Universitá degli studi di Cagliari

Dipartimento di Matematica e Informatica

Tesi di laurea magistrale

CHARACTERISTIC CLASSES AND WHITNEY'S IMBEDDING THEOREM

Candidato:
Marco Casula
Matr. 60/65/65111

Relatore:
Prof. Andrea Loi

Anno accademico 2022/2023

Contents

1 Introduction

Characteristic classes are cohomology classes associated to vector bundles. They measure in some way how a vectore bundle is twisted, or nontrivial. After a short recall of the main contents and notations in differential geometry and bundles in Section 2, we will define in Section 3 these classes from a geometrical meaning as in [1] and [3], focused on connections and curvature of a manifold and the Whitney's Imbedding Theorem, which states that *for any finite CW complex M, a bundle can be induced by a continuous map from M to the Grassmann manifold (of sufficiently high dimension)*, namely, we will define the characteristic classes by cohomology since it is homotopy-invariant.

The following Section 4 links the above meaning of characteristic classes with the one given from connections: in fact, the usual way to compute them is by using polynomials in the ring $H^*(M, \mathbb{F})$ (where $\mathbb{F}$ depends on the context), according to Weil homomorphism. Representatives of characteristic classes will be closed differential forms constructed from the curvature form of a connection. This section is concluded by some examples of explicit computations.

In the last part of this work (Section 5) we will expose characteristic classes as viewed in [5]: they satisfies there the naturality property of commuting with the pullback, and it is equivalent to definition of representatives given in the previous section. Indeed, this is a *topological approach*, involving properties of the bundle that are not a priori related to any connection or curvature.

There are four main kinds:

1. Stiefel-Whitney classes $w^i(E) \in H^i(M, \mathbb{Z}_2)$ for a real vector bundle $\pi : E \to M$.

2. Chern classes $c^i(E) \in H^{2i}(M, \mathbb{Z})$ for a complex vector bundle.

3. Pontrjagin classes $p^i(E) \in H^{4i}(M, \mathbb{Z})$ for a real vector bundle.

4. The Euler class $e(E) \in H^n(M, \mathbb{Z})$ for an oriented n-dimensional real vector bundle.

The Stiefel-Whitney and Chern classes are formally quite similar. Pontrjagin classes can be regarded as a refinement of Stiefel-Whitney classes when one takes $\mathbb{Z}$ rather than $\mathbb{Z}_2$ coefficients, and the Euler class is a further refinement in the orientable case.

For an q-dimensional vector bundle $\pi : E \to M$ to be trivial is equivalent to its classifying map $f : M \to G_q$ being nullhomotopic, but as with most things in homotopy theory it can be quite difficult to determine whether this is the case or not. Much more accessible is the weaker question of whether f

induces a nontrivial map on homology or cohomology, and this is precisely what characteristic classes measure. The Stiefel-Whitney classes w^i, like the other characteristic classes, satisfy the important naturality property that

$$w^i(f^*(E)) = f^*(w^i(E))$$

so in particular if f is a classifying map $f : M \to G_q$ for E we have $E = f^*(E_q)$ and $w^i(E) = w^i(f^*(E_q)) = f^*(w^i(E_q))$. Thus if f induces the trivial map on $\mathbb{Z}_2$ cohomology, then the Stiefel-Whitney classes of E are trivial. The converse statement is also true because the classes $w^i(E_q)$ generate the cohomology ring $H^*(G_q, \mathbb{Z}_2)$. In fact $H^*(G_q, \mathbb{Z}_2)$ is exactly the polynomial ring $\mathbb{Z}_2[w^1(E_q), \ldots, w^q(E_q)]$.

The vanishing of all the characteristic classes of a vector bundle is a necessary condition for it to be trivial, but it is not always sufficient, as there exist nontrivial vector bundles whose characteristic classes are all zero. Perhaps the simplest example is the tangent bundle of the sphere S^5. One reason why characteristic classes are not sufficient to determine when a vector bundle is trivial is that, except for the Euler class, they are *stable invariants*, meaning that taking the direct sum of a given bundle with a trivial bundle does not change the characteristic classes.

In conclusion, this work has the target to expose the two interpretations of characteristic classes as above, giving in particular the example of the complex projective space via both methods.

2 Preliminaries on Manifolds and Bundles

For this section we will refer to [2] (differential geometry) and [4] (bundles and CW-complexes).

2.1 Tensors and Forms

Let V be a real vector space of dimension n. Then a basis $B = \{b_1, \dots, b_n\}$ induces a dual basis $B^* = \{\beta^1, \dots, \beta^n\}$, i.e. $\beta_i : V \to \mathbb{R}$ is the (unique) linear functional such that $\beta^i(b_j) = \delta^i_j$. Hence, any $v \in V$ and $\omega \in V^*$ can be uniquely decomposed as

$$v = \sum_{i=1}^n v^i b_i, \qquad \omega = \sum_{j=1}^n \omega_j \beta^j \qquad \Longrightarrow \qquad \omega(v) = \sum_{i=1}^n \omega_i v^i$$

Definition 2.1. Let $V_1, \dots, V_k, W$ be vector spaces. A map

$$F : V_1 \times \cdots \times V_k \to W$$

is said to be *multilinear* (or *k-linear*) if it is linear on each component.

A (k,l)-*tensor* on V is a multilinear map

$$F : V^* \times \cdots \times V^* \times V \times \cdots \times V \to \mathbb{R}$$

We will denote $T^{(k,l)}(V)$ the set of all (k,l) tensors over V.

Remark 2.2.
- *For any $k, l \in \mathbb{N}$, $T^{(k,l)}(V)$ is a vector space with operations:*

$$(F+G)(\cdot) = F(\cdot) + G(\cdot), \qquad (\lambda F)(\cdot) = \lambda F(\cdot)$$

- *Elements of $T^k(V) := T^{(k,0)}$ are called* contravariant *tensors.*
- *Elements of $T^l(V^*) := T^{(0,l)}$ are called* covariant *tensors.*

We would like to define a multiplication of tensors, i.e. given any $F \in T^{(k,l)}(V)$, $G \in T^{(p,q)}(V)$, obtaining a tensor $F \otimes G \in T^{(k+p,l+q)}(V)$.

Definition 2.3. F, G as above, their *tensorial product* is the map

$$\begin{aligned} F \otimes G : V^* \times \cdots \times V^* \times V \times \cdots \times V &\to \mathbb{R} \\ (\omega^1, \dots, \omega^{k+p}, v_1, \dots, v_{l+q}) &\mapsto F(\omega^1, \dots, \omega^k, v_1, \dots, v_l) \cdot \\ &\quad \cdot G(\omega^{k+1}, \dots, \omega^{k+p}, v_{l+1}, \dots, v_{l+q}) \end{aligned}$$

Remark 2.4. *It is easy to prove the following properties:*

- *$F \otimes G$ is multilinear*
- $(F_1 + F_2) \otimes G = F_1 \otimes G + F_2 \otimes G$
- $F \otimes (G_1 + G_2) = F \otimes G_1 + F \otimes G_2$
- $\lambda(F \otimes G) = (\lambda F) \otimes G = F \otimes (\lambda G)$*, for any* $\lambda \in \mathbb{R}$
- $(F \otimes G) \otimes H = F \otimes (G \otimes H)$

Moreover, the tensorial product is not commutative in general.

Example 2.5. Let $\dim(V) = 2$ and the basis $B = \{b_1, b_2\}$, with dual $B^* = \{\beta^1, \beta^2\}$. Then $\beta^1 \otimes \beta^2 \neq \beta^2 \otimes \beta^1$, since

$$(\beta^1 \otimes \beta^2)(b_1, b_2) = \beta^1(b_1) \cdot \beta^2(b_2) = 1$$

$$(\beta^2 \otimes \beta^1)(b_1, b_2) = \beta^2(b_1) \cdot \beta^1(b_2) = 0$$

We can introduce a special class of covariant tensors, according to the following

Definition 2.6. Let F be a k-covariant vector, i.e. $F \in T^k(V^*)$. F is said to be a *k-form* (or an *alternating tensor*) if

$$F(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -F(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k), \qquad \forall i \neq j$$

Equivalently, F is a k-form if

$$v_i = v_j \quad \textit{(with } i \neq j\textit{)} \quad \Longrightarrow \quad F(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = 0$$

The set of k-forms on V is denoted by $\Lambda^k(V^*)$.

The k-forms form a subspace of $T^k(V^*)$, but it is not closed with respect to tensorial product.

Example 2.7. The product $\beta^1 \otimes \beta^2$ in Example 2.5 is not a 2-form since

$$1 = (\beta^1 \otimes \beta^2)(b_1, b_2) \neq -(\beta^1 \otimes \beta^2)(b_2, b_1) = 0$$

To multiplicate forms, we have to modify the tensorial product in the following way

Definition 2.8. Let $F \in T^k(V^*)$; its *alternatization* is the tensor $\mathrm{Alt}(F) \in \Lambda^k(V^*)$ defined by

$$\begin{aligned} \mathrm{Alt}(F) : V \times \cdots \times V &\to \mathbb{R} \\ (v_1, \ldots, v_k) &\mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \varepsilon(\sigma) F(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \end{aligned}$$

Given two forms $\omega \in \Lambda^k(V^*)$, $\eta \in \Lambda^h(V^*)$, their *wedge product* is

$$\omega \wedge \eta := \frac{(k+h)!}{k!\, h!} \mathrm{Alt}(\omega \otimes \eta) \in \Lambda^{k+h}(V^*)$$

Remark 2.9. *It is easy to prove the following properties:*

- $(\omega + \omega') \wedge \eta = \omega \wedge \eta + \omega' \wedge \eta$
- $\omega \wedge (\eta + \eta') = \omega \wedge \eta + \omega \wedge \eta'$
- $\lambda(\omega \wedge \eta) = (\lambda\omega) \wedge \eta = \omega \wedge (\lambda\eta)$, *for any* $\lambda \in \mathbb{R}$
- $(\omega \wedge \eta) \wedge \sigma = \omega \wedge (\eta \wedge \sigma) = \frac{(k+h+l)!}{k!\, h!\, l!} \mathrm{Alt}(\omega \otimes \eta \otimes \sigma)$
- $\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega$
- $\omega^k := \omega \wedge \cdots \wedge \omega = 0$ *if* $k > \dim V$

We are now able to compute the dimension of the spaces $T^{(k,l)}(V)$ and $\Lambda^k(V^*)$.

Lemma 2.10. *Let V be a vector space, $B = \{b_1, \ldots, b_n\}$ a basis for V and $B^* = \{\beta^1, \ldots, \beta^n\}$ its dual.*

The set of tensorial products

$$\left\{\, b_{i_1} \otimes \cdots \otimes b_{i_k} \times \beta^{j_1} \otimes \cdots \otimes \beta^{j_l} \,\right\}, \qquad i_\cdot, j_\cdot \in \{1, \ldots, n\}$$

is a basis for $T^{(k,l)}(V)$.

The set of wedge products

$$\left\{\, \beta^{i_1} \wedge \cdots \wedge \beta^{i_k} \,\right\}, \qquad 1 \leq i_1 < \cdots < i_k \leq n$$

is a basis for $\Lambda^k(V^*)$.

Hence

$$\dim\left(T^{(k,l)}(V)\right) = n^{k+l}, \qquad \dim\left(\Lambda^k(V^*)\right) = \binom{n}{k}$$

2.2 Differentiable Manifolds

Let M be a (differentiable) manifold, $\dim M = n$. Given a point $p \in M$, we define an equivalent relation on smooth functions in a neighborhood of p to be «$(f, U) \sim (g, V)$ if and only if $f|_W = g|_W$ for an open $W \subseteq U \cap V$, $p \in W$». Its elements are classes $[(f, U)]$ (a *germ of functions*) with $f \in C^\infty(U)$ and $U \subseteq M$ open set containing p. We will denote simply with $f \in C^\infty(U)$ the germ $[(f, U)]$, where the point p is omitted and is clear from the context.

Thank to the relation above, we can consider the tangent space T_pM in $p \in M$ of all C^∞ directional derivatives $v : C^\infty(U) \to \mathbb{R}$ with $U \subseteq M$ open, $p \in U$, such that $\forall f, g \in C^\infty(U)$, $\forall a, b \in \mathbb{R}$:

1. v is linear, i.e. $v(af + bg) = av(f) + bv(g)$
2. $v(f \cdot g) = v(f)g + fv(g)$,

Remark 2.11. *Let (U, φ) be a chart of M in p, $\varphi = (x^1, \dots, x^n)$ local coordinates. Then we can consider the maps*

$$\frac{\partial}{\partial x^i}|_p : C^\infty(U) \to \mathbb{R}$$
$$f \mapsto \frac{\partial}{\partial x^i}|_p(f) := \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}(\varphi(p))$$

We have that $B = \{ \frac{\partial}{\partial x^1} \mid {}_p, \dots, \frac{\partial}{\partial x^n}|_p \}$ is a basis for T_pM, so any $v \in T_pM$ can be written as $v = \sum_{i=1}^n v(x^i)\frac{\partial}{\partial x^i}|_p$. The dual of B is $B^* = \{ dx^1 \mid {}_p, \dots, dx^n|_p \}$. If not specifically written, we will consider these B, B^* as "canonical" basis for a given chart $(U, \varphi) = (U, x^i, \dots, x^n)$.

Example 2.12 (Differential of a function)**.** Consider $f : U \to \mathbb{R}$ a smooth function on $U \subseteq M$. Given a point $p \in U$, the linear map $df|_p = df_p : T_pM \to \mathbb{R}$, $v \mapsto df_p(v) = v(f)$ is called the *differential* of f. Taking the basis B, B^* of T_pM, T_p^*M respectively, we have

$$df_p = \sum_i df_p(\frac{\partial}{\partial x^i}|_p)\, dx^i|_p = \sum_i \frac{\partial}{\partial x^i}|_p(f)\, dx^i|_p = \sum_i \frac{\partial(f \circ \varphi^{-1})}{\partial x^i}|_{\varphi(p)}\, dx^i|_p$$

Hence, T_pM is a real vector space of dimension n, so it is possible to define over it tensors and forms.

Definition 2.13. Let M be a manifold, T_pM the tangent space at $p \in M$.

A *vector field* is a map $X : M \to TM$ such that $X(p) = X_p \in T_pM$ for any point $p \in M$. Then we can decompose $X_p = \sum_i X^i(p)\frac{\partial}{\partial x^i}|_p$; consequently,

X is said to be *smooth* if $X^i \in C^\infty(U)$ for any $i = 1, \dots, n$. The set of smooth vector field over M is denoted by $\mathfrak{X}(M)$.

A 1-*form* is a map $\omega : M \to T^*M$ such that $\omega(p) = \omega_p \in T_p^*M$ for any point $p \in M$. Then we can decompose $\omega_p = \sum_i \omega_i(p) dx^i|_p$; consequently, ω is said to be *smooth* if $\omega_i \in C^\infty(U)$ for any $i = 1, \dots, n$. The set of smooth 1-form over M is denoted by $\Lambda^1(M)$.

A k-*form* is a map $\omega : M \to \Lambda^k(T^*M)$ such that $\omega(p) = \omega_p \in \Lambda(T_p^*M)$ for any point $p \in M$, i.e. $\omega_p : T_pM \times \cdots \times T_pM \to \mathbb{R}$ alternating multilinear map. Given (as above) a decomposition

$$\omega_p = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \dots i_k}(p)\, dx^{i_1}|_p \wedge \cdots \wedge dx^{i_k}|_p$$

ω is said to be *smooth* if $\omega_{i_1 \dots i_k} \in C^\infty(U)$. The set of smooth k-forms over M is denoted by $\Lambda^k(M)$.

A (k, l)-*tensor field* is a map $F : M \to T^{(k,l)}(TM)$ such that $F(p) = F_p \in T^{(k,l)}(T_pM)$ for any point $p \in M$, i.e.

$$F_p : T_p^*M \times \cdots \times T_p^*M \times T_pM \times \cdots \times T_pM \to \mathbb{R}$$

multilinear map. Given (as above) a decomposition

$$F_p = \sum F^{j_1 \dots j_l}_{i_1 \dots i_k}(p) \frac{\partial}{\partial x^{i_1}}|_p \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}}|_p \otimes dx^{j_1}|_p \otimes \cdots \otimes dx^{j_l}|_p \qquad (1)$$

F is said to be *smooth* if $F^{j_1 \dots j_l}_{i_1 \dots i_k} \in C^\infty(U)$. The set of smooth (k, l)-tensor fields over M is denoted by $\mathfrak{T}^{(k,l)}(M)$.

Remark 2.14. $\mathfrak{X}(M), \Lambda^1(M), \Lambda^k(M)$ *and* $\mathfrak{T}^{(k,l)}(M)$ *are vector spaces with operations defined as in Remark 2.2.*

Remark 2.15. *Exterior differentiation of forms, i.e. a linear map*

$$d : \Lambda^k(M) \to \Lambda^{k+1}(M)$$

satisfies these relations together with the wedge product:

- $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta, \qquad k = \deg \omega$
- $d^2\omega = d(d\omega) = 0$
- *If* $f \in \Lambda^0(M) = C^\infty(M)$*, then* df *is the usual differential of a function.*

We conclude this subsection with an identification of tensors and multilinear functions over a vector space V, in the special case $V = T_pM$.

Proposition 2.16. *Let $F \in \mathfrak{T}^{(k,l)}(M)$ a tensor field. Then we can consider the map*

$$\tilde{F} : \Lambda^1(M) \times \cdots \times \Lambda^1(M) \times \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$$

defined by

$$\tilde{F}(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l)(p) := F_p(\omega^1(p), \ldots, \omega^k(p), Y_1(p), \ldots, Y_l(p))$$

We have that $\tilde{F}$ is smooth and $C^\infty(M)$-multilinear. Conversely, given any map

$$A : \Lambda^1(M) \times \cdots \times \Lambda^1(M) \times \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M)$$

smooth and $C^\infty(M)$-multilinear, then $\exists! F \in \mathfrak{T}^{(k,l)}(M)$ such that $\tilde{F} = A$.

Proof. Let $p \in M$ be a point, $(U, x^1, \ldots, x^n)$ chart in p and the usual basis B, B^* for T_pM, T_p^*M. Since F can be decomposed as in (1), we obtain

$$\begin{aligned}\tilde{F}(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l)(p) &= F_p(\omega^1(p), \ldots, \omega^k(p), Y_1(p), \ldots, Y_l(p)) \\ &= \sum F^{j_1 \ldots j_l}_{i_1 \ldots i_k}(p)\,(\omega^1_{i_1} \cdot \cdots \cdot \omega^k_{i_k} \cdot Y^{j_1}_1 \cdot \cdots \cdot Y^{j_l}_l)|_p\end{aligned}$$

and it is smooth and $C^\infty(M)$-multilinear.

Now consider a map A as in the statement. It sufficies to define F_p by

$$F_p(w^1, \ldots, w^k, v_1, \ldots, v_l) = A(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l)(p) \tag{2}$$

where ω^i are one-forms such that $\omega^i(p) = w^i$ and Y_j are vector fields such that $Y_j(p) = v_j$. It is easy to prove that $\tilde{F} = A$ and the uniqueness is given by the fact that the expression (2) does not change taking $\eta^i \in \Lambda^1(M)$ with $\eta^i(p) = \omega^i(p)$ and $Z_j \in \mathfrak{X}(M)$ with $Z_j(p) = Y_j(p)$. □

We will not distinguish between F and $\tilde{F}$, and they will be both denoted with F.

2.3 Riemannian Metrics

Definition 2.17. Let M be a manifold. A *riemannian metric* on M is a tensor $g \in \mathfrak{T}^{(0,2)}(M)$ that is symmetric and positive-definite, i.e.

$$g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$$

$C^\infty(M)$-bilinear such that

- $g(X,Y) = g(Y,X)$
- $g(X,X) \geq 0$, and $g(X,X) = 0 \iff X = 0$.

A *riemannian manifold* is a couple (M, g), where M is a differentiable manifold and g is a riemannian metric on it.

Locally, we can consider a chart $(U, x^1, \dots, x^n)$ in $p \in M$ and the usual basis B, B^*. Due to $C^\infty(M)$-linearity of g, it is determined by the values

$$g_{ij}(p) = g_p(\frac{\partial}{\partial x^i}|_p, \frac{\partial}{\partial x^j}|_p)$$

These (smooth) functions define a matrix $g|_U = (g_{ij})$, $i, j = 1, \dots, n$.

Definition 2.18. A *local basis* over an open $U \subseteq M$ is a set $\{E_1, \dots, E_n\}$, where $E_i \in \mathfrak{X}(U)$ have the feature that $\{E_1(p), \dots, E_n(p)\}$ is a basis for T_pM for any $p \in U$.

A local basis is said to be *orthonormal* if for any $p \in U$

$$g_p(E_i(p), E_j(p)) = \delta_{ij}$$

Is it now desirable to define a special class of curves on riemannian manifold analogously to straight lines in the euclidean space. To do that, we need to differentiate the tangent space TM.

Definition 2.19. Let M be a manifold; a *linear connection* over M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

denoted by $\nabla(X,Y) =: \nabla_X Y$, such that $\forall f, g \in C^\infty(M)$, $\forall X, X_i, Y, Y_j \in \mathfrak{X}(M)$, $\forall a, b \in \mathbb{R}$:

a) $\nabla_{fX_1+gX_2} Y = f\nabla_{X_1} Y + g\nabla_{X_2} Y$

b) $\nabla_X (aY_1 + bY_2) = a\nabla_X Y_1 + b\nabla_X Y_2$

c) $\nabla_X (fY) = f\nabla_X Y + X(f)Y$

$\nabla_X Y$ is also called the *covariant derivative* of Y in direction X.

Remark 2.20. *∇ is not a tensor since by (c) it is not $C^\infty(M)$-linear on the second component.*

Since $\nabla_X Y \in \mathfrak{X}(M)$, we can take a local basis $\{E_1, \dots, E_n\}$ on $U \subseteq M$ and decompose this vector field with respect to the basis. In particular, we can compute

$$\nabla_{E_i} E_j = \sum_{k=1}^{n} \Gamma_{ij}^k E_k$$

where $\Gamma_{ij}^k \in C^\infty(U)$ are uniquely determined by the basis. They are called the *Christoffel symbols* of the connection ∇. Given two generic vector fields $X, Y \in \mathfrak{X}(M)$, $X = \sum_{i=1}^n X^i E_i$ and $Y = \sum_{j=1}^n Y^i E_i$, the covariant derivative of Y in direction X becomes

$$\nabla_X Y = \sum_{k=1}^{n} \Big(X(Y^k) + \sum_i \sum_j \Gamma_{ij}^k X^i Y^j \Big) E_k$$

Remark 2.21. *It is well-known that for any chart (U, φ) of a manifold M there exists a riemannian metric g given by the pullback of the euclidean metric on $\mathbb{R}^n$. Moreover, it is also possible to define a linear connection over the same chart taking n^3 smooth functions as Christoffel symbols.*

It is possible to extend these two local quantities in the following way: recall that a partition of the unity *on M is a family of $C^\infty(M)$ positive maps $\{\phi_i\}_{i \in I}$ such that:*

1. *each ϕ_i has compact support;*
2. *the family of supports is locally finite;*
3. $\sum_i \phi_i(p) = 1, \qquad \forall p \in M.$

A partition of the unity $\{\phi_i\}_{i \in I}$ is subordinated *to a cover $\{U_a\}_{a \in A}$ if for each $i \in I$ there exists $a \in A$ such that* $\mathrm{supp}(\phi_i) \subseteq U_a$.

Using the following, a riemannian metric or a connection can be defined globally on a manifold.

Theorem 2.22. *Let M be a manifold, $\{U_a\}_{a \in A}$ open cover of M. Then there exists a countable partition of the unity $\{\phi_i\}$ subordinated to the cover.*

2.4 Vector Bundles

Let us recall some fundamental properties of vector bundles.

Definition 2.23. A *k-dimensional vector bundle* is $(E, M, \pi : E \to M)$ where E is a manifold (called the *total space*), M is another manifold (the *basis space*) and π is a smooth surjective function (the *projection*) such that:

1. $\forall p \in M$, $E_p := \pi^{-1}(p)$ is a k-dimensional vector space (the *fiber* over p);
2. $\forall p \in M$, $\exists U \subseteq M$ open, $p \in U$, $\exists \varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ diffeomorphism s.t.
$$\pi = \pi_U \circ \varphi$$
where $\pi_U : U \times \mathbb{R}^k \to U$ is the projection on the first factor (φ is a *local trivialization*);
3. $\forall p \in M$, $\varphi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is an isomorphism of vector spaces.

We will denote with $\mathrm{Vect}^n(M)$ the set of n-dimensional vector bundles over M.

Example 2.24. $E = M \times V$, where M is a manifold, V k-dimensional vector space, $\pi = \pi_M : M \times V \to M$ projection onto the first factor is the *trivial bundle.*

Example 2.25. If we let E be the quotient space of $I \times \mathbb{R}$ under the identifications $(0, t) \sim (1, -t)$, then the projection $I \times \mathbb{R} \to I$ induces a map $\pi : E \to S^1$ which is a 1-dimensional vector bundle, or *line bundle.*

More generally, consider the base space $M = G_q(\mathbb{R}^n)$ (the Grassmann manifold) and the trivial bundle $E = G_q(\mathbb{R}^n) \times \mathbb{R}^n$, i.e.
$$\pi : G_q(\mathbb{R}^n) \times \mathbb{R}^n \to G_q(\mathbb{R}^n)$$
is the canonical q-dimensional bundle. For $q = 1$ we get
$$(G_1(\mathbb{R}^n) \times \mathbb{R}^n, \pi, G_1(\mathbb{R}^n)) = (\mathbb{R}P^{n-1} \times \mathbb{R}^n, \pi, \mathbb{R}P^{n-1})$$
called the *canonical line bundle* (this construction also holds for the complex case).

Example 2.26. The *tangent bundle*
$$TM = \bigcup_{p \in M} T_pM = \{\ (p, v) : p \in M,\ v \in T_pM\ \}$$
and the projection $\pi(p, v) = p$. Given a chart $(U, x^1, \dots, x^n)$ in $p \in M$, for any $v \in T_pM$ we can decompose $v = \sum_i v^i \frac{\partial}{\partial x^i}|_p$, hence
$$\begin{aligned} \varphi : \pi^{-1}(U) &\to U \times \mathbb{R}^n \\ (p, v) &\mapsto (p, v^1, \dots, v^n) \end{aligned}$$
is a local trivialization.

From now we will consider vector bundles (sometimes called only "bundles") identified with the total space E, whether the basis and the projection are obvious from the context.

Definition 2.27. Let $(E, M, \pi : E \to M)$ be a vector bundle. A *section* of the bundle is a map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. A section is *smooth* if it is a C^∞ map between manifolds.

The set of all smooth sections on the bundle is denoted by

$$\Gamma(E) = \{ \text{ Smooth Sections of } E \ \}$$

Remark 2.28. *$\Gamma(E)$ is a vector space, in fact let $s, s' \in \Gamma(E)$, then $s(p), s'(p) \in E_p$ since*

$$\pi(s(p)) = \mathrm{id}_M(p) = p \quad \Longrightarrow \quad s(p) \in \pi^{-1}(p) = E_p$$

and the same implication holds for s'. Hence the sum $(s+s')(p) := s(p)+s'(p)$ and the product by real number $(\lambda s)(p) := \lambda \cdot s(p)$ are well defined as E_p is a vector space by Definition 2.23,(1).

Example 2.29. These sections will be widely used in the rest of this work:

- $\Gamma(TM) = \mathfrak{X}(M)$
- $\Gamma(T^*M) = \left\{ \omega : M \to T^*M \;\middle|\; \omega(p) \in T^*_pM \right\} = \Lambda^1(M)$
- $\Gamma(T^{(k,l)}(TM)) = \mathfrak{T}^{(k,l)}(M)$

Sections are used to differentiate the total space of a bundle, that was our target since we are interested in defining the analogous of straight lines (in euclidean spaces) in a generic riemannian manifold, using the feature that "straight lines have acceleration zero". The concept of acceleration involves operations between speed vectors (that lie in the tangent space, i.e. a particular bundle) that we are now able to introduce.

Definition 2.30. Let $(E, M, \pi : E \to M)$ be a bundle. A *connection* over E is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

such that the following properties are satisfied $\forall f, g \in C^\infty(M)$, $\forall X, X_1, X_2 \in \mathfrak{X}(M)$, $\forall s, s_1, s_2 \in \Gamma(E)$, $\forall a, b \in \mathbb{R}$:

1. $\nabla_{fX_1+gX_2} s = f\nabla_{X_1} s + g\nabla_{X_2} s$
2. $\nabla_X(as_1 + bs_2) = a\nabla_X s_1 + b\nabla_X s_2$

3. $\nabla_X(fs) = f\nabla_X s + X(f)s$

Remark 2.31. *If $E = TM$, then $\Gamma(E) = \mathfrak{X}(M)$ and connections over TM coincide with linear connections previously defined.*

More generally, it is possible to take as a starting point a linear connection ∇ over M and define from that a family of connections $\nabla^{(k,l)}$ (in the sense of Definition 2.30) over each bundle $T^{(k,l)}(TM)$ that coincide on TM, i.e. $\nabla^{(1,0)} = \nabla$.

We will not distinguish between the connections of that family (the superscript will be obvious from the context), and we will denote all of them simply with ∇.

Finally, we will give an equivalent definition of a connection in Section 3.3, removing the dependence on the vector field as first factor. It will be denoted as D instead of Δ.

2.4.1 Pullback and Universal principal Bundles

There are two particular types of bundles that will be used in the statement of Whitney's Theorem.

Proposition 2.32. *Given a map $f : A \to B$ and a bundle $\pi : E \to B$, then there exists a bundle $\pi' : E' \to A$ with a map $f' : E' \to E$ taking the fiber of E' over each point $a \in A$ isomorphically onto the fiber of E over $f(a)$, and such a bundle E' is unique up to isomorphism.*

$$\begin{tikzcd} E' \arrow[r, "f'"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

From the uniqueness statement it follows that the isomorphism type of E' depends only on the isomorphism type of E since we can compose the map f' with an isomorphism of E with another vector bundle over B. Often the bundle E' is written as $f^*(E)$ and called the *pullback* of E by f.

One can be more explicit about local trivializations in the pullback bundle $f^*(E)$: if E is trivial over a subspace $U \subset B$ then $f^*(E)$ is trivial over $f^{-1}(U)$ since linearly independent sections s_i of E over U give rise to independent sections $a \mapsto (a, s_i(f(a)))$ of $f^*(E)$ over $f^{-1}(U)$. In particular, the pullback of a trivial bundle is a trivial bundle.

Example 2.33. If f is a constant map, having image a single point $b \in B$, then $f^*(E)$ is just the product $A \times \pi^{-1}(b)$, a trivial bundle.

Example 2.34. For an n-dimensional vector bundle $E \to B$, we can consider the *Stiefel bundle* $V_k(E) \to B$, where points of $V_k(E)$ are k-tuples of orthogonal unit vectors in fibers of E. The fiber of $V_k(E)$ is the Stiefel manifold $V_k(\mathbb{R}^n)$ of orthonormal k-frames in $\mathbb{R}^n$ (cfr. (4), Section 3).

Remark 2.35. *The following properties hold for pullback bundles:*

1. $(fg)^*(E) \cong g^*(f^*(E))$
2. $\mathrm{id}_A^*(E) \cong E$
3. $f^*(E_1 \oplus E_2) \cong f^*(E_1) \oplus f^*(E_2)$
4. $f^*(E_1 \otimes E_2) \cong f^*(E_1) \otimes f^*(E_2)$

Then we come to the main technical result about pullbacks, that we state without proof:

Theorem 2.36. *Let $\pi : E \to B$ be a bundle and $f_0, f_1 : A \to B$ homotopic maps. Then the induced bundles $f_0^*(E)$ and $f_1^*(E)$ are isomorphic if A is compact Hausdorff or more generally paracompact.*

Finally, we show that there is a special n-dimensional vector bundle over particular spaces $E_n \to G_n$ with the property that all n-dimensional bundles over paracompact base spaces are obtainable as pullbacks of this single bundle. The basic idea is to use the Grassmann manifold $G_n(\mathbb{R}^k)$ of suitable dimension, since the inclusions

$$\mathbb{R}^k \subset \mathbb{R}^{k+1} \subset \dots \qquad \Longrightarrow \qquad G_n(\mathbb{R}^k) \subset G_n(\mathbb{R}^{k+1}) \subset \dots$$

Definition 2.37. We let $G_n(\mathbb{R}^\infty) = \cup_k G_n(\mathbb{R}^k)$, the set of all n-dimensional vector subspaces of the vector space $\mathbb{R}^\infty$, with the weak topology (i.e. a set in $G_n(\mathbb{R}^\infty)$ is open iff it intersect each $G_n(\mathbb{R}^k)$ in an open set).

In order to construct the total space of the desired bundle, define

$$E_n(\mathbb{R}^k) = \left\{ \, (l, v) \in G_n(\mathbb{R}^k) \times \mathbb{R}^k \;\middle|\; v \in l \, \right\}$$

so it can be proved that the projection $\pi : E_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ given by $\pi(l, v) = l$ is then a vector bundle, both for finite and infinite k (cfr. [5], pag. 28), and with the notation $G_n := G_n(\mathbb{R}^\infty)$ and $E_n := E_n(\mathbb{R}^\infty)$ the previous projection is the needed one.

Due to Theorem 2.36, vector bundles over a fixed base space are classified by homotopy classes of maps into G_n. Because of this, G_n is called the *classifying space* for n-dimensional vector bundles and $E_n \to G_n$ is called the *universal bundle.*

As an example of how a vector bundle could be isomorphic to a pullback $f^*(E_n)$, consider the tangent bundle to S^n. This is the vector bundle $\pi : E \to S^n$ where

$$E = \left\{ (x, v) \in S^n \times \mathbb{R}^{n+1} \;\middle|\; x \perp v \right\}$$

Each fiber $\pi^{-1}(x)$ is a point in $G_n(\mathbb{R}^{n+1})$, so we have a map

$$\begin{aligned} S^n &\to G_n(\mathbb{R}^{n+1}) \\ x &\mapsto \pi^{-1}(x) \end{aligned}$$

Via the inclusion $\mathbb{R}^{n+1} \to \mathbb{R}^\infty$ we can view this as a map $f : S^n \to G_n(\mathbb{R}^\infty) = G_n$, and E is exactly the pullback $f^*(E_n)$.

$$\begin{tikzcd} E = TS^n \arrow[r] \arrow[d, "\pi"] & E_n \arrow[d] \\ S^n \arrow[r, "f"] & G_n \end{tikzcd}$$

Remark 2.38. *The preceding constructions and results hold equally well for vector bundles over $\mathbb{C}$, with $G_n(\mathbb{C}^k)$ the space of n-dimensional $\mathbb{C}$-linear subspaces of $\mathbb{C}^k$, and so on.*

2.5 CW Complexes

We lastly introduce the class of manifolds that will be mainly used in the rest of this exposition.

Definition 2.39. Let X, Y be two topological spaces, $A \subseteq X$ closed. Let $f : A \to Y$ be continuous, $\sim$ equivalence relationship on $X \cup Y$ (disjoint union) given by

$$z_1 \sim z_2 \iff \begin{cases} z_1 = z_2 \\ z_1, z_2 \in A \quad \textit{and} \quad f(z_1) = f(z_2) \\ z_1 \in A, \quad z_2 = f(z_1) \\ z_2 \in A, \quad z_1 = f(z_2) \end{cases}$$

Then the quotient space $X \cup_f Y/\sim$ is said to be constructed from X *glued with* Y *along* f.

Definition 2.40. Let $D^n \subseteq \mathbb{R}^n$ be the closed unitary disc, $e^n = \mathrm{Int}(D^n)$ is called an *n-cell.* X topological space, $f : S^{n-1} \to X$, $X \cup_f D^n$ gluing an n-cell via f.

A 0-*dimensional CW complex* is a set of points with discrete topology.

An *n-dimensional CW complex* is a space of the form $X \cup_f e^n_I$ where:

- X is a k-dimensional CW complex, $k < n$;
- $e^n_I := \cup_{i \in I} e^n$ (disjoint) is a sum of n-cells, $|I| < \infty$.

The *n-skeleton* X^n is obtained from X^{n-1} by attaching n-cells e^n_I via maps

$$\varphi_I : S^{n-1} \to X^{n-1}$$

where $x \sim \varphi_I(x)$ for $x \in \partial D^n_I$. To each φ_I it corresponds a *characteristic map* (extending φ_I):

$$\phi_I : D^n_I \to X^n \to X$$

The *dimension* of the CW complex X is the biggest n with nonzero characteristic maps. X can be regarded as the union space $X = \cup_n X^n$ with the weak topology, i.e. $A \subseteq X$ is open if and only if $A \cap X^n$ is open (in X^n) for any n.

3 Characteristic Classes

The last few decades have seen the development, in different branches of mathematics, of the notion of a local product structure, i.e., fiber spaces and their generalizations. Characteristic classes are the simplest global invariants which measure the deviation of a local product structure from a product structure. They are intimately related to the notion of curvature in differential geometry. In fact, a real characteristic class is a "total curvature", according to a well-defined relationship. We will give in this section an exposition of the relations between characteristic classes and curvature and discuss some of their applications.

The simplest characteristic class is the Euler characteristic. If M is a finite cell complex, its Euler characteristic is defined by

$$\chi(M) = \sum_k (-1)^k \, \alpha_k = \sum_k (-1)^k \, b_k \tag{3}$$

where α_k is the number of k-cells and b_k is the k-dimensional Betti number of M. The equality of the last two expressions in (3) is known as the Euler-Poincaré formula.

Now let M be a compact oriented differentiable manifold of dimension n and let X be a smooth vector field on M with isolated zeroes. Each zero can be assigned a multiplicity. In his dissertation (1927) H. Hopf proved that

$$\chi(M) = \sum \textit{zeroes of } X$$

This gives a differential topological meaning to $\chi(M)$.

This idea can be immediately generalized. Instead of one vector field we consider k smooth vector fields $X_1, \ldots, X_k$. In the generic case the points on M where the exterior product $X_1 \wedge \cdots \wedge X_k = 0$, i.e., where the vectors are linearly dependent, form a $(k-1)$-dimensional submanifold. Depending on the parity of $n-k$, this defines a $(k-1)$-dimensional cycle, with integer coefficients $\mathbb{Z}$ or with coefficients $\mathbb{Z}_2$, whose homology class, and in particular the homology class $\mod 2$ in all cases, is independent of the choice of the k vector fields. Because the linear dependence of vector fields is expressed by "conditions", it is more proper to define the differential topological invariants so obtained as cohomology classes. This leads to the Stiefel-Whitney cohomology classes $w^i \in H^i(M, \mathbb{Z}_2)$, $1 \leq i \leq n-1$, $i = n-k+1$. The nth Stiefel-Whitney class corresponding to $k = 1$ or the Euler class has integer coefficients $w^n \in H^n(M, \mathbb{Z})$. It is related to $\chi(M)$ by

$$\chi(M) = \int_M \omega^n$$

where we write the pairing of homology and cohomology by an integral.

Whitney went much farther. He saw the great generality of the notion of a vector bundle over an arbitrary topological space M. He also saw the effectiveness of the principal bundles and the fact that the universal principal bundle

$$\pi : V_q(\mathbb{R}^{q+N}) := O(q+N)/O(N) \to O(q+N)/\{O(q) \times O(N)\} =: G_q(\mathbb{R}^{q+N}) \tag{4}$$

say, has the property

$$\pi_i(V_q(\mathbb{R}^{q+N})) = 0, \qquad 0 \leq i < N$$

where π_i is the ith homotopy group. The left-hand side of (4) is called a Stiefel manifold and can be regarded as the space of all orthonormal q-frames through a fixed point 0 of the euclidean space $\mathbb{R}^{q+N}$ of dimension $q+N$ and the right-hand side is the Grassmann manifold of all q-dimensional linear spaces through 0 in $\mathbb{R}^{q+N}$, while the mapping $\pi = (\pi_1, \ldots, \pi_{N-1})$ in (4) can be interpreted geometrically as taking the q-dimensional space spanned by the q vectors of the frame. Thus the universal principal bundle has the feature that its total space has a string of vanishing homotopy groups while its base space, the Grassmann manifold, has rich homological properties.

3.1 The Imbedding Theorem

The importance of the universal bundle lies in the following

Theorem 3.1 (Whitney-Pontrjagin)**.** *Let M be a finite cell complex. A vector bundle E of fiber dimension q over M can be induced by a continuous mapping $f : M \to G_q(\mathbb{R}^{q+N})$, $\dim M < N$, and f is defined up to a homotopy. Equivalently, the map*

$$\varphi : [M, G_n] \to \mathrm{Vect}^n(M), \qquad [f] \mapsto f^*(E_n)$$

where $[M, G_n]$ denotes the set of homotopy classes of maps $f : M \to G_n$ and G_n is the classifying space (cfr. Definition 2.37), is a bijection.

Proof. Suppose M being a CW complex. In particular, it is paracompact, and it is sufficient to prove the statement, due to Theorem 2.36.

The key observation is the following: for an n-dimensional vector bundle $\pi : E \to M$, an isomorphism $E \cong f^*(E_n)$ is equivalent to a map $g : E \to \mathbb{R}^\infty$ that is a linear injection on each fiber. To see this, suppose first that we have a map $f : M \to G_n$ and an isomorphism $E \cong f^*(E_n)$. Then we have a commutative diagram

$$\begin{array}{ccccc} E \cong f^*(E_n) & \xrightarrow{\tilde{f}} & E_n & \xrightarrow{p} & \mathbb{R}^\infty \\ \downarrow{\scriptstyle \pi} & & \downarrow & & \\ M & \xrightarrow{f} & G_n & & \end{array}$$

where $p(l, v) = v$. The composition across the top row is a map $g : E \to \mathbb{R}^\infty$ that is a linear injection on each fiber, since both $\tilde{f}$ and p have this property. Conversely, given a map $g : E \to \mathbb{R}^\infty$ that is a linear injection on each fiber, define $f : M \to G_n$ by letting $f(x)$ be the n-plane $g(\pi^{-1}(x))$. This clearly yields a commutative diagram as above.

To show surjectivity of the map φ, suppose $\pi : E \to M$ is an n-dimensional vector bundle. Let $\{U_a\}$ be an open cover of M such that E is trivial over each U_a. As viewed in Section 2, there is a countable sub-cover U_i and a partition of unity $\{\phi_i\}$ supported in U_i for each i. Let $g_i : \pi^{-1}(U_i) \to \mathbb{R}^n$ be the composition of a trivialization $\pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ with projection onto $\mathbb{R}^n$. The map $(\phi_i\pi)g_i$, $v \mapsto \phi_i(\pi(v))g_i(v)$ extends to a map $E \to \mathbb{R}^n$ that is zero outside $\pi^{-1}(U_i)$. Near each point of M only finitely many ϕ_i's are nonzero, and at least one is nonzero, so these extended $(\phi_i\pi)g_i$'s are the coordinates of a map $g : E \to (\mathbb{R}^n)^\infty = \mathbb{R}^\infty$ that is a linear injection on each fiber.

For injectivity, if we have isomorphisms $E \cong f_0^*(E_n)$ and $E \cong f_1^*(E_n)$ for two maps $f_0, f_1 : M \to G_n$, then these give maps $g_0, g_1 : E \to \mathbb{R}^\infty$ that are linear injections on fibers, as in the first paragraph of the proof. We claim g_0 and g_1 are homotopic through maps g_t that are linear injections on fibers. If this is so, then f_0 and f_1 will be homotopic via $f_t(x) = g_t(\pi^{-1}(x))$.

The first step in constructing a homotopy g_t is to compose g_0 with the homotopy $L_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by

$$L_t(x_1, x_2, \dots) = (1 - t)(x_1, x_2, \dots) + t(x_1, 0, x_2, 0, \dots)$$

For each t this is a linear map whose kernel is easily computed to be 0, so L_t is injective. Composing the homotopy L_t with g_0 moves the image of g_0 into the odd-numbered coordinates. Similarly we can homotope g_1 into the even-numbered coordinates. Still calling the new g's g_0 and g_1, let $g_t = (1 - t)g_0 + tg_1$. This is linear and injective on fibers for each t since g_0 and g_1 are linear and injective on fibers. $\square$

Since the Grassmann manifold plays a fundamental role, it would be good to have a better grasp on its topology. Here we show that $G_q(\mathbb{R}^\infty)$ has the

$$\begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & * & 1 & 0 \end{bmatrix}$$

Figure 1: Echelon form of a matrix

structure of a CW complex with each $G_q(\mathbb{R}^{q+N})$ a finite subcomplex. Recall that $G_q(\mathbb{R}^{q+N})$ is a real, closed manifold of dimension qN (in particular it is a Hausdorff space).

There is a nice description of the cells in the CW structure on $G_q(\mathbb{R}^{q+N})$ in terms of echelon form for matrices, i.e. matrices of the form in Figure 1, where the asterisks denote entries that are arbitrary numbers. It is well-known that any $q \times (q+N)$ matrix A can be put into an echelon form by a finite sequence of elementary row operations.

Assume that our given $q \times (q+N)$ matrix A has rank q. The shape of the echelon form is specified by which columns contain the special entries 1, say in the columns numbered $\sigma_1 < \cdots < \sigma_q$. This q-tuple is called the *Schubert symbol* $\sigma(A)$, and it depends only on the matrix A (i.e. on the q-plane spanned by the rows of A) and not on the particular reduction of A to echelon form. For example, the matrix in Figure 1 has symbol $\sigma = (3, 5, 6, 9)$.

Given a Schubert symbol σ one can consider the set $e(\sigma)$ of all q-planes in $\mathbb{R}^{q+N}$ having σ as their Schubert symbol. In terms of echelon forms, the various q-planes in $e(\sigma)$ are parametrized by the arbitrary entries in the echelon form. There are $\sigma_i - i$ of these entries in the ith row, for a total of $(\sigma_1 - 1) + \cdots + (\sigma_n - n)$ entries. Thus $e(\sigma)$ is homeomorphic to a euclidean space of this dimension, or equivalently an open cell.

Proposition 3.2. *The cells $e(\sigma)$ are the cells of a CW structure on $G_q(\mathbb{R}^{q+N})$.*

Proof. Our main task will be to find a characteristic map for $e(\sigma)$. This is a map from a closed ball of the same dimension as $e(\sigma)$ into $G_q(\mathbb{R}^{q+N})$ whose restriction to the interior of the ball is a homeomorphism onto $e(\sigma)$. From the echelon forms described above it is not clear how to do this, so we will use a slightly different sort of echelon form. We allow the special 1's to be arbitrary nonzero numbers and we allow the entries below these 1's to be nonzero. Then we impose the conditions that the rows are orthonormal and that the last nonzero entry in each row is positive. Let us call this an *orthonormal echelon form.* Once again there is a unique orthonormal echelon form for each q-plane l since if we let l_i denote the subspace of l spanned by the first i rows of the standard echelon form, or in other words $l_i = l \cap R^{\sigma_i}$, then there is a unique unit vector in l_i orthogonal to l_{i-1} and having positive σ_i-th coordinate.

The ith row of the orthonormal echelon form then belongs to the hemisphere H_i in the unit sphere $S^{\sigma_i-1} \subset \mathbb{R}^{\sigma_i} \subset \mathbb{R}^{q+N}$ consisting of unit vectors with non-negative σ_i-th coordinate. In the Stiefel manifold $V_q(\mathbb{R}^{q+N})$ let $E(\sigma)$ be the subspace of orthonormal frames $(v_1, \ldots, v_q) \in (S^{q+N-1})^q$ such that $v_i \in H_i$ for each i. We claim that $E(\sigma)$ is homeomorphic to a closed ball. To prove this the main step is to show that the projection $\pi : E(\sigma) \to H_1$, $\pi(v_1, \ldots, v_q) = v_1$, is a trivial fiber bundle. This is equivalent to finding a projection $p : E(\sigma) \to \pi^{-1}(v_0)$ which is a homeomorphism on fibers of π, where $v_0 = (0, \ldots, 0, 1) \in \mathbb{R}^{\sigma_1} \subset \mathbb{R}^{q+N}$, since the map $\pi \times p : E(\sigma) \to H_1 \times \pi^{-1}(v_0)$ is then a continuous bijection of compact Hausdorff spaces, hence a homeomorphism.

The map $p : \pi^{-1}(v) \to \pi^{-1}(v_0)$ is obtained by applying the rotation ρ_v of $\mathbb{R}^{q+N}$ that takes v to v_0 and fixes the $(q + N - 2)$-dimensional subspace orthogonal to v and v_0. This rotation takes H_i to itself for $i > 1$ since it affects only the first σ_1 coordinates of vectors in $\mathbb{R}^{q+N}$. Hence p takes $\pi^{-1}(v)$ onto $\pi^{-1}(v_0)$.

The fiber $\pi^{-1}(v_0)$ can be identified with $E(\sigma')$ for $\sigma' = (\sigma_2 - 1, \ldots, \sigma_q - 1)$. By induction on q this is homeomorphic to a closed ball of dimension $(\sigma_2 - 2) + \cdots + (\sigma_q - q)$, so $E(\sigma)$ is a closed ball of dimension $(\sigma_1 - 1) + \cdots + (\sigma_q - q)$. The boundary of this ball consists of points in $E(\sigma)$ having v_i in ∂H_i for at least one i. This too follows by induction since the rotation ρ_v takes ∂H_i to itself for $i > 1$.

The natural map $E(\sigma) \to G_q(\mathbb{R}^{q+N})$ sending an orthonormal q-tuple to the q-plane it spans takes the interior of the ball $E(\sigma)$ to $e(\sigma)$ bijectively. Since G_q has the quotient topology from V_q, the map $\operatorname{Int} E(\sigma) \to e(\sigma)$ is a homeomorphism. The boundary of $E(\sigma)$ maps to cells $e(\sigma')$ of G_n where σ' is obtained from σ by decreasing some σ_i's, so these cells $e(\sigma')$ have lower dimension than $e(\sigma)$.

To see that the maps $E(\sigma) \to G_q(\mathbb{R}^{q+N})$ for the cells $e(\sigma)$ are the characteristic maps for a CW structure on $G_q(\mathbb{R}^{q+N})$ we can argue as follows. Let X^i be the union of the cells $e(\sigma)$ in $G_q(\mathbb{R}^{q+N})$ having dimension at most i (the *ith scheleton* of $G_q(\mathbb{R}^{q+N})$). Suppose by induction on i that X^i is a CW complex with these cells. Attaching the $(i+1)$-cells $e(\sigma)$ of X^{i+1} to X^i via the maps $\partial E(\sigma) \to X^i$ produces a CW complex Y and a natural continuous bijection $Y \to X^{i+1}$. Since Y is a finite CW complex it is compact, and X^{i+1} is Hausdorff as a subspace of $G_q(\mathbb{R}^{q+N})$, so the map $Y \to X^{i+1}$ is a homeomorphism and X^{i+1} is a CW complex, finishing the induction. Thus we have a CW structure on $G_q(\mathbb{R}^{q+N})$. $\square$

Remark 3.3. *Since the inclusions $G_q(\mathbb{R}^{q+N}) \subset G_q(\mathbb{R}^{q+N+1})$ for varying N are inclusions of subcomplexes and $G_q(\mathbb{R}^\infty)$ has the weak topology with respect to these subspaces, it follows that we have also a CW structure on $G_q(\mathbb{R}^\infty)$.*

Example 3.4. $q = N = 2$, the Grassmann manifold $G(2,2)$ has six cells corresponding to the Schubert symbols

$$(1,2), \quad (1,3), \quad (1,4), \quad (2,3), \quad (2,4), \quad (3,4)$$

of dimensions 0, 1, 2, 2, 3, 4 respectively. In general the number of cells of $G(q,N)$ is $\binom{q+N}{q}$, the number of ways of choosing the q distinct numbers $\sigma_i \leq q+N$.

$$\sigma(A) = (1,3) \qquad \Longrightarrow \qquad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 \end{bmatrix}$$

Remark 3.5. *Similar constructions work to give CW structures on complex Grassmann manifolds $G_q(\mathbb{C}^{q+N})$, but here $e(\sigma)$ will be a cell of dimension $(2\sigma_1 - 2) + \cdots + (2\sigma_q - 2q)$. The hemisphere H_i is defined to be the subspace of the unit sphere $S^{2\sigma_i - 1} \subset \mathbb{C}^{\sigma_i} \cong \mathbb{R}^{2\sigma_i}$ consisting of vectors whose σ_i-th coordinate is real and nonnegative, so H_i is a ball of dimension $2\sigma_i - 2$. The transformation $\rho_v \in SU(q+N)$ is uniquely determined by specifying that it takes v to v_0 and fixes the orthogonal $(q+N-2)$-dimensional complex subspace, since an element of $U(2)$ of determinant 1 is determined by where it sends one unit vector.*

3.2 Definition and Examples of Characteristic Classes

From previous subsection, we are allowed to define a particular cohomology element:

Definition 3.6. Let $u \in H^i(G_q(\mathbb{R}^{q+N}), A)$ be a cohomology class with coefficients groups A, $f : M \to G_q(\mathbb{R}^{q+N})$ a continuous map. It follows from Theorem 3.1 that the pull-back $f^*u \in H^i(M, A)$ depends only on the bundle. It is called a *characteristic class corresponding to the universal class u.*

$$\begin{tikzcd} M \arrow[d, "f"] \arrow[dr, "f^*u"] & \\ G_q(\mathbb{R}^{q+N}) \arrow[r, "u"] & A \end{tikzcd}$$

Consider the classifying map $f : M \to G_n$ as in 2.37. Our target is to define for a real bundle $\pi : E \to M$, a special class $w^i(E) = w^i(f^*(E_n)) = f^*(w^i(E_n)) \in H^i(M, \mathbb{Z}_2)$ (the ith *Stiefel-Whitney class*), and for a complex bundle $\pi : E \to M$, a class $c^i(E) = c^i(f^*(E_n)) = f^*(c^i(E_n)) \in H^{2i}(M, \mathbb{Z})$ (the ith *Chern class*), but we first need to develop the theory of connections and curvature on a bundle.

However, we can show some examples of these classes in particular cases exhibited in [1], pag. 99-100.

Example 3.7. Consider all the q-dimensional linear spaces X through 0 in $\mathbb{R}^{q+N}$ satisfying the Schubert condition

$$\dim(X \cap \mathbb{R}^{i+N-1}) \geq i, \qquad 1 \leq i \leq q$$

where $\mathbb{R}^{i+N-1}$ is a fixed space of dimension $i + N - 1$ through 0. They form a cycle mod 2 of dimension $qN - 1$ in $G_q(\mathbb{R}^{q+N})$. The dual of its homology class is an element $\tilde{w}^i \in H^i(G_q(\mathbb{R}^{q+N}), \mathbb{Z}_2)$ and is called the ith universal Stiefel-Whitney class. Its image $w^i(E) = f^*\tilde{w}^i \in H^i(M, \mathbb{Z}_2)$, $1 \leq i \leq q$, is called the Stiefel-Whitney class of the bundle E.

Example 3.8. Similarly, consider the q-dimensional linear spaces X through 0 satisfying the condition

$$\dim(X \cap \mathbb{R}^{2k+N-2}) \geq 2k,$$

where $\mathbb{R}^{2k+N-2}$ is fixed. They form a cycle of dimension $qN - 4k$ with integer coefficients. The dual of its homology class is an element $\tilde{p}^k \in H^{4k}(G_q(\mathbb{R}^{q+N}), \mathbb{Z})$ and is called a universal Pontrjagin class. Its image $p^k(E) = f^*\tilde{p}^k \in H^{4k}(M, \mathbb{Z})$, $1 \leq k \leq \lfloor \frac{n}{4} \rfloor$, $n = \dim M$, is called a Pontrjagin class of E.

Example 3.9. It has been known that the complex Grassmann manifold

$$G_q(\mathbb{C}^{q+N}) = U(q + N)/U(q) \times U(N)$$

has simpler topological properties than the real ones. In fact, it is simply connected, has no torsion (i.e. no homology class of finite order), and its odd-dimensional homology classes are all zero. $G_q(\mathbb{C}^{q+N})$ can be regarded as the manifold of all q-dimensional linear spaces X through a fixed point 0 in the complex number space $\mathbb{C}^{q+N}$. Imitating Example 3.7, let $\mathbb{C}^{i+N-1}$ be a fixed space of dimension $i + N - 1$ through 0. Then all the X satisfying the condition

$$\dim(X \cap \mathbb{C}^{i+N-1}) \geq i \qquad 1 \leq i \leq q$$

form a cycle of real dimension $2(qN - 1)$ with coefficients $\mathbb{Z}$. As above, this defines the Chern classes $c^i(E) \in H^{2i}(M, \mathbb{Z})$, $1 \leq i \leq q$, of a complex vector bundle E and they are cohomology classes with integer coefficients.

3.3 Curvature and Connections

The characteristic classes are closely related to the notion of curvature in differential geometry. In this respect one could take as a starting point the theorem in plane geometry that the sum of angles of a triangle is equal to π. More generally, let D be a domain in a two-dimensional riemannian manifold, whose boundary ∂D is sectionally smooth. Then its Euler characteristic is given by the Gauss-Bonnet formula

$$2\pi\chi(D) = \sum_i (\pi - \alpha_i) + \int_{\partial D} k_g \, ds + \iint_D K \, dA$$

where at the right-hand side we have the sum of exterior angles at the corners, the integral of the geodesic curvature, and the last term is the integral of the gaussian curvature. They are respectively the point curvature, the line curvature and the surface curvature of the domain D, and the Gauss-Bonnet formula should be interpreted as expressing the Euler characteristic $\chi(D)$ as a *total curvature.*

The interpretation has a far-reaching generalization. Let $\pi : E \to M$ be a real vector bundle of fiber dimension q. Let $\Gamma(E)$ be the space of sections of E. We can reformulate the definition of connections on a bundle slightly differently from Section 2, in order to have independence with respect to vector fields:

Definition 3.10. Let $\Gamma(E)$ be the space of smooth sections of E, i.e. smooth mappings $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. A *connection* (or *covariant differential*) in E is a map

$$D : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

where T^*M is the cotangent bundle of M and the right-hand side stands for the space of sections of the tensor product bundle $T^*M \otimes E$, such that the following two conditions are satisfied:

1. $D(s_1 + s_2) = Ds_1 + Ds_2, \qquad \forall s_1, s_2 \in \Gamma(E);$
2. $D(fs) = df \otimes s + f\, Ds, \qquad \forall s \in \Gamma(E), \forall f \in C^\infty(M).$

Let s_i, $1 \leq i \leq q$ be a local frame field, i.e. be q sections defined in a neighborhood, which are everywhere linearly independent. Then we can write

$$Ds_i = \sum_i \nabla_i^j \otimes s_j \tag{5}$$

where $\nabla = (\nabla_i^j)$, $i, j = 1, \ldots, q$ is a matrix of one-forms, the *connection matrix.*

Remark 3.11. *What is the relation between Definitions 2.30 and 3.10? Consider a frame s_i as above, then we have, in the sense of 2.30, $\forall X \in \mathfrak{X}(M)$,*

$$\nabla_X s_i = \sum_{j=1}^{n} f_{ij} s_j, \qquad f_{ij} \in C^\infty(U)$$

Otherwise, in the sense of 3.10 we have

$$Ds_i = \sum_{j=1}^{n} \nabla_i^j \otimes s_j, \qquad \nabla_i^j \in \Lambda^1(U)$$

Then the two definitions are related by

$$f_{ij} = \nabla_i^j (X|_U)$$

Putting

$${}^t s = (s_1, \ldots, s_q), \qquad {}^t s = \textit{transpose of } s$$

we can write (5) as a matrix equation

$$Ds = \nabla \otimes s \tag{6}$$

The effect on the connection matrix under a change of the frame can be easily found. In fact, let

$$s' = gs$$

be a new frame field, where g is a nonsingular $(q \times q)$-matrix of c^∞-functions. Let ∇' be the connection matrix relative to the frame field s' so that

$$Ds' = \nabla' \otimes s'$$

Using the properties of D as expressed above, we find immediately

$$\nabla' g = dg + g\nabla \tag{7}$$

This is the equation for the change of the connection matrix under a change of the frame field.

Taking the exterior derivative of (7), we get

$$R' = gRg^{-1} \tag{8}$$

where

$$R = d\nabla - \nabla \wedge \nabla \in \Lambda^2(M)$$

and R' is defined in terms of ∇' by similar equation. R is a $(q \times q)$-matrix of two-forms and is called the *curvature matrix* relative to the frame field s.

Equation (8) shows that it undergoes a very simple transformation law under a change of the frame field. As a consequence it follows from (8) that $\mathrm{tr}(R^k)$ is a form of degree $2k$ globally defined in M. Moreover, $\mathrm{tr}(R^k)$ can be proved to be a closed form and the cohomology class $\{\mathrm{tr}(R^k)\} \in H^{2k}(M, \mathbb{R})$ (in the sense of de Rham's theorem) can be identified with a characteristic class of E.

Example 3.12. When the bundle $\pi : E \to M$ is oriented and has a riemannian structure, the structure group is reduced to $SO(q)$, and we can restrict our consideration to frame fields consisting of orthonormal frames. Then both connection and curvature matrices are anti-symmetric, and we have

$$R = -{}^tR = (R_{ij}), \qquad R_{ij} + R_{ji} = 0$$

If q is even, the *pfaffian*

$$\mathrm{Pf}(R) = \frac{(-1)^r}{2^q \pi^r r!} \sum_i \varepsilon_{i_1,\ldots,i_q} R_{i_1 i_2} \wedge \cdots \wedge R_{i_{q-1} i_q}, \quad r = \frac{q}{2}$$

represents the Euler class, i.e.

$$\{\mathrm{Pf}(R)\} = \omega^q(E) \tag{9}$$

Formula (9) is essentially the high-dimensional Gauss-Bonnet Theorem.

3.4 Principal bundles

We will develop the fundamental notions of a connection in a principal bundle with a Lie group as structure group. We begin by a review and an explanation of our notation on Lie groups. All manifolds and mappings are C^∞.

Let G be a Lie group of dimension r. A *left translation* $L_a : G \to G$ is defined by $L_a(s) = as$, where $a \in G$ is fixed. Let e be the unit element of G and T_e the tangent space at e. A tangent vector $X_e \in T_e$ generates a *left-invariant vector field* given by $X_s = (L_s)_* X_e$. If T_e^* is the cotangent space at e and $\omega_e \in T_e^*$, we get a *left-invariant one-form* (or *Maurer-Cartan form*) ω_s by the pull-back

$$\omega_s = (L_s^{-1})^* \omega_e \qquad or \qquad L_s^* \omega_s = \omega_e$$

Let ω_e^i, $1 \le i \le r$, be a basis in T_e^*. Then $\omega^i = \omega_s^i \in T_s^*$ are everywhere linearly independent and we have

$$d\omega^i = \frac{1}{2} \sum_{j,k} c_{jk}^i \omega^j \wedge \omega^k, \qquad c_{jk}^i + c_{kj}^i = 0 \tag{10}$$

It is easily proved that c^i_{jk} are constants, the *constants of structure of* G. Equations (10) are known as the Maurer-Cartan structure equations.

Let $X_i = (X_i)_s \in T_s$ be a dual basis to ω^i. The X_i are left-invariant vector fields or, what is the same, linear differential operators of the first order (cfr. Definition 2.13 and Remark 2.14). Dual to (10) are the equations of Lie:

$$[X_i, X_j] = -\sum_k c^k_{ij} X_k \tag{11}$$

The tangent space T_e has an algebra structure given by the bracket. It is called *the Lie Algebra* of G and will be denoted by $\mathfrak{g}$.

For a fixed $a \in G$ the inner automorphism $s \mapsto asa^{-1}$ leaves e fixed and induces a linear mapping

$$\mathrm{ad}(a) : \mathfrak{g} \to \mathfrak{g}$$

called the *adjoint mapping.*

Lemma 3.13. *We have:*

$$\mathrm{ad}(ab) = \mathrm{ad}(a)\,\mathrm{ad}(b), \qquad \forall a, b \in G$$

$$\mathrm{ad}(a)[X, Y] = [\mathrm{ad}(a)X, \mathrm{ad}(a)Y], \qquad \forall X, Y \in \mathfrak{g}$$

Proof. It follows from the chain rule for the differential, i.e.

$$\begin{aligned}\mathrm{ad}(ab)X = X_{ab} &= (L_{ab})_* X_e (R_{b^{-1}a^{-1}})_* \\ &= (L_a \circ L_b)_* X_e (R_{b^{-1}} \circ R_{a^{-1}})_* \\ &= (L_a)_* (L_b)_* X_e (R_{b^{-1}})_* (R_{a^{-1}})_* \\ &= (\mathrm{ad}(a) \circ \mathrm{ad}(b))X\end{aligned}$$

and same arguments hold for the second identity. □

Let M be a manifold. It will be desirable to consider $\mathfrak{g}$-valued exterior differential forms in M. As $\mathfrak{g}$ has an algebra structure, such forms can be multiplied. In fact, every $\mathfrak{g}$-valued form is a sum of terms $X \otimes \omega$, where ω is an exterior differential form and $X \in \mathfrak{g}$. We define

$$[X \otimes \omega, Y \otimes \eta] := [X, Y] \otimes (\omega \wedge \eta)$$

Distributivity in both factors then defines the multiplication of any two $\mathfrak{g}$-valued forms. Interchange of order of multiplication follows the rule

$$[X \otimes \omega, Y \otimes \eta] = (-1)^{rs+1}[Y \otimes \eta, X \otimes \omega]$$

with $r = \deg \omega, s = \deg \eta$. This notion allows us to write the Maurer-Cartan equations (10) in a simple form. The expression

$$\omega = \sum_i (X_i)_e \otimes \omega_s^i$$

defines a left-invariant $\mathfrak{g}$-valued one-form in G, which is independent of the choice of the basis. It is *the* Maurer-Cartan form of G.

Lemma 3.14. *Using* (10) *and* (11) *we have*

$$d\omega = -\frac{1}{2}[\omega, \omega] \tag{12}$$

This writes the Maurer-Cartan equation in a basis-free form.

Proof.

$$\begin{aligned}
[\omega, \omega] &= \sum_{i,j} [(X_i)_e \otimes \omega_s^i, (X_j)_e \otimes \omega_s^j] \\
&= \sum_{i,j} \left([(X_i)_e, (X_j)_e] \otimes \omega_s^i \wedge \omega_s^j\right) \\
&= \sum_{i,j} \left(-\sum_k c_{ij}^k (X_k)_e \otimes \omega_s^i \wedge \omega_s^j\right) \\
&= -2 \sum_k (X_k)_e \otimes \left(\frac{1}{2} \sum_{i,j} c_{ij}^k \omega_s^i \wedge \omega_s^j\right) \\
&= -2 \sum_k (X_k)_e \otimes d\omega_s^k = -2 d\omega
\end{aligned}$$

□

Exterior differentiation of (12) gives the Jacobi identity:

$$[\omega, [\omega, \omega]] = 0$$

What we have discussed for left translation naturally holds also for right translations. In particular, we have a right-invariant one-form α in G. Under the mappings $s \mapsto s^{-1}$, ω goes into $-\alpha$. We derive therefore from (12)

$$d\alpha = \frac{1}{2}[\alpha, \alpha]$$

If we denote by ds the identity endomorphism in T_s and consider it as an element of $T_s \otimes T_s^*$, i.e. the tensor (1,1) given by

$$\begin{aligned} ds : T_s \times T_s^* &\to \mathbb{R} \\ (v, \omega) &\mapsto \omega(\mathrm{id}(v)) = \omega(v) \end{aligned}$$

then we can write

$$\omega = (L_{s^{-1}})_* ds = s^{-1} ds \tag{13}$$

where $(L_{s^{-1}})_*$ acts only on the first factor T_s in the tensor product $T_s \otimes T_s^*$; the last expression is a convenient abbreviation. In the same way we can write $\alpha = ds\, s^{-1}$.

Example 3.15. $G = GL(q, \mathbb{R})$. We can regard it as the group of all nonsingular $(q \times q)$-matrices X with real elements. Then $\mathfrak{g}$ is the space $M_q(\mathbb{R})$ of all matrices of order q, and $\omega = X^{-1} dX$. Thus the notation in (13) has in this case a concrete meaning. The Maurer-Cartan equation is

$$d\omega = -\omega \wedge \omega$$

Definition 3.16. A *principal fiber bundle* with a group G is a mapping

$$\pi : P \to M$$

which satisfies the following conditions:

1. G acts freely on P to the left, i.e. there is an action $G \times P \to P$ given by $(a, z) \mapsto az = L_a z \in P$ such that $az \neq z$ when $a \neq e$ (the action is called *transitive*);

2. $M = P/G$;

3. P is locally trivial, i.e. there is an open covering $\{ U, V, \dots \}$ of M such that to each member U of the covering there is a chart $\varphi_U : \pi^{-1}(U) \to U \times G$, with $\varphi_U(z) = (\pi(z) = x, s_U(z))$, satisfying

$$s_U(az) = a s_U(z), \qquad \forall z \in \pi^{-1}(U), a \in G \tag{14}$$

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, "\varphi"] & M \\ M \times G \arrow[ur, "\pi_M"] & \end{tikzcd} \qquad \begin{tikzcd} \pi^{-1}(U) \arrow[r, "\pi"] \arrow[d, "\varphi|_U"] & U \subseteq M \\ U \times G \arrow[ur, "\pi_U"] & \end{tikzcd}$$

Suppose $z \in \pi^{-1}(U \cap V)$. By (14) we have also

$$s_V(az) = as_V(z)$$

so that

$$s_U(az)^{-1}s_V(az) = s_U(z)^{-1}s_V(z)$$

is independent of a and depends only on $x = \pi(z)$. We put

$$s_U(z)^{-1}s_V(z) = g_{UV}(x)$$

or

$$s_U g_{UV} = s_V \tag{15}$$

The g_{UV} are mappings of $U \cap V$ into G and satisfy the relations

$$\begin{aligned} g_{UV}g_{VU} &= e \qquad \text{in } U \cap V \\ g_{UV}g_{VW}g_{WU} &= e \qquad \text{in } U \cap V \cap W \end{aligned}$$

They are called the *transition functions* of the bundle. It is well-known that the bundle, the principal bundle or any of its associated bundles, can be constructed from the transition functions. In particular, given a covering $\{U_a\}$ and a system of transition functions satisfying (15), one can construct a corresponding principal G-bundle taking as total space the quotient of $\cup_a U_a \times G$ (disjoint union) with the identifications

$$(p, a) \sim (p, g_{ab}(p) \cdot a) \in U_a \cap U_b \times G, \qquad \forall p \in U_a \cap U_b$$

Example 3.17. Suppose $P \to M$ is an n-dimensional vector bundle. Then the bundle $F(P) \to M$ of n-frames is a principal $GL(n, \mathbb{R})$-bundle.

Now let $\pi : P \to M$ be a principal G-bundle. For $z \in P$ the map $G \to P$ given by $g \mapsto g \cdot z$ induces an injection $\mu_z : \mathfrak{g} \to T_z$ and the quotient space is naturally identified with $T_{\pi(z)}$. That is, we have an exact sequence

$$0 \to \mathfrak{g} \xrightarrow{\mu_z} T_z \xrightarrow{\pi_*} T_{\pi(z)} \to 0 \tag{16}$$

The vectors in the image of μ_z are called *vertical* and we want to single out a complement in T_z of *horizontal* vectors, i.e. we want to split the exact sequence (16). This is equivalent to a linear map $\phi_z : T_z \to \mathfrak{g}$ such that

$$\phi_z \circ \mu_z = \mathrm{id}_{\mathfrak{g}}$$

Therefore it is natural to define a connection in P simply to be a 1-form $\phi \in \Lambda^1(P, \mathfrak{g})$ such that the condition above holds for all $z \in P$.

However, we want a further condition on ϕ. If we denote with $H_z \subseteq T_z$ the subspace of horizontal vectors, by (14) each fiber of P is the group manifold G defined up to left translations.

We can give then the following:

Definition 3.18. A *connection* in a principal G-bundle $\pi : P \to M$ is a 1-form $\phi \in \Lambda^1(P, \mathfrak{g})$ satisfying:

1. $\phi_z \circ \mu_z = \mathrm{id}$
2. $H_{gz} = (L_g)_* H_z$

Example 3.19. Let $\mathbb{C}P^1$ be the complex projective space of (complex) dimension 1 and let (z_0, z_1) be homogeneous coordinates, i.e. given the cover $\{U, V\}$ as

$$U = \{\ (z_0, z_1)\ \mid\ z_0 \neq 0\ \}$$
$$V = \{\ (z_0, z_1)\ \mid\ z_1 \neq 0\ \}$$

each point on $\mathbb{C}P^1$ is uniquely determined by the ratios z_1/z_0 over U and by z_0/z_1 over V. The complex-valued 1-form

$$\phi = \frac{\bar{z}_0\, dz_0 + \bar{z}_1\, dz_1}{|z_0|^2 + |z_1|^2}$$

where as usual the bar denotes complex conjugation and $|z|^2 = z\bar{z}$, is a connection.

We will give other two definitions of a connection, which are equivalent to the previous one:

Definition 3.20 (Connection - bis)**.** This is the dual of the first definition, by giving instead of $H_z \in T_z$ its annihilator V_z^* in the cotangent space T_z^*. This in turn is equivalent to giving a $\mathfrak{g}$-valued one-form ϕ in P which restricts to $ds_U s_U^{-1}$ on a fiber, i.e. locally

$$\phi(z) = ds_U s_U^{-1} + \nabla_U(x, s_U, dx)$$

such that

$$\phi(az) = \mathrm{ad}(a)\phi(z)$$

The last condition is equivalent to condition (2) in the first definition. It implies that locally

$$\phi(z) = ds_U s_U^{-1} + \mathrm{ad}(s_U)\nabla_U(x, dx) \tag{17}$$

where $\nabla_U(x, dx)$ is a $\mathfrak{g}$-valued one-form in U. Thus the second definition of a connection is the existence of a $\mathfrak{g}$-valued one-form in P, which has the local expression (17).

Definition 3.21 (Connection - ter)**.** When we express the condition that in $\pi^{-1}(U \cap V)$ the right-hand side of (17) is equal to the corresponding expression with the subscript V, we get

$$\nabla_U = dg_{UV} g_{UV}^{-1} + \mathrm{ad}(g_{UV})\nabla_V \qquad \textit{in } U \cap V \tag{18}$$

where the first term at the right-hand side is the pull-back of the right-invariant form in G under g_{UV}. Hence a connection in P is given by a $\mathfrak{g}$-valued one-form ∇_U in every member U of an open covering $\{ U, V, \dots \}$ of M, such that in $U \cap V$ the equation (18) holds. This is essentially the classical definition of a connection.

We wish to take the exterior derivative of (17). For this purpose we need the following lemma:

Lemma 3.22. *Let ∇ be a $\mathfrak{g}$-valued one-form in U. Let $s \in G$ and let $\alpha = ds\, s^{-1}$ be the right-invariant $\mathfrak{g}$-valued one-form in G. Then, in $U \times G$, we have*

$$d(\mathrm{ad}(s)\nabla) = \mathrm{ad}(s)d\nabla + [\mathrm{ad}(s)\nabla, \alpha]$$

We put

$$R_U = d\nabla_U - \frac{1}{2}[\nabla_U, \nabla_U]$$

$$\Phi = d\phi - \frac{1}{2}[\phi, \phi] \tag{19}$$

Applying the Lemma we get by exterior differentiation of (17),

$$\Phi = \mathrm{ad}(s_U) R_U \tag{20}$$

Thus Φ is a $\mathfrak{g}$-valued two-form in P, which has the local expression (20). Alternately, we have in $U \cap V$,

$$R_U = \mathrm{ad}(g_{UV}) R_V$$

Either Φ or R_U will be called the *curvature form* of the connection.

Exterior differentiation of (19) gives the Bianchi Identity:

$$d\Phi = -[\Phi, \phi] = [\phi, \Phi] \tag{21}$$

In particular, $d\Phi$ vanishes on sets of horizontal vectors.

Remark 3.23. *Let X, Y be horizontal vector fields on P. Then by Definition* (19) *we have*

$$\Phi(X, Y) = -\frac{1}{2}\phi([X, Y])$$

which gives an alternative definition of curvature.

Remark 3.24. *The definition of curvature given above is based on the following idea: consider the product bundle $M \times G \to M$ and the connection ϕ given at $(x, g) \in M \times G$ by*

$$\phi_{(x,g)} = (L_{g^{-1}} \circ \pi)_*$$

where $\pi : M \times G \to G$ is the projection and $L_{g^{-1}} : G \to G$ is the left translation by g^{-1}. This is called the flat connection *(or the* Maurer-Cartan connection*) of $M \times G$. It has the feature*

$$d\phi = \frac{1}{2}[\phi, \phi]$$

Hence, defining the curvature form Φ as in (19) *we obtain that the flat connection ϕ has curvature form $\Phi = 0$, and its name is then meaningful. In this way, the curvature measures in some sense how much the choosen connection differs from the flat one.*

Generalizing this special case, a connection ϕ in a principal G-bundle $\pi : P \to M$ is called flat if the curvature form vanishes, that is, $\Phi = 0$.

One of the most important cases of this general theory is when $G = GL(q, \mathbb{R})$. As discussed above, s_U is now a nonsingular matrix of order q, ∇_U, ϕ are matrices of one-forms, and R_U, Φ are matrices of two-forms. Equation (17) becomes a matrix equation

$$\phi = (ds_U + s_U \nabla_U) s_U^{-1} \tag{22}$$

Let σ_U (resp. σ_V) be the one-rowed matrix formed by the first row of s_U (resp. s_V). Then (15) gives, by taking the first rows of both sides,

$$\sigma_U g_{UV} = \sigma_V$$

This is the equation for the change of the associated vector bundle E, defined as the bundle of the first row vectors of the matrices representing the elements of $GL(q, \mathbb{R})$. Moreover, equating the right-hand side of (22) with the corresponding expression with the subscript V, we get

$$(ds_U + s_U \nabla_U) g_{UV} = ds_V + s_V \nabla_V \tag{23}$$

On taking the first rows of both sides of (23), we have

$$D\sigma_U g_{UV} = D\sigma_V$$

where we put

$$D\sigma_U = d\sigma_U + \sigma_U \nabla_U$$

Applying to a section of E, we can identify this with the operator D in (6). Thus we have shown that the connection in a vector bundle defined in Subsection 3.3 is included as a special case of our general theory (cfr. [3], pag. 56, Exercise 8).

Another important case is the bundle (4) discussed before, which is a principal bundle with the group $O(q)$. This bundle plays a fundamental role in the study of submanifolds in euclidean space. As remarked above, its importance in bundle theory arises from the fact that it is a universal bundle when N is large. We will describe a canonical connection on it. Let $\mathbb{R}^{q+N}$ be the euclidean space of dimension $q+N$. Let

$$e_A = (e_{A,1}, \ldots, e_{A,q+N}), \qquad 1 \leq A \leq q+N$$

be an orthonormal frame, so that the matrix

$$X = (e_{AB})$$

is orthogonal. $O(q+N)$ can be identified with the space of all orthonormal frames e_A (or all orthogonal matrices X). Let

$$de_A = \sum_B \alpha_{AB} e_B$$

Then, if $\alpha = (\alpha_{AB})$, we have

$$\alpha = dX\, X^{-1} = -{}^t\alpha$$

The Stiefel manifold $V_q(\mathbb{R}^{q+N}) = O(q+N)/O(N)$ can be identified with the manifold of all orthonormal frames $e_1, \ldots, e_q$ and the Grassmann manifold $G(q, N) = O(q+N)/\{O(q) \times O(N)\}$ with the q-planes spanned by $e_1, \ldots, e_q$. The matrix

$$\alpha = (\alpha_{ij}), \qquad i, j = 1, \ldots, q \tag{24}$$

defines a connection in the bundle (4), as easily verified.

4 Weil Homomorphism

The local expression (20) of the curvature form Φ prompts us to introduce functions $F(X_1,\dots,X_h)$, $X_i \in \mathfrak{g}$, $i = 1,\dots,h$, which are real or complex valued and satisfy the conditions:

1. F is h-linear and remains unchanged under any permutation of its arguments;

2. F is "invariant", i.e.

$$F(\mathrm{ad}(a)X_1,\dots,\mathrm{ad}(a)X_h) = F(X_1,\dots,X_h), \qquad \forall a \in G \tag{25}$$

To the h-linear function $F(X_1,\dots,X_h)$ there corresponds the polynomial

$$F(X) = F(X,\dots,X), \qquad X \in \mathfrak{g}$$

of which $F(X_1,\dots,X_h)$ is the *complete polarization*. We will call $F(X)$ an *invariant polynomial*. All invariant polynomial under G form a ring, to be denoted by $I(G)$.

The invariance condition (25) implies its "infinitesimal form"

$$\sum_{1\le i\le h} F(X_1,\dots,[Y,X_i],\dots,X_h) = 0, \qquad Y, X_i \in \mathfrak{g}$$

More generally, if Y is a $\mathfrak{g}$-valued one-form and X_i is a $\mathfrak{g}$-valued form of degree m_i, $1 \le i \le h$, we have

$$\sum_{1\le i\le h} (-1)^{m_1+\dots+m_{i-1}} F(X_1,\dots,[Y,X_i],\dots,X_h) = 0 \tag{26}$$

Now consider a principal G-bundle $\pi : P \to M$ on a differentiable manifold M, and suppose ϕ is a connection in P with curvature form $\Phi \in \Lambda^2(P,\mathfrak{g})$. Then for $h \ge 1$ we have

$$\Phi^h = \Phi \wedge \dots \wedge \Phi \in \Lambda^{2h}(P, \mathfrak{g}\times\dots\times\mathfrak{g})$$

and it follows from (20) that if F is an invariant polynomial of degree h, we have the form of degree $2h$:

$$F(\Phi^h) =: F(\Phi) = F(R_U)$$

The left-hand side shows that it is globally defined in P, while the right-hand side shows that it is a form in M. Moreover, by the Bianchi identity (21) and by (26), we have

$$\frac{1}{h} dF(\Phi) = F([\phi,\Phi],\Phi,\dots,\Phi) = 0$$

Hence $F(\Phi)$ is closed and its cohomology class $[F(\Phi)]$ is an element of $H^{2h}(M, \mathbb{R})$. We shall prove that this class depends only on F and is independent of the choice of the connection.

Lemma 4.1. *Let ϕ_0, ϕ_1 be $\mathfrak{g}$-valued one-forms and let $F \in I(G)$ be an invariant polynomial of degree h. Let*

$$\phi_t = \phi_0 + t\alpha, \qquad \alpha = \phi_1 - \phi_0$$

$$\Phi_t = d\phi_t - \frac{1}{2}[\phi_t, \phi_t]$$

Then

$$F(\Phi_1) - F(\Phi_0) = hd \int_0^1 F(\alpha, \Phi_t, \dots, \Phi_t)\, dt$$

Proof. To prove the lemma we first find

$$\frac{1}{h}\frac{d}{dt}F(\Phi_t) = F(d\alpha - [\phi_t, \alpha], \Phi_t, \dots, \Phi_t)$$

On the other hand,

$$dF(\alpha, \Phi_t, \dots, \Phi_t) = F(d\alpha, \Phi_t, \dots, \Phi_t) - (h-1)F(\alpha, [\phi_t, \Phi_t], \Phi_t, \dots, \Phi_t)$$

The invariance of F implies, by (26),

$$F([\phi_t, \alpha], \Phi_t, \dots, \Phi_t) - (h-1)F(\alpha, [\phi_t, \Phi_t], \Phi_t, \dots, \Phi_t) = 0$$

It follows that

$$\frac{1}{h}\frac{d}{dt}F(\Phi_t) = dF(\alpha, \Phi_t, \dots, \Phi_t)$$

and the lemma follows by integrating this equation with respect to t. □

Corollary 4.2. *Let ϕ_0, ϕ_1 be two connections in the bundle $\pi : P \to M$ and let $F \in I(G)$. Then $F(\phi_0)$ and $F(\phi_1)$, as closed forms in M, are cohomologous in M.*

Corollary 4.3. *Let ϕ be a connection in the bundle $\pi : P \to M$ and let $F \in I(G)$. Then $F(\phi)$ is a coboundary in P. More precisely, let*

$$\Phi_t = t\, d\phi - \frac{1}{2}t^2[\phi, \phi] = t\phi + \frac{1}{2}(t - t^2)[\phi, \phi]$$

Then

$$F(\Phi) = hd \int_0^1 F(\phi, \Phi_t, \dots, \Phi_t)\, dt \tag{27}$$

By putting

$$\omega_P(F) = [F(\Phi)], \qquad F \in I(G)$$

where the right-hand side denotes the cohomology class represented by the closed form $F(\Phi)$, we have defined a mapping

$$\omega_P : I(G) \to H^*(M, \mathbb{R})$$

It is clearly a ring homomorphism and is called the *Weil homomorphism.*

Theorem 4.4 (Naturality). *Let $f : N \to M$ be a smooth map. Then*

$$\omega_{f^*P} = f^* \circ \omega_P$$

Proof. If ϕ is a connection in $\pi : P \to M$ with curvature form Φ then clearly $f^*\phi$ is a connection in $f^*P \to N$ with curvature form $f^*\Phi$. Therefore since

$$f^*F(\Phi^h) = F((f^*\Phi)^h)$$

the theorem is proved. □

We are now ready to complete Definition 3.6:

Definition 4.5. Let $\omega_P : I(G) \to H^*(M, \mathbb{R})$ be the Weil homomorphism. Given an element (polynomial) $F \in I(G)$, the image $\omega_P(F)$ is the *characteristic class* of P corresponding to F.

In the case that G is a compact connected Lie group, the Weil homomorphism has a simple geometric interpretation, which we will state without proof. There is a universal principal bundle $\pi_0 : E_G \to B_G$ with group G such that we have the bundle map

$$\begin{tikzcd} P \arrow[r] \arrow[d, "\pi"] & E_G \arrow[d, "\pi_0"] \\ M \arrow[r, "f"] & B_G \end{tikzcd}$$

where f is defined up to a homotopy. B_G is called the classifying space with the group G. The following diagram is commutative:

$$\begin{tikzcd} & I(G) \arrow[dl, "\omega_0"'] \arrow[d, "\omega_P"] \\ H^*(B_G, \mathbb{R}) \arrow[r, "f^*"] & H^*(M, \mathbb{R}) \end{tikzcd}$$

and ω_0 is an isomorphism. In other words, the invariant polynomials can be identified with the cohomology classes of the classifying space and the Weil homomorphism gives the representatives of characteristic classes by closed differential forms constructed from the curvature forms of a connection.

We put

$$TF(\phi) = h \int_0^1 F(\phi, \Phi_t, \ldots, \Phi_t)\, dt$$

so that (27) can be written

$$\pi^* F(R_U) = F(\Phi) = d(TF(\phi))$$

T will be called the *transgression operator*; it enables $F(\Phi)$ to be written as a coboundary in a canonical way, by the use of a connection.

One application of the transgression operator is the following description of the de Rham ring of P.

Theorem 4.6 (Chevalley). *Let G be a compact connected semi-simple group of rank r (dimension of maximal torus in G). Let $\pi : P \to M$ be a principal G-bundle over a compact manifold M and ϕ a connection in P. Then the ring $I(G)$ of invariant polynomials is generated by elements $F_1, \ldots, F_r$ and the de Rham ring of P can be given as the quotient ring*

$$H^*(P, \mathbb{R}) = A/dA$$

where

$$A = \Lambda[TF_1(\phi), \ldots, TF_r(\phi)]$$

is the ring of polynomials in $TF_1(\phi), \ldots, TF_r(\phi)$ with coefficients which are forms in M.

4.1 Examples

For geometrical applications we will describe in detail the Weil homomorphism for some of the classical groups:

Example 4.7. $G = GL(q, \mathbb{C}) = \{\, X \mid \det X \neq 0 \,\}$, where X is a matrix of order q with complex elements. The coefficients $F_i(X)$, $i = 1, \ldots, q$ in the polynomial in t defined by

$$\det\left(tI_q + \frac{i}{2\pi} X\right) = t^q + F_1(X) t^{q-1} + \cdots + F_q(X)$$

where I_q is the unit matrix, are invariant polynomials.

Suppose $\pi : E \to M$ be a complex vector bundle and ϕ be a connection, with the curvature form Φ, so that Φ is a matrix of two-forms. Then we have

$$[F_i(\Phi)] = c^i(E) \in H^{2i}(M, \mathbb{Z})$$

Notice that the coefficients are here so chosen that the corresponding classes have integer coefficients.

Remark 4.8. *By the above Corollary 4.2 it suffices to establish this result in the classifying space $B_G = G_q(\mathbb{C}^{q+N})$ (N sufficiently large), with its connection defined in a similar way as the one in* (24) *for the real Grassmann manifold. In other words it is sufficient to consider the universal bundle with its universal connection. The same remark applies in the identification in the next two cases.*

Example 4.9. $G = GL(q, \mathbb{R}) = \{ X \mid \det X \neq 0 \}$, where X is a matrix of order q with real elements. We put

$$\det \left(tI_q - \frac{1}{2\pi} X \right) = t^q + E_1(X) t^{q-1} + \cdots + E_q(X)$$

Let $\pi : E \to M$ be a real vector bundle and ϕ be a connection, with the curvature form Φ. Then $[E_{2k+1}(\Phi)] = 0$, for $1 \leq k \leq \lfloor \frac{n}{4} \rfloor$ and

$$[E_{2k}(\Phi)] = p^k(E) \in H^{4k}(M, \mathbb{Z})$$

is the kth Pontrjagin class of E.

Example 4.10. $G = SO(2r)$. A representative of the Euler class was given by formula (9), Section 3.

4.2 The Complex Projective Space

Suppose we consider a topological space M with a principal $GL(n, \mathbb{C})$-bundle $\xi : E \to M$ and a $GL(m, \mathbb{C})$-bundle $\eta : F \to M$. Then we can define the *Whitney sum*

$$(\xi \oplus \eta) : E \oplus F \to M$$

in terms of transition functions as follows: first let

$$\oplus : GL(n, \mathbb{C}) \times GL(m, \mathbb{C}) \to GL(n+m, \mathbb{C})$$

be the homomorphism taking a pair of matrices (A, B) to the matrix

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

Now choose a covering $\{U_a\}_{a\in\Sigma}$ of M such that both E and F have trivializations over U_a, $a \in \Sigma$, and let $\{g_{ab}\}$ and $\{h_{ab}\}$ be the corresponding transition functions for E and F respectively. Then $\xi \oplus \eta : E \oplus F \to M$ is the bundle with transition functions $\{g_{ab} \oplus h_{ab}\}$. Notice that if E and F are differentiable then also $E \oplus F$ is.

Notice that $GL(1,\mathbb{C}) = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is the multiplicative group of non-zero complex numbers. Moreover, $GL(1,\mathbb{C})$-bundles are in 1-1 correspondence with 1-dimensional complex vector bundles (also called *complex line bundles*).

Example 4.11 (The canonical line bundle on the complex projective space $\mathbb{C}P^n$). Here $\mathbb{C}P^n$ is defined as the quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ under the action of $\mathbb{C}^*$ given by

$$\lambda \cdot (z_0, z_1, \ldots, z_n) = (\lambda \cdot z_0, \ldots, \lambda \cdot z_n)$$

It is easy to see that the natural projetion

$$\pi : E := \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$$

is a principal $\mathbb{C}^*$-bundle. The associated complex line bundle is by definition the canonical line bundle. If we let the total Chern class be the sum

$$c(E) = 1 + c^1(E) + \cdots + c^n(E) \in H^*(M,\mathbb{C})$$

then we can easily compute the Chern class c^1 for the case $q = 1$. By Example 4.7 we have

$$\det\left(tI_q + \frac{i}{2\pi}X\right) = t^q + c^1(X)t^{q-1} + \cdots + c^q(X)$$

Hence for $q = 1$ we have the identity

$$t + \frac{i}{2\pi}X = t + c^1(X) \qquad \Longrightarrow \qquad c^1(\Phi) = \frac{i}{2\pi}\Phi$$

Here $\mathbb{C}P^1$ is given with the canonical orientation determined by the 2-form $dx \wedge dy$ where $z = x + iy = z_1/z_0$ is the complex coordinate in the Riemann sphere $\mathbb{C}P^1$ with homogeneous coordinates (z_0, z_1). Consider then the connection

$$\phi = \frac{\bar{z}_0\, dz_0 + \bar{z}_1\, dz_1}{|z_0|^2 + |z_1|^2}$$

given in Example 3.19. Since $\mathbb{C}^*$ is abelian the curvature form is given by

$$\Phi = d\phi$$

Now let $U = \mathbb{C}P^1 \setminus \{(0,1\} = \{\ (z_0, z_1)\ \mid\ z_0 \neq 0\ \}$ and use the local coordinate $z = z_1/z_0$. Then $z_1 = z_0 z$ and $dz_1 = z\,dz_0 + z_0\,dz$. Hence

$$\begin{aligned}\phi &= \frac{\bar{z}_0\,dz_0 + \bar{z}_0\bar{z}(z\,dz_0 + z_0\,dz)}{|z_0|^2(1+|z|^2)} \\ &= \frac{(\frac{dz_0}{z_0} + |z|^2\frac{dz_0}{z_0} + \bar{z}\,dz)}{1+|z|^2} = \frac{dz_0}{z_0} + \frac{\bar{z}}{1+|z|^2}\,dz\end{aligned}$$

Therefore

$$\Phi = d\phi = d(\frac{dz_0}{z_0} + \frac{\bar{z}}{1+|z|^2}\,dz) = d(\frac{\bar{z}}{1+|z|^2}\,dz) = \frac{d\bar{z}\wedge dz}{(1+|z|^2)^2}$$

It follows that in U, $c^1(\Phi)$ is given by

$$c^1(\Phi) = \frac{i}{2\pi}\frac{d\bar{z}\wedge dz}{(1+|z|^2)^2}$$

5 Topological approach

Until now we have considered characteristic classes from a *geometrical* point of view, based on the notion of connection and curvature of a manifold. One could also be interested in defining them in a *topological* way, by taking Chern classes, Stiefel-Whitney classes, and the Euler class from an axiomatic point of view. In this sense it is remarkable the fact that these two different approaches actually coincide, as we will show in this Section.

Furthermore, we will assume without mention that all base spaces of vector bundles are CW complexes, so that the results of Section 3 apply.

5.1 Definition of the Stiefel-Whitney Classes and Chern Classes as polynomials

Here is the basic result about Stiefel-Whitney classes giving their most essential properties, which can be regarded as axioms:

Theorem 5.1. *There is a unique sequence of functions $w^1, w^2, \dots$ assigning to each real vector bundle $\pi : E \to M$ a class $w^i(E) \in H^i(M, \mathbb{Z}_2)$, depending only on the isomorphism type of E, such that*

(SW1) $w^i(f^*(E)) = f^*(w^i(E))$ *for a pullback* $f^*(E)$.

(SW2) $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$ *where* $w = 1 + w^1 + w^2 + \cdots \in H^*(M, \mathbb{Z}_2)$.

(SW3) $w^i(E) = 0$ *if* $i > \dim E$.

(SW4) For the canonical line bundle $E \to \mathbb{R}P^\infty$, $w^1(E)$ is a generator of $H^1(\mathbb{R}P^\infty, \mathbb{Z}_2)$.

The sum $w(E) = 1 + w^1(E) + w^2(E) + \dots$ is the *total Stiefel-Whitney class*. Note that (SW3) implies that the sum $1 + w^1(E) + w^2(E) + \dots$ has only finitely many nonzero terms, so this sum does indeed lie in $H^*(M, \mathbb{Z}_2)$. Condition (SW2) is just a compact way of writing the relations

$$w^n(E_1 \oplus E_2) = \sum_{i+j=n} w^i(E_1) \smile w^j(E_2), \qquad w^0 = 1$$

and it is sometimes called the *Whitney sum formula*.

Property (SW4) can be viewed as a nontriviality condition. If this were dropped we could set $w^i(E) = 0$ for all E and all $i > 0$, and the first three condition would be satisfied with a non-interesting case.

For complex vector bundles there are analogous *Chern classes*:

Theorem 5.2. *There is a unique sequence of functions $c^1, c^2, \ldots$ assigning to each complex vector bundle $\pi : E \to M$ a class $c^i(E) \in H^{2i}(M, \mathbb{Z})$, depending only on the isomorphism type of E, such that*

(C1) $c^i(f^*(E)) = f^*(c^i(E))$ for a pullback $f^*(E)$.

(C2) $c(E_1 \oplus E_2) = c(E_1) \smile c(E_2)$ where $c = 1 + c^1 + c^2 + \cdots \in H^*(M, \mathbb{Z})$.

(C3) $c^i(E) = 0$ if $i > \dim E$.

(C4) For the canonical line bundle $E \to \mathbb{C}P^\infty$, $c^1(E)$ is a generator of $H^2(\mathbb{C}P^\infty, \mathbb{Z})$ specified in advance.

Same considerations hold for the complex case, but here the condition (C4) is now also a normalization condition, since replacing each function c^i by $k_i c^i$ (product with a fixed integer k) gives new functions satisfying (C1)-(C3).

Proposition 5.3. *The Stiefel-Whitney classes (respectively Chern classes) defined in Section 3 satisfy (SW1)-(SW4) (resp. (C1)-(C4)).*

Proof. We study the complex case (the real one is analogous).

The (C1) condition is the "natural property" satisfied due to Theorem 4.4.

To show (C2) let E_1, E_2 be two bundles over M of dimension n, m respectively. Clearly $\Gamma(E_1 \oplus E_2) = \Gamma(E_1) \times \Gamma(E_2)$. Thus if we get ϕ_1, ϕ_2 two connections on E_1, E_2 with curvature matrices Φ_1, Φ_2, then

$$\phi = \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}$$

represents a connection on $E_1 \oplus E_2$ whose curvature matrix is

$$\Phi = \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix}$$

(see introduction to Subsection 4.2). Hence

$$\begin{aligned} c(E_1 \oplus E_2) &= \det(tI_{n+m} + \frac{i}{2\pi}\Phi) \\ &= \det(tI_n + \frac{i}{2\pi}\Phi_1) \cdot \det(tI_m + \frac{i}{2\pi}\Phi_2) = c(E_1) \smile c(E_2) \end{aligned}$$

Property (C3) is a consequence of the definition of wedge product: let $F \in I(G)$ an invariant polynomial of degree $i > \dim E$. Then

$$\Phi^i = \Phi \wedge \cdots \wedge \Phi = 0$$

by Remark 2.9. Thus $F(\Phi^i) = 0$ and also its image via Weil homomorphism is zero.

Finally, (C4) will be shown in the last example of this work. $\square$

The topological approach introduced in this section can be used to compute the characteristic classes differently from Section 3. In particular, we apply it to the complex projective space, that has been investigated in Example 4.11.

Example 5.4. Consider the unit sphere $S^2 \subset \mathbb{R}^3$ and the stereographic projection as in Figure 2. It is easy to see that the inverse of this projection is the diffeomorphism

$$\begin{aligned} g : \mathbb{C}P^1 &\to S^2 \\ z = x + iy &\mapsto \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{x^2 + y^2 - 1}{1 + x^2 + y^2} \right) \end{aligned}$$

Our idea is to obtain a generator of $H^2(\mathbb{C}P^1)$ as a pullback via g of a generator of $H^2(S^2)$.

From Mayer-Vietoris sequence for cohomology, a generator of the latter can be taken to be a bump n-form on S^n. So the generator of $H^2(S^2)$ is represented by

$$\sigma = \frac{1}{4\pi}(u_1\, du_2\, du_3 - u_2\, du_1\, du_3 + u_3\, du_1\, du_2) = \frac{1}{4\pi}\frac{du_1\, du_2}{u_3}$$

by removing the du_3 dependence on the set where $u_3 \neq 0$ since

$$u_1^2 + u_2^2 + u_3^2 = 1 \qquad \Longrightarrow \qquad u_1\, du_1 + u_2\, du_2 + u_3\, du_3 = 0$$

Hence the desired pullback is $g^*\sigma$ given by

$$g^*\sigma = \frac{1}{4\pi}\frac{du_1\, du_2}{u_3}$$

where now

$$u_1 = \frac{2x}{1 + x^2 + y^2} \qquad u_2 = \frac{2y}{1 + x^2 + y^2} \qquad u_3 = \frac{x^2 + y^2 - 1}{1 + x^2 + y^2}$$

In terms of $z = x + iy$, it can be written as

$$g^*\sigma = -\frac{i}{\pi}\frac{dx\, dy}{(1 + x^2 + y^2)^2} = -\frac{i}{2\pi}\frac{dz\, d\bar{z}}{(1 + |z|^2)^2}$$

Lastly, recall that the standard orientation in the unit sphere $S^{n-1} \subset \mathbb{R}^n$ is the following one: if σ is a generator of $H^{n-1}(S^{n-1})$ and $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$

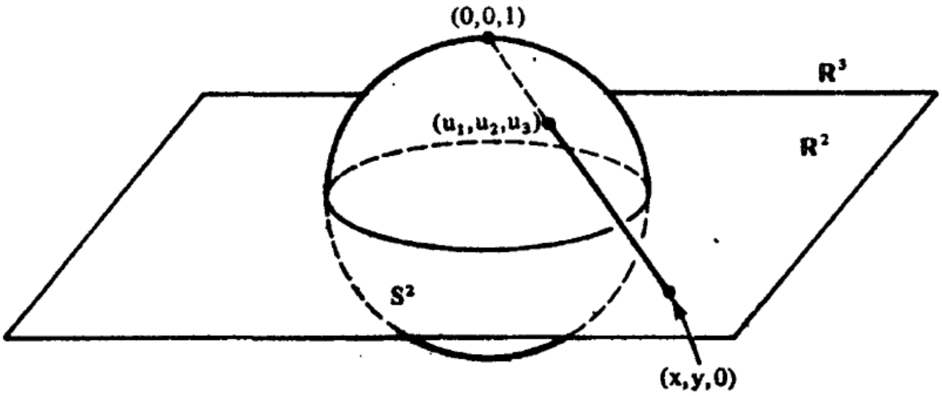

Figure 2: Stereographic projection $S^2 \to \mathbb{C}^1$

is a deformation retraction, then σ is positive on S^{n-1} if and only if $dr \cdot \pi^*\sigma$ is positive on $\mathbb{R}^n \setminus \{0\}$. Hence the standard orientation on $\mathbb{C}P^1$ is given locally by $dx\,dy$, so the positive generator in $H^2(\mathbb{C}P^1)$ is

$$\alpha = -g^*\sigma = \frac{i}{2\pi}\frac{dz\,d\bar{z}}{(1+|z|^2)^2} = -\frac{i}{2\pi}\frac{d\bar{z}\,dz}{(1+|z|^2)^2}$$

as shown in Example 4.11.

References

[1] Shiing-shen Chern (1979), *Complex Manifold Without Potential Theory, (with an appendix on the geometry of characteristic classes)*, Second edition, Springer Verlag.

[2] John M. Lee (2018), *Introduction to Riemannian Manifolds*, Second edition, Springer.

[3] Johan L. Dupont (1977), *Curvature and Characteristic Classes*, Springer Verlag.

[4] Dale Husemoller (1993), *Fibre Bundles*, Third edition, Springer Verlag.

[5] Allen Hatcher (2017), *Vector Bundles and K-Theory*, Version 2.2.

[6] Michael Spivak (1999), *A Comprehensive Introduction to Differential Geometry*, Vol. 2, Third edition, Publish or Perish INC.