Dual Kähler metrics

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(joint work with Roberto Mossa and Fabio Zuddas)

Differential Geometry Workshop in Lerici
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Aim of the talk: *Extend the concept of duality between an Hermitian symmetric spaces to more general Kähler metrics*

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4. Cartan-Hartogs domains and their duals

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An **HSSNT** is a Kähler manifold (M, g) such that for all $p \in M$ the geodesic symmetry:

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Up to homotheties, (M, g) is biholomorphically isometric to a bounded symmetric domain $\Omega \subset \mathbb{C}^n$ centred at the origin $0 \in \mathbb{C}^n$ equipped with the Kähler metric g_Ω whose associated Kähler form is

$$\omega_\Omega = -\frac{i}{2\pi}\partial\bar{\partial} \log N_\Omega$$

where

$$N_\Omega(z,\overline{z}) = (V(\Omega)K_\Omega(z,\overline{z}))^{-\frac{1}{\gamma}}$$

is the **generic norm**, $V(\Omega)$ is the **Euclidean volume** of Ω, γ the **genus** of Ω and K_Ω its **Bergman kernel**.

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Thus

$$g_\Omega = \frac{1}{\gamma} g_B$$

where g_B is the **Bergman metric** with associated Kähler form given by

$$\omega_B = \frac{i}{2\pi} \partial\bar{\partial} \log K_\Omega$$

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and the dimension n of Ω can be written as

$$n = r + \frac{r(r-1)}{2}a + rb$$

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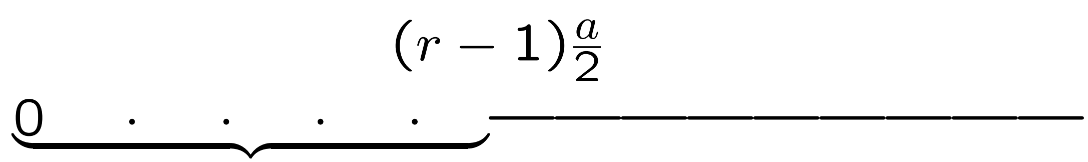
More precisely we have

$$W(\Omega) = \left\{0, \frac{a}{2}, 2\frac{a}{2}, \ldots, (r-1)\frac{a}{2}\right\} \cup \left((r-1)\frac{a}{2}, \infty\right).$$



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$$K_{\mathbb{C}H^n}(z, \bar{z}) = \frac{1}{(1-|z|^2)^{n+1}},\ N_{\mathbb{C}H^n}(z, \bar{z}) = 1 - |z|^2$$

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is a strictly plurisubharmonic on all $\mathbb{C}^n$ and so $\frac{i}{2\pi}\partial\bar{\partial}\log N^*_\Omega$ is a Kähler form on $\mathbb{C}^n$, where $N^*_\Omega(z, \bar{z}) = N_\Omega(z, -\bar{z})$.

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Morever, $\mathbb{C}^n$ can be compactified to a compact Kähler manifold (Ω^*, g_{Ω^*}), the **compact dual of** (Ω, g_Ω), with Kähler form ω_{Ω^*} such that

$$\omega_{\Omega^*|\mathbb{C}^n} = \frac{i}{2\pi}\partial\bar{\partial} \log N^*_\Omega$$

$$0 \in \Omega \overset{HC}{\subset} \mathbb{C}^n \overset{\text{Borel}}{\hookrightarrow} \Omega^* \overset{BW}{\to} \mathbb{C}P^N$$

where BW is a Kähler embedding, i.e. $BW^*(g_{FS}) = g_{\Omega^*}$ and $\Omega^* = \mathbb{C}^n \sqcup H$, where

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Example. $(\mathbb{C}^n, g_{\Omega^*|\mathbb{C}^n})$ *admits a Fubini-Study compactification by taking* $\Psi = BW_{|\mathbb{C}^n} : (\mathbb{C}^n, g_{\Omega^*|\mathbb{C}^n}) \to (\mathbb{C}P^N, g_{FS}),\ P = BW(\Omega^*)$.

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$$-\log N_{\mathbb{C}H^n}(z, \bar{z}) = -\log(1 - |z|^2) \longrightarrow \log N^*_{\mathbb{C}H^n}(z, \bar{z}) = +\log(1 + |z|^2)$$

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• *for all* $p \in \Omega$ *and* $v \in T_p\Omega$ *there exists a totally geodesic Kähler embedding of the polydisk* $\Delta^r := \mathbb{C}H^1 \times \cdots \times \mathbb{C}H^1$ *(*r*-times) passing through* p *and tangent to* v **(polydisk theorem)***.*

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- *for all* $p \in \Omega^*$ *and* $v \in T_p\Omega^*$ *there exists a totally geodesic Kähler embedding of the dual polydisk* $(\Delta^r)^* := \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ *(*r*-times) passing through* p *and tangent to* v **(dual polydisk theorem)**.

• (Ω, ω_Ω) *is a* symplectic dual *of* $(\Omega^*, \omega_{\Omega^*})$, *i.e. there exists a smooth diffeomorphism*

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(called a **symplectic duality***) such that* $\varphi_\Omega^* \omega_0 = \omega_\Omega$ *and* $\varphi_\Omega^* \omega_{\Omega^* | \mathbb{C}^n} = \omega_0$ *(A. J. Di Scala, L., Adv. Math. 2008).*

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• *Let* (g, X) *be a* **KRS*** *on a complex manifold* M*. If there exists a holomorphic isometry of* (M, g) *into* (Ω, g_Ω) *(resp.* (Ω^*, g_{Ω^*})*, then* g *is KE (L. R. Mossa, PAMS 2023).*

$^*Ric_g = \lambda\, g + L_X g$, where X is the real part of a holomorphic vector field.

- *let* Ω *be a bounded symmetric domain of rank* ≥ 2 *and let*

$$f : (\mathbb{C}H^n, g_{hyp}) \to (\Omega, g_\Omega) \subset \mathbb{C}^n \stackrel{\textit{Borel}}{\hookrightarrow} \Omega^* \stackrel{BW}{\to} \mathbb{C}P^N$$

be a holomorphic isometric embedding. Then $f(\mathbb{C}H^n)$ *is an irreducible component of* $BW^{-1}(H) \cap \Omega$ *with* H *hyperplane of* $\mathbb{C}P^N$ *(S. T. Chan, N. Mok, Math. Z. 2017)*

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$\bullet(\Omega, \alpha g_\Omega) \stackrel{\varphi_{\alpha,\Omega}}{\rightarrow} (\mathbb{C}P^\infty, g_{FS})$ *iff* $\alpha \in W(\Omega) \setminus \{0\}$ *(L. , M. Zedda, Math Ann. 2011)*

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• $(\Omega^*, \alpha g_{\Omega^*}) \stackrel{\varphi_{\alpha,\Omega^*}}{\rightarrow} (\mathbb{C}P^{N_\alpha}, g_{FS})$ *iff* $\alpha \in \mathbb{Z}^+$ *(for* $\alpha = 1$ *one gets the BW embedding)*

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• (Ω, g_Ω) *and* (Ω^*, g_{Ω^*}) *are not* **relatives**, *i.e. they do not share a common non trivial Kähler submanifold (A. J. Di Scala, L. , Ann. Sc. Norm. Super. Pisa 2010)*

• *any holomorphic isometry* $(\Omega_1, g_{\Omega_1}) \to (\Omega_2, g_{\Omega_2})$ *between bounded symmetric domains with* Ω_1 *irreducible of rank* ≥ 2 *is totally geodesic (N. Mok, J. Eur. Math. Soc. 2012)*

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Let (M, g) *be a* **compact Kähler manifold** *and* $[\omega] = c_1(L)$*, with* L *ample line bundle on* M*. The* **Kempf distortion function** *(or* **density function** *in Donaldson's terminology) is defined by*

$$T_{\alpha g}(x) = \sum_{j=1}^{N_\alpha} h_\alpha(s_j(x), s_j(x)),\ \ x \in M, \alpha \in \mathbb{Z}^+$$

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where $s_0, \ldots, s_{N_\alpha}$, $N_\alpha + 1 = \dim H^0(L^{\otimes\alpha})$ *is an orthonormal basis w.r.t*

$$\langle s, t\rangle_\alpha = \int_M h_\alpha(s,t)\frac{\omega^n}{n!}, s, t \in H^0(L^{\otimes\alpha}),\ \mathit{Ric}(h_\alpha) = \alpha\omega$$

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- $a_0 = 1$ *and any* $a_j(x)$ *is a polynomial of the curvature and its covariant derivatives at* x *of the metric* g *(Z. Lu, Amer. J. Math. 2000).*

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- the coefficients a_j of MME expansion of Rawnsley's ϵ function of the metric g_Ω are constants; the coefficients a_j^* of TYCZ expansion of Kempf distortion function of the metric g_{Ω^*} are constants and

Balanced metrics on HSS

• αg_Ω *is balanced iff* $\alpha > \gamma - 1$ *(L. , M. Zedda, Math. Z. 2012)*

• αg_{Ω^*} *is balanced iff it is projectively induced iff* $\alpha \in \mathbb{Z}^+$ *(C. Arezzo, L. , Comm. Math. Phys. 2004)*

• *the coefficients* a_j *of MME expansion of Rawnsley's* ϵ *function of the metric* g_Ω *are constants; the coefficients* a_j^* *of TYCZ expansion of Kempf distortion function of the metric* g_{Ω^*} *are constants and*

$a_j = (-1)^j a_j^*$ *(L. , M. Zedda, Manuscripta Math. 2015)*

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Among all the potentials the Calabi's diastasis function (Calabi, Ann. Math. 1952) is characterized by

$$D^g_0(z) = \sum_{|I|,|J|\geq 0} a_{IJ} z^I \bar z^J,\ a_{J0} = a_{0J} = 0$$

Let (U, g) *and* (U^*, g^*) *be complex domains of* $\mathbb{C}^n$ *containing the origin with real analytic Kähler forms* $\omega = \frac{i}{2\pi}\partial\bar{\partial}D_0^g$ *and* $\omega^* = \frac{i}{2\pi}\partial\bar{\partial}D_0^{g^*}$*.*

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Definition. We say that (U^*, g^*) is the Kähler dual of (U, g) and g^* is a Kähler metric dual to g (and viceversa) if

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Example. *Let (Ω, g_Ω) be a bounded symmetric domain. Then its Kähler dual is given by $(\mathbb{C}^n, g_{\Omega^*|\mathbb{C}^n})$.*

Example. *Consider the compact flag manifold* $\frac{SU(3)}{S(U(1)^3)}$.

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One can write Calabi's diastasis function for the general $SU(3)$*-invariant Kahler metric* g *using Alekseevsky-Perelomov coordinates centred at a point* p*:*

$$D_0^g(z) = c_1 \log \Delta_1(z) + c_2 \log \Delta_2(z), \ z \in \mathbb{C}^3, \ c_1, c_2 \in \mathbb{R}^+$$

$$\Delta_1(z) = \log[1 + |z_1|^2 + |z_2|^2 + |z_1|^2|z_3|^2 + z_2\bar{z}_1\bar{z}_3 + \bar{z}_2 z_1 z_3]$$

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Thus the metric g does not admit a dual.

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Conjecture: We believe that if a homogeneous Kähler metric on a flag manifold admits a Kähler dual then the flag manifold is an HSSCT.

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Cartan-Hartogs domains (*CH* domains in the sequel) are a 1-parameter family of noncompact nonhomogeneous domains of $\mathbb{C}^{n+1}$, given by:

$$M_{\Omega,\mu} := \left\{(z,w) \in \Omega \times \mathbb{C} \mid |w|^2 < N_\Omega^\mu(z,\bar{z})\right\} \subset \mathbb{C}^{n+1}$$

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We endow $M_{\Omega,\mu}$ with the complete Kähler metric $g_{\Omega,\mu}$ whose associated Kähler form is given by

$$\omega_{\Omega,\mu} = -\frac{i}{2}\partial\bar{\partial}\log\left(N_\Omega^\mu(z,\bar{z}) - |w|^2\right)$$

(A. Wang, W. Yin, L. Zhang, and W. Zhang, Asian J. Math.2004)

Remark A CH domain is homogeneous iff Ω has rank one, i.e. $\Omega = \mathbb{C}H^n$ is the unit ball in $\mathbb{C}^n$ and $\mu = 1$. In this case $M_{\Omega,\mu} = \mathbb{C}H^{n+1}$ and $g_{\Omega,\mu} = g_{hyp}$.

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(4) (CH-polydisk theorem) The Cartan-Hartogs polydisk theo-

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Theorem.(R. Mossa, M. Zedda, Ann. Mat. Pura Appl. 2022) There exists a symplectic duality between $(M_{\Omega,\mu}, g_{\Omega,\mu})$ and $(\mathbb{C}^{n+1}, \omega^*_{\Omega,\mu})$, i.e. a smooth diffeomorphism $\varphi : M_{\Omega,\mu} \to \mathbb{R}^{2n+2}$ such that $\varphi^*\omega_0 = \omega_{\Omega,\mu}$ and $\varphi^*\omega^*_{\Omega,\mu} = \omega_0$, iff $(\mathbb{C}^{n+1}, g^*_{\Omega,\mu}) = (\mathbb{C}^{n+1}, g_{FS})$ iff $\Omega = \mathbb{C}H^n, \mu = 1$.

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(e) $\left(\mathbb{C}^{n+1}, g^*_{\Omega,\mu}\right)$ admits a Fubini–Study compactification

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• Let (g, X) be a KRS on a complex manifold M and Ω_i, $i = 1, 2$ be Cartan domains. If there exists a holomorphic isometry of (M, g) into $(M_{\Omega_1,\mu_1}, g_{\Omega_1,\mu_1})$ and into $(\mathbb{C}^{n+1}, g^*_{\Omega_2,\mu_2})$, with $\mu_1, \mu_2 \in \mathbb{Q}^+$, then g is KE

The Bergman metric on Cartan-Hartogs domains

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Theorem.(L., R. Mossa, F. Zuddas, 2024) *Let* $M_{\Omega,\mu}$ *be a CH domain and* $\tilde{g}_{\Omega,\mu}$ *be its Bergman metric, Then the Kähler dual* $(U^*, \tilde{g}^*_{\Omega,\mu})$ *can be defined* ($U^* \neq \mathbb{C}^{n+1}$ *in general). Moreover the following conditions are equivalent*

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Thank you for your attention!