times \\ &\times\left(\sum_{i,j}\partial_j f\,\partial_{\overline{k}}f\,dz^j\wedge d\overline{z}^k\right)\wedge\left(\sum_{r_1<\cdots<r_{m-1}}\bigwedge_{i=1}^{m-1}dz^{r_i}\wedge d\overline{z}^{r_i}\right) = \\ &= (\sqrt{-1})^m\,m!\,\left(\sum_j \partial_j f\,\partial_{\overline{j}}f\,\bigwedge_i dz^i\wedge d\overline{z}^i\right) = \\ &= \left(\sum_j|\partial_j f|^2_{\mathbb{C}}\right)(\sqrt{-1})^m\,m!\,\bigwedge_i dz^i\wedge d\overline{z}^i = \left(\sum_j|\partial_j f|^2_{\mathbb{C}}\right)\omega^m\end{aligned}$$

but since in normal coordinates we have for all $j,\ k$

$$g(\frac{\partial}{\partial x^k},\ \frac{\partial}{\partial x^j}) = g(\frac{\partial}{\partial y^k},\ \frac{\partial}{\partial y^j}) = 2\delta_{kj}\ , \quad g(\frac{\partial}{\partial x^k},\ \frac{\partial}{\partial y^j}) = 0$$

we then have

$$\sqrt{-1}\,\partial f\wedge\overline{\partial}f\wedge\omega^{m-1} = \frac{1}{2m}||\nabla f||^2_g\,\omega^m \tag{$*1$}$$

Notice that ($*1$) is independent of the choice of coordinates.

ii) By the Leibniz rule and the Kähler condition

$$\partial(f\,\overline{\partial}f \wedge \sum_i (\omega')^{m-i} \wedge \omega^{i-1}) =$$
$$= \partial f \wedge \overline{\partial}f \wedge \sum_i (\omega')^{m-i} \wedge \omega^{i-1} + f\,\partial\overline{\partial}f \wedge \sum_i (\omega')^{m-i} \wedge \omega^{i-1}$$

Thus, Stokes' theorem gives

$$\int_M \sqrt{-1}\,\partial f \wedge \overline{\partial}f \wedge \sum_i (\omega')^{m-i} \wedge \omega^{i-1} = \\ = -\int_M \sqrt{-1}\,f\,\partial\overline{\partial}f \wedge \sum_i (\omega')^{m-i} \wedge \omega^{i-1} \tag{$*2$}$$

iii) Let $1 \leq a \leq m$. By the Multinomial Theorem, in special coordinates we have

$$(\omega')^a = (\sqrt{-1})^a\, a! \sum_{r_1<\cdots<r_a} \bigwedge_{i=1}^{a} g'_{r_i\overline{r_i}}\, dz^{r_i} \wedge d\overline{z}^{r_i} =$$
$$= (\sqrt{-1})^a\, a! \sum_{r_1<\cdots<r_a} \prod_{j=1}^{a} g'_{r_j\overline{r_j}} \bigwedge_{i=1}^{a} dz^{r_i} \wedge d\overline{z}^{r_i}$$

Hence, for any b such that $a + b \leq m$

$$(\omega')^a \wedge \omega^b = (\sqrt{-1})^{a+b}\,(a+b)! \sum_{l_1<\cdots<l_{a+b}} \alpha_{l_1\ldots l_{a+b}} \bigwedge_{i=1}^{a+b} dz^{l_i} \wedge d\overline{z}^{l_i}$$

where the coefficients $\alpha_{l_1\ldots l_{a+b}}$ are sums of products of the $g'_{c\overline{c}}$'s, hence positive. Consequently, we obtain as in **i)**

$$\sqrt{-1}m\,\partial f \wedge \overline{\partial}f \wedge (\omega')^{m-i} \wedge \omega^{i-1} = \left(\sum_j |\partial_j f|^2_{\mathbb{C}}\, \alpha_{1\ldots\hat{j}\ldots n}\right) \omega^m$$

Although this expression is coordinate-dependent, it tells us that there is a non-negative map $\beta_i \in C^\infty(M)$ such that

$$\sqrt{-1}\,\partial f \wedge \overline{\partial}f \wedge (\omega')^{m-i} \wedge \omega^{i-1} = \beta_i\,\omega^m \tag{$*3$}$$

We can now estimate $||\varphi||_{L^2}$. Being φ smooth and M compact of real dimension $2m \geq 2$, we can apply the Poincaré inequality (see [Heb99] p. 40 Theorem 2.10)

$$||\varphi - \tilde{\varphi}||_{L^2} \leq D\, ||\nabla\varphi||_{L^2} \tag{P}$$

where $\tilde{\varphi} := \frac{\int_M \varphi\, dV_g}{\text{vol}(M)}$ and $D > 0$ is a constant independent of φ. Then combining (P), ($*1$), ($*2$), ($*3$), ($E9$) and $\varphi \leq -1$ results into

$$\begin{aligned}
||\varphi||_{L^2} - \frac{||\varphi||_{L^1}}{\sqrt{\text{vol}(M)}} &\leq ||\varphi||_{L^2} - \frac{|\int_M \varphi dV_g|}{\sqrt{\text{vol}(M)}} = \\
&= ||\varphi||_{L^2} - ||\tilde{\varphi}||_{L^2} \leq ||\varphi - \tilde{\varphi}||_{L^2} \leq \\
&\leq D\, ||\nabla\varphi||_{L^2} = D' \left(\int_M ||\nabla\varphi||_g^2 \, \frac{\omega^m}{2m} \right)^{\frac{1}{2}} = \\
&= D' \left(\int_M \sqrt{-1}\, \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^{m-1} \right)^{\frac{1}{2}} \leq \\
&\leq D' \left(\int_M \sqrt{-1}\, \partial\varphi \wedge \overline{\partial}\varphi \wedge \sum_i \, (\omega')^{m-i} \wedge \omega^{i-1} \right)^{\frac{1}{2}} = \qquad (D15) \\
&= D' \left(- \int_M \sqrt{-1}\, \varphi\, \partial\overline{\partial}\varphi \wedge \sum_i \, (\omega')^{m-i} \wedge \omega^{i-1} \right)^{\frac{1}{2}} = \\
&= D' \left(\int_M (-\varphi)\, (-\sqrt{-1})\, \partial\overline{\partial}(-\varphi) \wedge \sum_i \, (\omega')^{m-i} \wedge \omega^{i-1} \right)^{\frac{1}{2}} = \\
&= D'' \left(\int_M |\varphi|\, (e^{CF} - 1)\, \frac{\omega^m}{m!} \right)^{\frac{1}{2}} \leq D'''\, ||\varphi||_{L^1}^{\frac{1}{2}}
\end{aligned}$$

where $D''' > 0$ depends only on $m, \ \sup_M(F)$. From ($D15$) and ($D14$), we have a constant C_2 depending only on $m, \ \sup_M(F)$ such that

$$||\varphi||_{L^2} \leq C_2 \tag{$D16$}$$

For $p > 2$, we will make use of the following Sobolev inequality on compact manifolds (see [Heb99] p. 32 Theorem 2.6)

$$||u||_{L^{\frac{2m}{m-1}}} \leq B\, ||u||_{H_1^2} \tag{S}$$

where $u \in C^{\infty}(M)$ and $B > 0$ is a constant independent of u. Combining (S), ($*1$), ($*2$), ($*3$), ($E9$) and $-\varphi \geq 1$, we have for all $p \geq 1$

$$
\begin{aligned}
&\Big(\int_M |(-\varphi)^{\frac{p+1}{2}}|^{\frac{2m}{m-1}} \frac{\omega^m}{m!}\Big)^{\frac{m-1}{m}} - \int_M |(-\varphi)^{\frac{p+1}{2}}|^2 \frac{\omega^m}{m!} \leq \\
&\leq B^2 \int_M ||\nabla\left((-\varphi)^{\frac{p+1}{2}}\right)||_g^2 \frac{\omega^m}{m!} = \\
&= B' \int_M \sqrt{-1}\,\partial\left((-\varphi)^{\frac{p+1}{2}}\right) \wedge \overline{\partial}\left((-\varphi)^{\frac{p+1}{2}}\right) \wedge \omega^{m-1} = \\
&= B''\,(p+1)^2 \int_M (-\varphi)^{p-1} \sqrt{-1}\,\partial(-\varphi) \wedge \overline{\partial}(-\varphi) \wedge \omega^{m-1} \leq \\
&\leq B''\,(p+1)^2 \int_M \sqrt{-1}\,(-\varphi)^{p+1}\,\partial\varphi \wedge \overline{\partial}(-\varphi) \wedge \omega^{m-1} \leq \qquad (D17)\\
&\leq B''\,(p+1)^2 \int_M (-\varphi)^{p+1} \sqrt{-1}\,\partial\varphi \wedge \overline{\partial}\varphi \wedge \sum_i (\omega')^{m-i} \wedge \omega^{i-1} = \\
&= B''\,(p+1)^2 \int_M (-\varphi)^{p+1}\,(-\sqrt{-1})\,\partial\overline{\partial}(-\varphi) \wedge \sum_i (\omega')^{m-i} \wedge \omega^{i-1} = \\
&= B''\,(p+1)^2 \int_M (-\varphi)^{p+1}\,(e^{CF}-1)\,\omega^m \leq \\
&\leq B'''(p+1)^2 \int_M (-\varphi)^{p+1}\,\omega^m
\end{aligned}
$$

where $B''' > 0$ depends only on M, $\sup_M(F)$. It follows that for a constant C_3 that depends only on M, $\sup_M(F)$

$$||-\varphi||_{L^{\frac{(p+1)m}{m-1}}} \leq (C_3\,(p+1))^{\frac{2}{p+1}} ||-\varphi||_{L^{p+1}} \qquad (D18)$$

Set $p_0 = 1$ and for any $i \geq 1$

$$p_i := \frac{m}{m-1}(p_{i-1}+1) - 1$$

Then for all i: $p_i \geq 1$ and $p_i < p_{i+1}$. By repeated applications of ($D18$), for all i

$$||-\varphi||_{L^{p_i+1}} \leq \prod_{j=0}^{i-1} (C_3\,(p_j+1))^{\frac{2}{p_j+1}} ||-\varphi||_{L^2} \qquad (D19)$$

Being M compact, we can pass to the limit in $(D19)$ obtaining

$$||-\varphi||_{C^0} = \lim_{i\to\infty} ||-\varphi||_{L^{p_i+1}} \leq \left(\prod_{j=0}^{\infty} (C_3\,(p_j+1))^{\frac{2}{p_j+1}}\right) ||-\varphi||_{L^2} \quad (D20)$$

Observe that

$$\prod_{j=0}^{\infty} (C_3\,(p_j+1))^{\frac{2}{p_j+1}} = e^{\log(\prod_{j=0}^{\infty} (C_3\,(p_j+1))^{\frac{2}{p_j+1}})} =$$

$$= e^{\sum_{j=0}^{\infty} 2\frac{\log(C)+\log(p_j+1)}{p_j+1}}$$

and by the choice of the p_i 's

- $|\frac{\log(C)}{p_{j+1}+1}\frac{p_j+1}{\log(C)}| = \frac{m-1}{m} < 1$
- $|\frac{\log(p_{j+1}+1)}{p_{j+1}+1}\frac{p_j+1}{\log(p_j+1)}| < 1$

Hence, by the ratio test we have the convergence of both the series

$$\sum_{j=0}^{\infty} 2\frac{\log(C)}{p_j+1}, \quad \sum_{j=0}^{\infty} 2\frac{\log(p_j+1)}{p_j+1}$$

which implies the convergence of $\sum_{j=0}^{\infty} 2\frac{\log(C)+\log(p_j+1)}{p_j+1}$. Consequently, $(D20)$ yields the desired estimate

$$||-\varphi||_{C^0} \leq C_4 \quad (D21)$$

3.2.4 Third Order Estimates

We look for an estimate of the mixed third-order derivatives $\partial_i\, \partial_{\bar{j}}\, \partial_k\, \varphi$. We follow [Yau78] for this exposition. In order to proceed smoothly, we denote

$$\varphi_{i\bar{j}k} := \partial_i\, \partial_{\bar{j}}\, \partial_k\, \varphi$$

and similarly for the other derivatives. Consider the smooth function

$$S = \sum_{i,j,k,r,s,t} g^{i\bar{r}}\, g^{s\bar{j}}\, g^{k\bar{t}}\, \varphi_{i\bar{j}k}\, \varphi_{\bar{r}s\bar{t}}$$

which does not depend on the choice of coordinates, being the contraction of a tensor through the metric. Choosing normal coordinates, we see

$$S = \sum_{i,j,k} |\varphi_{i\bar{j}k}|^2_{\mathbb{C}}$$

so S is real and non-negative. To simplify the next calculations, we introduce the following equivalence relations. Let $A,\ B \in C^{\infty}(M)$. Then

- $A \sim B \iff |A - B| \leq C_1\, \sqrt{S} + C_2$
- $A \approx B \iff |A - B| \leq C_3\, S + C_4\, \sqrt{S} + C_5$

where $C_1,\ C_2,\ C_3,\ C_4,\ C_5$ are constants that can be estimated independently of $A,\ B$. We compute in special coordinates

$$\begin{aligned}\Delta' S = \sum (g')^{a\bar{b}}\, (&-(g')^{i\bar{p}}\, (g')^{q\bar{r}}\, (g')^{s\bar{j}}\, (g')^{k\bar{t}}\, \varphi_{q\bar{p}\bar{b}}\, \varphi_{\bar{i}j\bar{k}} \varphi_{\bar{r}s\bar{t}} + \\ &- (g')^{i\bar{r}}\, (g')^{p\bar{j}}\, (g')^{s\bar{q}}\, (g')^{k\bar{t}}\, \varphi_{p\bar{q}\bar{b}}\, \varphi_{\bar{i}j\bar{k}} \varphi_{\bar{r}s\bar{t}} + \\ &- (g')^{i\bar{r}}\, (g')^{s\bar{j}}\, (g')^{k\bar{p}}\, (g')^{q\bar{t}}\, \varphi_{\bar{p}q\bar{b}}\, \varphi_{\bar{i}j\bar{k}} \varphi_{\bar{r}s\bar{t}} + \\ &+ g^{i\bar{r}}\, g^{s\bar{j}}\, g^{k\bar{t}}\, \varphi_{i\bar{j}k\bar{b}}\, \varphi_{\bar{r}s\bar{t}} + g^{i\bar{r}}\, g^{s\bar{j}}\, g^{k\bar{t}}\, \varphi_{i\bar{j}k}\, \varphi_{\bar{r}s\bar{t}\bar{b}})_a \sim \qquad (E10) \\ \sim \sum (g')^{a\bar{b}}\, [&(-2(g')^{i\bar{p}}\, (g')^{q\bar{r}}\, (g')^{s\bar{j}}\, (g')^{k\bar{t}}\, \varphi_{q\bar{p}\bar{b}}\, \varphi_{\bar{i}j\bar{k}} \varphi_{\bar{r}s\bar{t}} + \\ &- (g')^{i\bar{r}}\, (g')^{p\bar{j}}\, (g')^{s\bar{q}}\, (g')^{k\bar{t}}\, \varphi_{p\bar{q}\bar{b}}\, \varphi_{\bar{i}j\bar{k}} \varphi_{\bar{r}s\bar{t}}) + \\ &g^{i\bar{r}}\, g^{s\bar{j}}\, g^{k\bar{t}}\, (\varphi_{i\bar{j}k\bar{b}}\, \varphi_{\bar{r}s\bar{t}} + \varphi_{i\bar{j}k}\, \varphi_{\bar{r}s\bar{t}\bar{b}})]_a\end{aligned}$$

where the sum is over $i, j, k, p, q, r, s, t, a, b$. Then, expanding ($E10$), we obtain

$$\begin{aligned}
\Delta' S \sim \sum (g')^{a\bar{b}} \, [2(g')^{i\bar{c}} \, (g')^{d\bar{p}} \, (g')^{q\bar{r}} \, (g')^{s\bar{j}} \, (g')^{k\bar{t}} \, \varphi_{\bar{c}da} \, \varphi_{q\bar{p}\bar{b}} \, \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}} + \\
+ 2(g')^{i\bar{p}} \, (g')^{q\bar{c}} \, (g')^{d\bar{r}} \, (g')^{s\bar{j}} \, (g')^{k\bar{t}} \, \varphi_{\bar{c}da} \, \varphi_{q\bar{p}\bar{b}} \, \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}} + \\
+ 2(g')^{i\bar{p}} \, (g')^{q\bar{r}} \, (g')^{d\bar{j}} \, (g')^{s\bar{c}} \, (g')^{k\bar{t}} \, \varphi_{\bar{c}da} \, \varphi_{q\bar{p}\bar{b}} \, \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}} + \\
+ 2(g')^{i\bar{p}} \, (g')^{q\bar{r}} \, (g')^{s\bar{j}} \, (g')^{k\bar{c}} \, (g')^{d\bar{t}} \, \varphi_{\bar{c}da} \, \varphi_{q\bar{p}\bar{b}} \, \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}} + \\
- 2(g')^{i\bar{p}} \, (g')^{q\bar{r}} \, (g')^{s\bar{j}} \, (g')^{k\bar{t}} \, (\varphi_{q\bar{p}\bar{b}a} \, \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}} + \\
+ \varphi_{q\bar{p}\bar{b}} \, \varphi_{i\bar{j}ka} \, \varphi_{\bar{r}s\bar{t}} + \varphi_{q\bar{p}\bar{b}} \, \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}a}) + \\
+ (g')^{i\bar{c}} \, (g')^{d\bar{r}} \, (g')^{p\bar{j}} \, (g')^{s\bar{q}} \, (g')^{k\bar{t}} \, \varphi_{\bar{c}da} \, \varphi_{q\bar{j}\bar{b}} \, \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}} + \\
+ (g')^{i\bar{r}} \, (g')^{d\bar{j}} \, (g')^{p\bar{c}} \, (g')^{s\bar{q}} \, (g')^{k\bar{t}} \, \varphi_{\bar{c}da} \, \varphi_{q\bar{j}\bar{b}} \, \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}} + \\
+ (g')^{i\bar{r}} \, (g')^{p\bar{j}} \, (g')^{d\bar{q}} \, (g')^{s\bar{c}} \, (g')^{k\bar{t}} \, \varphi_{\bar{c}da} \, \varphi_{q\bar{j}\bar{b}} \, \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}} + \\
+ (g')^{i\bar{r}} \, (g')^{p\bar{j}} \, (g')^{s\bar{q}} \, (g')^{k\bar{c}} \, (g')^{d\bar{t}} \, \varphi_{\bar{c}da} \, \varphi_{q\bar{j}\bar{b}} \, \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}} + \\
- (g')^{i\bar{r}} \, (g')^{p\bar{j}} \, (g')^{s\bar{q}} \, (g')^{k\bar{t}} \, (\varphi_{p\bar{q}\bar{b}a} \, \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}} + \\
+ \varphi_{p\bar{q}\bar{b}} \, \varphi_{i\bar{j}ka} \, \varphi_{\bar{r}s\bar{t}} + \varphi_{p\bar{q}\bar{b}} \, \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}a}) + \\
- (2(g')^{i\bar{c}} \, (g')^{d\bar{r}} \, (g')^{s\bar{j}} \, (g')^{k\bar{t}} \, \varphi_{\bar{c}da} + (g')^{i\bar{r}} \, (g')^{d\bar{j}} \, (g')^{s\bar{c}} \, (g')^{k\bar{t}} \, \varphi_{\bar{c}da}) \times \\
\times (\varphi_{i\bar{j}k\bar{b}} \, \varphi_{\bar{r}s\bar{t}} + \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}\bar{b}}) \, + (g')^{i\bar{r}} \, (g')^{s\bar{j}} \, (g')^{k\bar{t}} \times \\
\times (\varphi_{i\bar{j}k\bar{b}a} \, \varphi_{\bar{r}s\bar{t}} \, + \varphi_{i\bar{j}k\bar{b}} \, \varphi_{\bar{r}s\bar{t}a} \, + \varphi_{i\bar{j}ka} \, \varphi_{\bar{r}s\bar{t}\bar{b}} \, + \varphi_{i\bar{j}k} \, \varphi_{\bar{r}s\bar{t}\bar{b}a})]
\end{aligned} \qquad (E11)$$

where the sum is over $i,\ j,\ k,\ p,\ q,\ r,\ s,\ t,\ a,\ b,\ c,\ d$. Differentiating ($E2$) with respect to ∂_s , we get

$$\begin{aligned}
\sum_{i,j} (g')^{i\bar{j}} \, \varphi_{i\bar{j}k\bar{l}s} \, \sim \sum (g')^{i\bar{t}} \, (g')^{n\bar{j}} \, \varphi_{n\bar{t}s} \, \varphi_{i\bar{j}k\bar{l}} + \\
+ F_{k\bar{l}s} + \left(\sum (g')^{t\bar{j}} \, (g')^{i\bar{n}} \, \varphi_{t\bar{n}\bar{l}} \, \varphi_{i\bar{j}k} \right)_s
\end{aligned} \qquad (E12)$$

where the two last sums are over all indices except $s,\ \bar{l}$. We rewrite and expand ($E12$) as

$$
\begin{aligned}
\sum_{a,b} (g')^{a\bar{b}} \varphi_{i\bar{j}k\bar{b}a} &= \sum_{a,b} (g')^{a\bar{b}} \varphi_{a\bar{b}i\bar{j}k} \sim \\
&\sim \sum (g')^{a\bar{b}} (g')^{q\bar{b}} \varphi_{q\bar{p}k} \varphi_{a\bar{b}i\bar{j}} + F_{i\bar{j}k} + \\
&\quad + \left(\sum (g')^{p\bar{b}} (g')^{a\bar{q}} \varphi_{p\bar{q}\bar{j}} \varphi_{a\bar{b}i} \right)_k = \\
&= \sum (g')^{a\bar{b}} (g')^{q\bar{b}} \varphi_{q\bar{p}k} \varphi_{a\bar{b}i\bar{j}} + F_{i\bar{j}k} + \\
&\quad - \sum (g')^{p\bar{c}} (g')^{d\bar{b}} (g')^{a\bar{q}} \varphi_{\bar{c}dk} \varphi_{p\bar{q}\bar{j}} \varphi_{a\bar{b}i} + \\
&\quad - \sum (g')^{p\bar{b}} (g')^{a\bar{c}} (g')^{d\bar{q}} \varphi_{\bar{c}dk} \varphi_{p\bar{q}\bar{j}} \varphi_{a\bar{b}i} + \\
&\quad + \sum (g')^{p\bar{b}} (g')^{a\bar{q}} \varphi_{p\bar{q}\bar{j}k} \varphi_{a\bar{b}i} + \\
&\quad + \sum (g')^{p\bar{b}} (g')^{a\bar{q}} \varphi_{p\bar{q}\bar{j}} \varphi_{a\bar{b}ik}
\end{aligned} \tag{E13}
$$

and we insert $(E13)$ together with $(E2)$ into $(E11)$, finding

$$
\begin{aligned}
\Delta' S \sim &\sum (g')^{a\bar{b}} [2(g')^{i\bar{c}} (g')^{d\bar{p}} (g')^{q\bar{r}} (g')^{s\bar{j}} (g')^{k\bar{t}} \varphi_{\bar{c}da} \varphi_{q\bar{p}\bar{b}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}} + \\
&+ 2(g')^{i\bar{p}} (g')^{q\bar{c}} (g')^{d\bar{r}} (g')^{s\bar{j}} (g')^{k\bar{t}} \varphi_{\bar{c}da} \varphi_{q\bar{p}\bar{b}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}} + \\
&+ 2(g')^{i\bar{p}} (g')^{q\bar{r}} (g')^{d\bar{j}} (g')^{s\bar{c}} (g')^{k\bar{t}} \varphi_{\bar{c}da} \varphi_{q\bar{p}\bar{b}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}} + \\
&+ 2(g')^{i\bar{p}} (g')^{q\bar{r}} (g')^{s\bar{j}} (g')^{k\bar{c}} (g')^{d\bar{t}} \varphi_{\bar{c}da} \varphi_{q\bar{p}\bar{b}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}}] + \\
&- 2 \sum (g')^{i\bar{p}} (g')^{q\bar{r}} (g')^{s\bar{j}} (g')^{k\bar{t}} \times \\
&\times [F_{q\bar{p}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}} + (g')^{t\bar{b}} (g')^{a\bar{n}} \varphi_{t\bar{n}\bar{p}} \varphi_{a\bar{b}q} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}} + \\
&+ (g')^{a\bar{b}} \varphi_{q\bar{p}\bar{b}} \varphi_{i\bar{j}ka} \varphi_{\bar{r}s\bar{t}} + (g')^{a\bar{b}} \varphi_{q\bar{p}\bar{b}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}a}] + \\
&+ \sum (g')^{i\bar{c}} (g')^{d\bar{r}} (g')^{p\bar{j}} (g')^{s\bar{q}} (g')^{k\bar{t}} \varphi_{\bar{c}da} \varphi_{q\bar{j}\bar{b}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}} + \\
&+ \sum (g')^{i\bar{r}} (g')^{d\bar{j}} (g')^{p\bar{c}} (g')^{s\bar{q}} (g')^{k\bar{t}} \varphi_{\bar{c}da} \varphi_{q\bar{j}\bar{b}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}} + \\
&+ \sum (g')^{i\bar{r}} (g')^{p\bar{j}} (g')^{d\bar{q}} (g')^{s\bar{c}} (g')^{k\bar{t}} \varphi_{\bar{c}da} \varphi_{q\bar{j}\bar{b}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}} + \\
&+ \sum (g')^{i\bar{r}} (g')^{p\bar{j}} (g')^{s\bar{q}} (g')^{k\bar{c}} (g')^{d\bar{t}} \varphi_{\bar{c}da} \varphi_{q\bar{j}\bar{b}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}} + \\
&- \sum (g')^{i\bar{r}} (g')^{p\bar{j}} (g')^{s\bar{q}} (g')^{k\bar{t}} (F_{p\bar{q}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}} + \\
&+ (g')^{t\bar{b}} (g')^{a\bar{n}} \varphi_{t\bar{n}\bar{q}} \varphi_{a\bar{b}\bar{p}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}} + \\
&+ (g')^{a\bar{b}} \varphi_{p\bar{q}\bar{b}} \varphi_{i\bar{j}ka} \varphi_{\bar{r}s\bar{t}} + (g')^{a\bar{b}} \varphi_{p\bar{q}\bar{b}} \varphi_{i\bar{j}k} \varphi_{\bar{r}s\bar{t}a}) +
\end{aligned} \tag{E14}
$$

$$+\ 2\mathrm{Re}[\sum (g')^{i\bar r}\,(g')^{s\bar j}\,(g')^{k\bar t}\,\varphi_{\bar r s\bar t}\,((g')^{a\bar p}\,(g')^{q\bar b}\,\varphi_{q\bar p\bar b}\,\varphi_{a\bar b i\bar j}\ + F_{i\bar j k} +$$
$$-\ (g')^{p\bar c}\,(g')^{d\bar b}\,(g')^{a\bar q}\,\varphi_{\bar c dk}\,\varphi_{p\bar q\bar j}\,\varphi_{a\bar b\bar i}\ - (g')^{p\bar b}\,(g')^{a\bar c}\,(g')^{d\bar q}\,\varphi_{\bar c dk}\,\varphi_{p\bar q\bar j}\,\varphi_{a\bar b\bar i} +$$
$$+\,(g')^{p\bar b}\,(g')^{a\bar q}\,\varphi_{p\bar q\bar j k}\,\varphi_{a\bar b\bar i}\ + (g')^{p\bar b}\,(g')^{a\bar q}\,\varphi_{p\bar q\bar j}\,\varphi_{a\bar b\bar i k})] +$$
$$+\sum (g')^{a\bar b}\,(g')^{i\bar r}\,(g')^{s\bar j}\,(g')^{k\bar t}\,(\varphi_{i\bar j k\bar b}\,\varphi_{\bar r s\bar t a}\ + \varphi_{i\bar j k a}\,\varphi_{\bar r s\bar t\bar b})$$

Insert the diagonal expression for the coefficients $g^{x\bar y}$ into ($E14$), and obtain

$$\begin{aligned}
\Delta' S \approx &\sum (1+\varphi_{a\bar a})^{-1}\,(1+\varphi_{i\bar i})^{-1}\,(1+\varphi_{j\bar j})^{-1}\,(1+\varphi_{k\bar k})^{-1}\times\\
&\qquad\times (1+\varphi_{p\bar p})^{-1}\,(1+\varphi_{q\bar q})^{-1}\,(2\,\varphi_{\bar i pa}\,\varphi_{q\bar p\bar a}\,\varphi_{i\bar j k}\,\varphi_{\bar q j\bar k} +\\
&\qquad + 2\,\varphi_{\bar k pa}\,\varphi_{q\bar i\bar a}\,\varphi_{i\bar j k}\,\varphi_{\bar q j\bar p} + 2\,\varphi_{\bar p qa}\,\varphi_{j\bar q\bar a}\,\varphi_{i\bar j k}\,\varphi_{\bar i p\bar k}) +\\
&-2\mathrm{Re}[\sum (1+\varphi_{a\bar a})^{-1}\,(1+\varphi_{i\bar i})^{-1}\,(1+\varphi_{j\bar j})^{-1}\,(1+\varphi_{k\bar k})^{-1}\times\\
&\qquad\times (1+\varphi_{p\bar p})^{-1}\,(\varphi_{p\bar i\bar a}\,\varphi_{i\bar j ka}\,\varphi_{\bar p j\bar k} + \varphi_{j\bar p\bar a}\,\varphi_{i\bar j k}\,\varphi_{\bar i p\bar k a} +\\
&\qquad + \varphi_{\bar i pa}\,\varphi_{i\bar j k}\,\varphi_{\bar p j\bar k\bar a})] +\\
&+\sum (1+\varphi_{a\bar a})^{-1}(1+\varphi_{i\bar i})^{-1}(1+\varphi_{j\bar j})^{-1}(1+\varphi_{k\bar k})^{-1}\times\\
&\qquad\times (|\varphi_{i\bar j ka}|^2_{\mathbb{C}} + |\varphi_{i\bar j k\bar a}|^2_{\mathbb{C}}) =\\
=&\sum (1+\varphi_{a\bar a})^{-1}(1+\varphi_{i\bar i})^{-1}(1+\varphi_{j\bar j})^{-1}(1+\varphi_{k\bar k})^{-1}\times\\
&\qquad\times [\,|\varphi_{\bar i j\bar k a}\ - \sum_p \varphi_{\bar i p\bar k}\,\varphi_{\bar p ja}\,(1+\varphi_{p\bar p})^{-1}|^2_{\mathbb{C}} +\\
&\qquad + |\varphi_{i\bar j ka}\ - \sum_p (\varphi_{\bar p ia}\,\varphi_{p\bar j k} + \varphi_{\bar p ik}\,\varphi_{p\bar j a})\,(1+\varphi_{p\bar p})^{-1}|^2_{\mathbb{C}}\,]
\end{aligned} \tag{E15}$$

On the other hand, ($E6$) does not depend on the chosen coordinates. In special coordinates, since we have estimated $\Delta\varphi$, we find a constant C' independent of φ such that

$$\Delta'(\Delta\varphi) \geq \sum_{i,j,k} (1+\varphi_{i\bar i})^{-1}\,(1+\varphi_{k\bar k})^{-1}\,|\varphi_{k\bar i\bar j}|^2_{\mathbb{C}} - C' \tag{D22}$$

Combining ($E15$) and ($D22$), for some positive constants $C_7,\ C_8,\ C_9$ independent of φ

$$\Delta'(S + C_7\,\Delta\varphi) \geq C_8\,S - C_9 \tag{D23}$$

where we estimated $\sqrt{S}$ by means of S because

$$2\sqrt{S} \leq (\sqrt{S})^2 + 1 = S + 1$$

Let $p \in M$ be a point of maximum for $S + C_7\, \Delta\varphi$. Then $(D23)$ reads

$$0 = \Delta'(S + C_7\, \Delta\varphi)(p) \geq C_8\, S(p) - C_9 \tag{$D24$}$$

For any $q \in M$, by $(D24)$ and the estimate of $\Delta\varphi$

$$\begin{aligned} C_8\, S(q) &= C_8\, (S(q) + C_7\, \Delta\varphi(q)) - C_8\, C_7\, \Delta\varphi(q) \leq \\ &\leq C_8\, (S(p) + C_7\, \Delta\varphi(p)) + C_{10} \leq \\ &\leq C_9 + C + +C_{10} \end{aligned}$$

which implies that there is a constant C_{11} independent of φ such that

$$\sup_M(S) \leq C_{11} \tag{$D25$}$$

We can then use $(D25)$ to estimate the mixed third-order derivatives of φ.

3.2.5 Uniqueness of the solution

Here we discuss the result that will provide the uniqueness of the g-positive smooth solution of $(M\text{-}A)$. We follow [Yau78].

Theorem 4. Let $\varphi,\ \phi$ be two g-positive C^2-solutions of $(M\text{-}A)$. Then $\varphi - \phi$ is constant.

Proof. By $(M\text{-}A)$, we have

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}} + \phi_{i\bar{j}})$$

which can be rearranged as

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}} + (\phi - \varphi)_{i\bar{j}})\, \det(g_{i\bar{j}} + \varphi_{i\bar{j}})^{-1} = 1$$

As in Lemma 16, being $\varphi,\ \phi$ both g-positive, we can construct holomorphic coordinates that at a point diagonalize the matrices

$$(g'_{i\bar{j}}) := (g_{i\bar{j}} + \varphi_{i\bar{j}})\,, \quad (g''_{i\bar{j}}) := (g_{i\bar{j}} + \varphi_{i\bar{j}} + (\phi - \varphi)_{i\bar{j}})$$

Since $(E8)$ shows that $(M\text{-}A)$ does not depend on the choice of coordinates, the above relation reads

$$1 = \prod_i \left(g''_{i\bar{i}}\right) \prod_j \left(g'_{j\bar{j}}\right)^{-1} = \prod_k \left(g''_{k\bar{k}}\,(g')^{k\bar{k}}\right)$$

By the arithmetic-geometric mean inequality, we have

$$\begin{aligned} \frac{m + \Delta'(\phi - \varphi)}{m} &= \frac{\sum (g')^{k\bar{k}}\,(\phi - \varphi)_{k\bar{k}} + 1}{m} = \\ &= \frac{\sum (g')^{k\bar{k}}\,((\phi - \varphi)_{k\bar{k}} + g'_{k\bar{k}})}{m} \geq \\ &\geq \left(\prod_k \left(g''_{k\bar{k}}\,(g')^{k\bar{k}}\right) \right)^{\frac{1}{m}} = 1 \end{aligned}$$

from which we deduce that $\Delta'(\phi - \varphi) \geq 0$. By Theorem 3: $\phi - \varphi$ is constant. □

3.2.6 The continuity method

As anticipated, we solve (M-A) as in [Yau78] by applying the continuity method. Fix $\alpha \in (0,\ 1)$. Consider W_k , $k \geq 3$, the subset of $[0,\ 1]$ made of the parameters t such that

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j}\right)\ (\det(g_{i\bar{j}}))^{-1} = \mathrm{vol}(M)\ \left(\int_M e^{tF}\right)^{-1}\ e^{tF} \qquad (*_t)$$

has a g-positive solution in $C^{k+1,\alpha}(M)$. Then $0 \in W_k$, because $\varphi \equiv 0$ is a solution of

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j}\right)\ (\det(g_{i\bar{j}}))^{-1} = 1 \qquad (*_0)$$

and $(g_{i\bar{j}})$ is positive definite being g a Kähler metric. Notice that (M-A) corresponds to $(*_1)$, so we want to show that $1 \in W_k$. Being $[0,\ 1]$ connected, it suffices to prove that W_k is open and closed in $[0,\ 1]$.

We begin by proving that W_k is open. Consider the open subspace of $C^{k+1,\alpha}(M)$

$$A := \{\ \eta \in C^{k+1,\alpha}(M) : \eta \text{ is } g\text{-positive}\ \}$$

and its subset

$$A_0 := \{\ \psi \in A : \int_M \psi\, dV_g = 0\ \}$$

Furthermore, consider the affine subspace of $C^{k-1,\alpha}(M)$

$$B := \{\ f \in C^{k-1,\alpha}(M) : \int_M f\, dV_g = \mathrm{vol}(M)\ \}$$

Consider the map

$$G\colon A \to B : \varphi \mapsto \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j}\right)\ (\det(g_{i\bar{j}}))^{-1}$$

Then G is independent of the choice of coordinates (recall the well-definition of the Ricci form) and well-defined (recall Remark 16). Notice that if we set

$$f_t := \frac{\mathrm{vol}(M)}{\int_M e^{tF}}\, e^{tF}$$

then $f_t \in B$ because F is smooth, and

$$t \in W_k \iff f_t \in \mathrm{Im}(G)$$

Hence, we want to show that for any $t_0 \in W_k$ there is $\epsilon > 0$ such that

$$|t - t_0| < \epsilon \implies f_t \in B$$

To do this, we prepare to use the Inverse Function Theorem in Banach spaces (see [Sch69], p. 15 Theorem 1.20). Let $\phi \in A$ and pick any segment $\phi + s\eta$ through ϕ. We compute by Jacobi's formula

$$\begin{aligned} &\frac{d}{ds}\, G(\phi + s\eta)\big|_{s=0} = \\ &\quad = \det\left(g_{i\bar{j}} + \frac{\partial^2 \phi}{\partial z^i \partial \overline{z}^j}\right) (\det(g_{i\bar{j}}))^{-1} \sum_{a,b} (g_\phi)^{a\bar{b}}\, \frac{\partial}{\partial z^a \partial \overline{z}^b} \left(\frac{d(\phi + s\eta)}{ds}\bigg|_{s=0} \right) = \\ &\quad = \det(g^{\phi}_{i\bar{j}})\, (\det(g_{i\bar{j}}))^{-1}\, \Delta_\phi \eta \end{aligned}$$

where we used the subscript ϕ to denote the usual objects defined using ϕ. This proves that G is Frechet-differentiable. Observe the following facts:

- Since B is an affine space, its tangent space at any point is its underlying vector space

  $$B_0 := \{\, f \in C^{k-1,\alpha}(M) : \int_M f\, dV_g = 0 \,\}$$

  On the other hand, being A open in a vector space, its tangent space at any point is $C^{k+1,\alpha}(M)$ itself.

- Integrating by parts, we see that a necessary condition for the equation

  $$\Delta_\phi \eta = f$$

  to have a weak solution is that $\int_M f\, dV_\phi = 0$. In a compact Kähler manifold, the converse is true (see [Szé14], p. 33 Theorem 2.12).

- By elliptic regularity, any weak solution of $\Delta_\phi \eta = f$ is in $C^{k+1,\alpha}(M)$ when $f \in C^{k-1,\alpha}(M)$. Being $k \geq 3$, a weak solution is also a classical solution.

Thus, the differential of G at ϕ is surjective if and only if for any $f \in B_0$ the equation

$$\Delta_\phi \eta = (\det(g^\phi_{i\bar{j}}))^{-1} \det(g_{i\bar{j}})\, f \tag{L}$$

has a (weak) solution, and this condition is equivalent to

$$0 = \int_M (\det(g^\phi_{i\bar{j}}))^{-1} \det(g_{i\bar{j}})\, f \, dV_\phi = \int_M f \, dV_g$$

But the latter is always verified on B_0, so the differential of G is surjective at any point. Moreover, by Theorem 3 any two solutions of the same equation (L) differ by a constant, which implies that

the differential of G is bijective at the points of A_0

Also notice that $\text{Im}(G) = G(A_0)$, since for any $\psi \in A$

$$\psi - \frac{\int_M \psi \, dV_g}{\text{vol}(M)} \in A_0 \,, \quad G(\psi) = G\left(\psi - \frac{\int_M \psi \, dV_g}{\text{vol}(M)}\right)$$

Finally, let $t_0 \in W_k$. Then $f_{t_0} \in G(A_0)$, and by the Inverse Function Theorem, there is an open neighborhood U of f_{t_0} in B which is contained in $\text{Im}(G)$. On a compact manifold, we have the continuous inclusion

$$C^k(M) \hookrightarrow C^{k-1,\alpha}(M) \tag{I}$$

Consequently, the map

$$\gamma \colon \mathbb{R} \to C^{k-1,\alpha}(M) : t \mapsto f_t$$

is continuous. Therefore, there is $\epsilon > 0$ such that

$$\gamma(t_0 - \epsilon, \ t_0 + \epsilon) \subseteq U$$

that is, for $|t - t_0| < \epsilon \ : \ f_t \in B$.

We now prove that W_k is closed. Let $\{ t_q \}_q \subseteq W_k$ be a sequence that converges to t_0. Consider the sequence $\{ \varphi_q \}_q$ of solutions of $(*_q)$. Up to normalizing, assume

$$\int_M \varphi_q \, dV_g = 0$$

Fix q. Differentiating $(*_q)$ with respect to $\frac{\partial}{\partial z^p}$, Jacobi's formula results into

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_q}{\partial z^i \partial \bar{z}^j} \right) \sum_{a,b} g_q^{a\bar{b}} \frac{\partial^2}{\partial z^a \partial \bar{z}^b} \left(\frac{\partial \varphi_q}{\partial z^p} \right) =$$

$$= \mathrm{vol}(M) \left(\int_M e^{t_q F} \right)^{-1} \frac{\partial}{\partial z^p} \left(e^{t_q F} \det(g_{i\bar{j}}) \right)$$

where we used the subscript q to denote the metric defined by φ_q. Note that

$$\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi_q}{\partial z^i \partial \bar{z}^j} \right) \sum_{a,b} g_q^{a\bar{b}} \frac{\partial^2}{\partial z^a \partial \bar{z}^b}$$

defines a second order operator T. Since its coefficients are given by sums and products of the coefficients $g_{i\bar{j}}$ with the second-order derivatives of φ_q, the following properties are satisfied:

- T is uniformly elliptic, thanks to the second order estimates for φ_q and φ_q being g-positive.
- The coefficients of T are in $C^{0,\alpha}(M)$ (recall $k \geq 3$ and (I)). Moreover, they are uniformly bounded due to the third order estimate for φ_q.

On the other hand, the right-hand side of the equation is a smooth map. We can then apply the Schauder estimate (see [Szé14], p. 32 Theorem 2.10) from which we have a constant C' independent of $\frac{\partial \varphi_q}{\partial z^p}$ such that

$$|| \frac{\partial \varphi_q}{\partial z^p} ||_{C^{2,\alpha}} \leq C'(K_1 + || \frac{\partial \varphi_q}{\partial z^p} ||_{L^1})$$

where K_1 is an upper bound for the α-Hölder norm of the right-hand side of the equation. Using the mean value theorem, we can estimate $|| \frac{\partial \varphi_q}{\partial z^p} ||_{L^1}$ by the second-order estimate. Therefore, we have a constant C'' independent of φ_q for which

$$|| \frac{\partial \varphi_q}{\partial z^p} ||_{C^{2,\alpha}} \leq C''$$

A similar kind of estimate for $||\frac{\partial \varphi_q}{\partial \bar{z}^p}||_{C^{2,\alpha}}$ is obtained by differentiating accordingly $(*_q)$. Consequently, we can proceed by bootstrapping: the recently found estimates imply that

the coefficients of T are uniformly bounded in $C^{1,\alpha}(M)$

then by the Schauder estimate we have the bounds independent of φ_q

$$||\frac{\partial \varphi_q}{\partial z^p}||_{C^{3,\alpha}} \leq D_1 \ , \quad ||\frac{\partial \varphi_q}{\partial \bar{z}^p}||_{C^{3,\alpha}} \leq D_2$$

and these estimates imply that

the coefficients of T are uniformly bounded in $C^{2,\alpha}(M)$

We iterate this process, together with elliptic regularity, to obtain that $\varphi_q \in C^{k+2,\alpha}(M)$ and estimate $||\varphi_q||_{C^{k+2,\alpha}}$ independently of φ_q. Then

$\{\ \varphi_q\}_q$ is uniformly bounded in $C^{k+2,\alpha}(M)$

Since M is compact, we have the compact inclusion

$$C^{k+2,\alpha}(M) \hookrightarrow C^{k+1,\alpha}(M)$$

Thus, up to a subsequence, there is $\varphi \in C^{k+1,\alpha}(M)$ such that $\varphi_q \to \varphi$ in $C^{k+1,\alpha}(M)$ as $t_q \to t_0$. By continuity of $G,\ \gamma$, which we introduced to show that W_k is open, we have that

φ is a solution to $(*_{t_0})$

Furthermore, the convergence also gives for φ the same a priori estimates that hold for any φ_q. Denoting the tensor g' constructed using φ, by Remark 18

$g,\ g'$ are uniformly equivalent

so φ is g-positive, proving that $t_0 \in W_k$.

3.3 Conclusion

We have therefore proved that (M-A) has a g-positive C^k-solution for any $k \geq 3$. We can also require that each of these functions has zero integral over M. Then, by Theorem 4, these solutions actually coincide, resulting in a smooth g-positive solution for (M-A) and proving Calabi's conjecture.

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