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Kähler Immersions of Kähler–Einstein Manifolds into Infinite Dimensional Complex Space Forms

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Submitted for Ph.D. Examination

December 9th, 2009

Abstract

This thesis consists of three main results. In the first one we describe all Kähler immersions of a bounded symmetric domain into the infinite dimensional complex projective space in terms of the Wallach set of the domain. In the second one we exhibit an example of complete and non-homogeneous Kähler-Einstein metric with negative scalar curvature which admits a Kähler immersion into the infinite dimensional complex projective space. As last, we prove that the complex hyperbolic space is the only Cartan domain which admits a Kähler immersion into the indefinite complex Euclidean space of finite index.

Declaration

I declare that to the best of my knowledge the contents of this thesis are original and my work except where indicated otherwise.

Acknowledgements

First of all, I would like to thank my supervisor Andrea Loi for introducing me to mathematical research and for the large amount of time dedicated to me.

I would like to express my gratitude to Antonio J. Di Scala for his several suggestion throughout all my research.

I wish to thank Prof. Ernst Heintze for accepting me as visiting student and to all the professors and colleagues of the mathematics department of Augsburg Universität. In particular, special gratitude is due to a friend, Andreas Kollross, for all the precious suggestions at the very beginning of my work.

I would like also to thank all the colleagues of the mathematics department in Cagliari, in particular Debora Impera for joining me in all the "extra" mathematical activities, Roberto Mossa for the interesting mathematical discussions and Fabio Zuddas for his several encouragements and hints.

Furthermore, I would like to thank my family for being so chaotic that when I enter at home everything else it is easy to be forgotten.

I owe a very special thanks to Stefania Casu and Marta Corona for being friends, confidants and supporters in the hardest part of the past three years.

I want also to express my gratitude to Daria Uccheddu for having been there when I needed and for understanding without any explanation.

Finally, I wish to thank all my friends that helped me in many ways in this period, above all Elisabetta Mereu for all the lunchtimes together and Alessio Masala, Salvatore Pili and Eleonora Porcu for the beautiful weekend trips.

Contents

Introduction

In contrast with Riemannian geometry where a well–known theorem due to John Nash (cfr. [37]) states that any Riemannian manifold can be isometrically immersed into the real Euclidean space $\mathbb{R}^N$ for sufficiently large N, a Kähler manifold does not always admit an isometric and holomorphic (from now on *Kähler*) immersion into the complex Euclidean space $\mathbb{C}^N$ (not even if N is allowed to be infinite). For example, a compact manifold cannot be holomorphically immersed into $\mathbb{C}^N$ for any value of N, since every holomorphic function from a connected compact set into $\mathbb{C}$ is constant. But even if we consider noncompact manifolds, there are still many obstructions to the existence of such an immersion. For istance, if a Kähler manifold admits a Kähler immersion into $\mathbb{C}^N$, its metric is forced to be a real analytic Kähler metric, being the pull–back via a holomorphic map of the flat metric on $\mathbb{C}^N$. Other less trivial obstructions can be found in the seminal paper of Eugenio Calabi (see Chapter 2 or [9]). Calabi gives a complete answer to the problem of existence and uniqueness of a Kähler immersion of a Kähler manifold into $\mathbb{C}^N$, for $N \leq \infty$, and more generally into any other finite or infinite dimensional complex space form (cfr. Example 1.3.1). The criterion Calabi provides in order to find out if a particular manifold admits a Kähler immersion into a complex space form is based on the study of a particular Kähler potential, christened by Calabi *diastasis function*. The diastasis function is not always explicitely given, thus although Calabi's criterion is theoretically impeccable, most of the time it is of difficult applicability. Hence, it is an open problem of high interest classifying Kähler manifolds admitting a Kähler immersion into complex space forms, in particular when the metric is a Kähler–Einstein metric (see [17], [19], [24], [25], [30], [33], [36], [43], [50], [51] and [52]), that is the case we are interested in this thesis.

If the ambient space is the *finite* dimensional complex projective space, a classification is given only for low codimension. More precisely, Shiing-Shen Chern and Kazumi Tsukada (cfr. [17], [50]) show that if the codimension is less or equal than two, a Kähler–Einstein manifold Kähler immersed into $\mathbb{C}\mathrm{P}^N$ is either totally geodesic or the quadric. For general codimension, the only known examples of Kähler–Einstein manifolds admitting a Kähler immersion into $\mathbb{C}\mathrm{P}^N$ are homogeneous and it is conjecturally true these are the only ones (see Section 3.2 or e.g. [3], [17], [43] and [50]). Hence, it is natural to ask what happens when N is allowed to be *infinite*, namely we are concerned with the following question:

Question: *does there exist a nonhomogeneous and complete Kähler–Einstein manifold which can be Kähler immersed into* $\mathbb{C}\mathrm{P}^\infty$*?*

In this thesis we give a positive answer to this question (cfr. Section 4.4). More precisely, we consider a bounded symmetric domain Ω of rank $r \neq 1$ and genus γ endowed with a multiple of its Bergman metric g_B and we take the family of Cartan–Hartogs domains depending on the parameter $\mu > 0$ and based on Ω, defined by

$$\mathrm{M}_\Omega(\mu) = \left\{(z, w) \in \Omega \times \mathbb{C},\ |w|^2 < \mathrm{K}^{\mu/\gamma}(z, z)\right\},$$

where $\mathrm{K}(z, z)$ is the Bergman kernel of Ω. The first result of this thesis is the following:

Theorem 1 (Theorem 4.1.1). *There exists a continuous family of homothetic, complete, nonhomogeneous and projectively induced Kähler-Einstein metrics on each Cartan–Hartogs domain based on an irreducible bounded symmetric domain of rank* $r \neq 1$.

The proof of Theorem 1 is based on recent results (cfr. [41], [55]) about Einstein metrics on Cartan–Hartogs domains and on the following theorem which is the second main result of this thesis:

Theorem 2 (Theorem 4.1.2). *Let Ω be an irreducible bounded symmetric domain endowed with its Bergman metric g_B. Then (Ω, cg_B) admits an equivariant Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ if and only if $c\gamma$ belongs to $W(\Omega) \setminus \{0\}$, where γ denotes the genus of Ω.*

Here $W(\Omega)$ denotes the Wallach set of Ω, i.e. the set of all $\lambda \in \mathbb{C}$ such that there exists a Hilbert space $\mathcal{H}_\lambda$ whose reproducing kernel is $\mathrm{K}^{\frac{\lambda}{\gamma}}$ (see Section 4.3 or [2]).

In the case the ambient space is the finite dimensional Euclidean space $\mathbb{C}^N$ or the finite dimensional hyperbolic space $\mathbb{C}\mathrm{H}^N$, Masaaki Umehara (cfr. Section 3.1 or [51]) shows that the only Kähler–Einstein manifolds Kähler immersed into these spaces are totally geodesic. This result cannot be extended to the infinite dimensional case. In fact, in [9] Calabi provides an explicit Kähler immersion of the hyperbolic space $\mathbb{C}\mathrm{H}^n$ into the infinite dimensional flat space $\ell^2(\mathbb{C})$ (cfr. Section 3.3). Although, we believe that this is the only exception (Conjecture 3.3.4). In [19] Andrea Loi and Antonio José Di Scala show that the hyperbolic space is characterized among bounded symmetric domains to be the only one admitting a Kähler immersion into $\ell^2(\mathbb{C})$ (cfr. Section 3.3). In this thesis we extend this result to the case when the ambient space is the indefinite complex Euclidean space $\mathbb{C}^{r,s} = (\mathbb{C}^{r+s}, g_{r,s})$, r, $s \in \mathbb{N} \cup \{\infty\}$, $\{r, s\} \neq \{\infty, \infty\}$, where $g_{r,s}$ is the indefinite Kähler metric on $\mathbb{C}^{r+s}$ (cfr. Section 5.2). More precisely, we prove that the hyperbolic space $\mathbb{C}\mathrm{H}^n$ can be characterized among irreducible bounded symmetric domains as the only one which admits a Kähler immersion into $\mathbb{C}^{\infty,s}$, $s < \infty$, as expressed in the following theorem, the third and last result of this thesis:

Theorem 3 (Theorem 5.1.1). *Let (Ω, g_B) be a Cartan domain. Assume that there exists a local Kähler immersion (Ω, g_B) into $\mathbb{C}^{r,s}$, then $r = \infty$, $s \in \mathbb{N}$ and $(\Omega, g_B) = (\mathbb{C}\mathrm{H}^n, (n+1)g_{hyp})$.*

The thesis is divided into five chapters organized as follows.

The first three chapters are dedicated to recall preliminary notions and to summarize known results about complex and Kähler manifolds.

The first one introduces Kähler–Einstein manifolds and describes the problematic relative to the existence and uniqueness of Kähler–Einstein metrics on compact complex manifolds, according to the sign of the scalar curvature. In the last section it also provides some examples of Kähler–Einstein metrics, both in the compact and noncompact case, used in the rest of the thesis.

The second chapter presents the work of E. Calabi on the existence of a Kähler immersion of a complex manifold into complex space forms. The first section is dedicated to the definition and the basic properties of the diastasis function, while the second one describes Calabi's criterion.

The third chapter illustrates known results on the problem of classifying complex manifolds which admits a Kähler immersion into complex space forms when the metric is a Kähler–Einstein metric. The first three sections of this chapter give an outline of the problem in the case when the ambient space is finite dimensional, dividing the exposition in M. Umehara's work on Kähler immersion of Kähler–Einstein manifods into $\mathbb{C}^N$ and $\mathbb{C}\mathrm{H}^N$ and various results and conjectures on Kähler immersions of Kähler–Einstein manifolds into $\mathbb{C}\mathrm{P}^N$. The last section is devoted to the infinite dimensional case and in particular summarizes Calabi's result on Kähler immersion between complex space forms and A. J. Di Scala and A. Loi's characterization of $\mathbb{C}\mathrm{H}^n$ among the bounded symmetric domains to be the only one admitting a Kähler immersion into $\ell^2(\mathbb{C})$.

The last two chapters contain our results.

Chapter 4 is divided into three sections and it is dedicated to the proof of Theorem 1 and 2. In the first section we prove a general fact about projectively induced metrics in relation with the value of the constant that multiplies the metric. More precisely we show that *a Kähler manifold* (M, cg) *admits a local Kähler immersion into* $\mathbb{C}\mathrm{P}^\infty$ *for all positive value of* c *if and only if* (M, g) *does into* $\ell^2(\mathbb{C})$. The proof is based on a theorem of Bochner (Theorem 3.3.2 or [6, Th. 14]) and on some considerations on the diastasis function of (M, cg). The second section introduces the definition of the Wallach set $W(\Omega)$ of a bounded symmetric domain Ω and provides a very useful characterization of the diastasis of (Ω, g_B). All these elements enable us to prove Theorem 2. Last section is dedicated to the definition of a Cartan–Hartogs domain $(M_\Omega(\mu), cg(\mu))$ and to a description of the Kähler metric $g(\mu)$ in terms of its diastasis function. In particular we prove the following proposition, which completely determines when a Cartan–Hartogs domain $(M_\Omega(\mu), cg(\mu))$ is projectively induced: *the metric* $cg(\mu)$ *is projectively induced if and only if the metric* $(c+m)\frac{\mu}{\gamma}g_B$ *on* Ω *is projectively induced for every integer* $m \geq 0$. The proof of this proposition is based on Calabi's criterion applied to the diastasis of $cg(\mu)$ and of $(c+m)\frac{\mu}{\gamma}g_B$ respectively. By the work of Guy Roos, An Wang, Weiping Yin, Liyou Zhang and Wenjuan Zhang (see [41], [55] or Section 4.4), there exists a particular value μ_0 of μ such that $g(\mu_0)$ is a complete Kähler–Einstein metric which is homogeneous if and only if the rank of Ω is equal to 1. These facts together with Theorem 2 allows us to prove Theorem 1.

The last chapter of the thesis is divided into two sections and it is dedicated to the proof of Theorem 3. The first section provides the basic definitions of the indefinite Euclidean space and of its metric. The second one is devoted to the proof of Theorem 3, based on an extension of Calabi's criterion when the ambient space is the indefinite Euclidean space (Lemma 5.2.1), together with the particular structure of the diastasis function of Cartan domains (Proposition 4.3.2).

Chapter 1

Preliminaries

The first section of this chapter recalls definitions and standard properties of Kähler and Kähler–Einstein metrics on complex manifolds (see [35] for details and further results). In the second one we give an overview of the problematic relative to the existence and uniqueness of a Kähler–Einstein metric on compact complex manifolds, according to the sign of the scalar curvature (although we do not need this result we include it in this thesis for completeness). In the last section we provide some examples of Kähler–Einstein metrics (both in the compact and noncompact case) needed in the rest of the thesis.

1.1 Kähler metrics

Consider a n-dimensional complex manifold M and let us fix the following notations: denote by $\mathrm{Hol}(M)$ the space of holomorphic functions on M, by $\Omega^r(M, \mathbb{K})$ the space of r-forms on M with values on a field $\mathbb{K}$ and by $\Omega^{p,q}(M) \subset \Omega^{p+q}(M)$ the space of (p,q)-forms on M. Let also $H^{p,q}_{\bar\partial}(M)$ be the cohomology class of $\bar\partial$-closed (p,q)-forms on M.

Proposition 1.1.1 ($\partial\bar\partial$–Lemma)**.** *Let $\alpha \in \Omega^2(M, \mathbb{R}) \cap \Omega^{1,1}(M)$. If α is closed then for all $x \in M$ there exists an open set $U \ni x$ and a function $u \in C^\infty(U, \mathbb{R})$ such that*

$$\alpha_{|U} = i\partial\bar\partial u.$$

Proof. By Poincaré Lemma there exists a real 1-form τ such that $\alpha = d\tau$ locally. Since $\Omega^1(M,\mathbb{R}) \subset \Omega^1(M,\mathbb{C})$, we have $\tau = \tau^{1,0} + \tau^{0,1}$ and

$$\begin{aligned}\alpha =&(\partial + \bar\partial)(\tau^{1,0} + \tau^{0,1}) = \partial\tau^{1,0} + \bar\partial\tau^{0,1} + \bar\partial\tau^{1,0} + \partial\tau^{0,1} \\ =&\bar\partial\tau^{1,0} + \partial\tau^{0,1},\end{aligned}$$

where the last equality holds for $\alpha \in \Omega^{1,1}(M)$.
By Dolbeault Lemma $\bar\partial\tau^{0,1} = 0 \Rightarrow \exists\, f \in C^\infty(U,\mathbb{C})$ such that $\tau^{0,1} = \bar\partial f$, thus we have

$$\alpha_{|U} = \bar\partial\partial\bar f + \partial\bar\partial f = \partial\bar\partial(f - \bar f) = i\partial\bar\partial u,$$

where $u = 2\,\mathrm{Im}(f)$. $\square$

A Hermitian metric on M is a Riemannian metric g which is an isometry with respect to the almost complex structure J of M, i.e. $g(JX,JY) = g(X,Y)$, for all X, $Y \in \mathfrak{X}(M)$ (throughout this thesis $\mathfrak{X}(M)$ will denote the space of smooth vector fields on M). A Hermitian manifold is a couple (M,g) where M is a complex manifold and g a Hermitian metric on M.
Let (M,g) be a Hermitian manifold and let ω be the fundamental form associated to g, i.e. $\omega(X,Y) = g(X,JY)$ for all X, $Y \in \mathfrak{X}(M)$.

Definition 1.1.2. *A complex manifold M endowed with a Hermitian metric g is Kähler if and only if the fundamental form ω associated to g is closed.*

By $\partial\bar\partial$–Lemma Kähler manifolds are characterized by the existence around each point $x \in M$ of a neighbourhood $U \ni x$ and a smooth map $\Phi : U \to \mathbb{R}$ such that $\omega_{|U} = \frac{i}{2}\partial\bar\partial\Phi$ (see e.g. [35]). The function Φ is called a *Kähler potential* for the metric g and it is univocally determined up to the addition of the real part of a holomorphic function. In fact, if Φ' is another Kähler potential, in local coordinates we have

$$\frac{\partial^2(\Phi - \Phi')}{\partial z_\alpha \partial \bar z_\beta} = 0,$$

that implies

$$\Phi = \Phi' + f + \bar f, \quad \text{for } f \in \mathrm{Hol}(M).$$

Observe that in local coordinates one has

$$\omega = i \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}\, dz_\alpha \wedge d\bar{z}_\beta,$$

where

$$g_{\alpha\bar{\beta}} = g\left(\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta}\right) = \frac{1}{2}\frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta}.$$

Let Ric be the Ricci curvature of a Hermitian manifold (M, g) and let ρ be the Ricci form associated to Ric, i.e. if J is the almost complex structure on M,

$$\rho(X, Y) = \mathrm{Ric}(X, JY) \quad \text{for } X,\, Y \in \mathfrak{X}(M).$$

The nice feature of the Kähler metric is that the Ricci form has a very simple expression in terms of the metric tensor, i.e.

$$\rho = -i\partial\bar{\partial} \log \det(g_{\alpha\bar{\beta}}),$$

(we refer the reader to [48]).

1.2 Kähler–Einstein metrics

A Kähler metric g on a complex manifold M is Einstein if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\rho = \lambda\omega, \tag{1.1}$$

where ω is the fundamental form associated to g. The pair (M, g), where M is a complex manifold and g a Kähler–Einstein metric is said a Kähler–Einstein manifold.

Let M be a complex manifold endowed with a Kähler–Einstein metric g. In a neighbourhood U of $x \in M$ we have

$$\partial\bar{\partial}\left(\log\left(\det(g_{\alpha\bar{\beta}})\right) + \frac{\lambda}{2}\Phi\right) = 0,$$

and hence

$$\log\left(\det(g_{\alpha\bar{\beta}})\right) = -\frac{\lambda}{2}\Phi + f + \bar{f},$$

for some $f \in \mathrm{Hol}(U)$. Thus a Kähler metric g is Einstein if locally it satisfies the Monge-Ampère equation

$$\det(g_{\alpha\bar{\beta}}) = e^{-\frac{\lambda}{2}(\Phi+f+\bar{f})}. \tag{1.2}$$

Consider a compact Kähler manifold (M, ω) and let ρ be its associated Ricci form. The form $\rho/2\pi$ represents the first Chern class $c_1(M)$ of M (cfr. [16]). In [11] and [12], E. Calabi asks if given a closed (1,1)-form $\tilde{\rho}$ which represents the first Chern class of M, one can find a Kähler metric $\tilde{\omega}$ on M whose Ricci tensor is $\tilde{\rho}$. In [12] he shows that this metric is unique on each Kähler class (i.e. a cohomology class of type $(1,1)$ that contains a form that is positive definite). The existence of $\tilde{\omega}$ on each Kähler class is known as Calabi's Conjecture and it has been solved by S. T. Yau in 1976 (cfr. [57]).

Calabi's Conjecture has some immediate consequences on the existence and uniqueness of Kähler–Einstein metrics on compact complex manifolds. Let (M, ω) be a compact Kähler–Einstein manifold. According to the sign of λ, the first Chern class $c_1(M)$ must either vanish or have a representative which is negative or positive definite (we write in that cases $c_1(M) < 0$ and $c_1(M) > 0$ respectively). When $c_1(M)$ vanishes, M admits an unique Ricci-flat metric on each Kähler class. In fact, $c_1(M)$ contains the 0 form as representative and Calabi's Conjecture guarantees the existence of a Kähler form associated to 0 on each Kähler class $[\omega]$. That metric is known as Calabi-Yau metric.

When $c_1(M) < 0$, T. Aubin [4] and S. T. Yau [57] independently, prove that there exists a unique (up to homotheties) Kähler–Einstein metric depending only on the complex structure of M. Furthermore, the group of holomorphic transformations of M is finite and contained in the group of isometries of M.

When $c_1(M) > 0$ the existence of a Kähler–Einstein metric is not guaranteed. The problem has been largely studied by G. Tian (see e.g. [44], [46] and [47]). The simplest examples of Kähler–Einstein metrics with positive first Chern class are given by the Hermitian symmetric space of compact type (see Example 1.3.3 below). In [33] Y. Matsushima proves that when $c_1(M) > 0$ if two Einstein forms ω_1, ω_2 are cohomologous then they are isometric, namely there exists $F \in \mathrm{Aut}(M)$ such that $F^*\omega_2 = \omega_1$. Notice that X. X. Chen and G. Tian (cfr. [15]) generalise this result for extremal metrics.

1.3 Examples

Example 1.3.1 (Complex space forms)**.** A complex space form is a finite or infinite dimensional Kähler manifold of constant holomorphic sectional curvature, that if we assume to be complete and simply connected, then up to homotheties can be of three types, according to the sign of the holomorphic sectional curvature:

(i) the complex Euclidean space $\mathbb{C}^N$ of complex dimension $N \leq \infty$, endowed with the flat metric denoted by g_0. Here $\mathbb{C}^\infty$ denotes the Hilbert space $\ell^2(\mathbb{C})$ consisting of sequences w_j, $j = 1, 2, \dots$, $w_j \in \mathbb{C}$ such that $\sum_{j=1}^{+\infty} |w_j|^2 < +\infty$. Fixed a coordinate system $(z_1, \dots, z_N)$, a globally defined Kähler potential $\Phi \colon \mathbb{C}^N \to \mathbb{R}$ for g_0 is given by

$$\Phi(z) = \sum_{j=1}^{N} |z_j|^2. \tag{1.3}$$

Observe that the flat metric is trivially Ricci-flat, since $\det(g_{\alpha\bar\beta}) = 1$;

(ii) the complex projective space $\mathbb{C}\mathrm{P}^N$ of complex dimension $N \leq \infty$, with the Fubini-Study metric g_b of holomorphic sectional curvature $4b$ for $b > 0$. Let $[Z_0, \dots, Z_N]$ be homogeneous coordinates, $p = [1, 0, \dots, 0]$ and $U_0 = \{Z_0 \neq 0\}$. Define affine coordinates $z_1, \dots, z_N$ on U_0 by $z_j = Z_j/Z_0$. The Fubini-Study metric g_{FS} can be described on U_0 by the Kähler potential

$$\Phi(z) = \frac{1}{b} \log(1 + b\sum_{j=1}^{N} |z_j|^2), \quad \text{(for } b > 0\text{)}. \tag{1.4}$$

In this case g_b is Einstein with Einstein constant $\lambda = 2b(N+1)$;

(iii) the complex hyperbolic space $\mathbb{C}\mathrm{H}^N$ of complex dimension $N \leq \infty$, namely the unit ball $B \subset \mathbb{C}^N$ given by $B = \left\{(z_1, \dots, z_N) \in \mathbb{C}^N,\ \sum_{j=1}^{N} |z_j|^2 < 1\right\}$, endowed with the hyperbolic metric g_b of constant holomorphic sectional curvature $4b$, for

$b < 0$. Fixed a coordinates system around a point $p \in \mathbb{C}\mathrm{H}^N$, the hyperbolic metric is described by the globally defined Kähler potential Φ given by

$$\Phi(z) = \frac{1}{b}\log(1 + b\sum_{j=1}^{N}|z_j|^2), \quad (\text{for } b < 0). \tag{1.5}$$

In this case g_b is Einstein with Einstein constant $\lambda = 2b(N+1)$.

When it is not otherwise specified, we assume b to be respectively 0, 1 and -1 and we denote g_1 by g_{FS} and g_{-1} by g_{hyp}. Furthermore, in order to simplify the notation we write $\mathbb{C}^N$, $\mathbb{C}\mathrm{P}^N$ and $\mathbb{C}\mathrm{H}^N$ instead of $(\mathbb{C}^N, g_0)$, $(\mathbb{C}\mathrm{P}^N, g_{FS})$ and $(\mathbb{C}\mathrm{H}^N, g_{hyp})$.

Example 1.3.2 (Bergman metric)**.** We refer the reader to [27] for details and further results on the Bergman metric. Let D be a bounded domain of $\mathbb{C}^n$ and consider the Hilbert space

$$L^2_{hol}(D) = \left\{ f \in \mathrm{Hol}(D), \int_D |f|^2 d\mu < \infty \right\},$$

where $d\mu$ denotes the Lebesgue measure on $\mathbb{R}^{2n} = \mathbb{C}^n$. Consider the inner product on $L^2_{hol}(D)$ given by

$$(f, g) = \int_D f(\zeta)\overline{g(\zeta)}d\mu(\zeta).$$

Given a orthonormal basis $\{\varphi_j\}$ on $L^2_{hol}(D)$, the series

$$\mathrm{K}(z,\zeta) = \sum_{j=0}^{\infty} \varphi_j(z)\overline{\varphi_j(\zeta)}$$

converges uniformly on each compact subset of $D \times D$ and it is independent of the choice of the orthonormal basis. The function K so defined is the *Bergman kernel* of D, also called *reproducing kernel* for its reproducing property

$$f(z) = \int_D \mathrm{K}(z,\zeta)f(\zeta)d\mu(\zeta), \quad f \in L^2_{hol}(D). \tag{1.6}$$

The Kähler metric

$$\omega_B = \frac{i}{2}\partial\bar\partial \log \mathrm{K}(z,z),$$

is the Bergman metric on D. It is important to remark that the group of automorphism $\mathrm{Aut}(D)$ of D, i.e. biholomorphism $f\colon D \to D$, is contained in the group of isometries

$\mathrm{Isom}(D, g_B)$, that is if $F \in \mathrm{Aut}(D)$ then $F^* g_B = g_B$. If $\mathrm{Aut}(D)$ also acts transitively, i.e. D is homogeneous, then g_B is Einstein and $\mathrm{Ric}_{g_B} = -2g_B$, i.e. the Einstein constant is -2 (cfr. [29, p. 163]). Observe that the Bergman metric and the hyperbolic metric on $\mathbb{C}\mathrm{H}^n$ are homothetic, more precisely one has $(n+1)g_{hyp} = g_B$.

Example 1.3.3 (Hermitian symmetric spaces)**.** A connected complex manifold S with a Hermitian structure is said to be a Hermitian symmetric space if each point $p \in S$ is an isolated fixed point of an involutive holomorphic isometry s_p of S. The Hermitian structure of a Hermitian symmetric space is Kählerian. Let $A(S)$ be the set of holomorphic isometries of S and $A_0(S)$ its identity component. S is said to be of the *compact* or *noncompact type* according to the type of the Riemannian pair $(A_0(S), K)$, where K is the isotropy subgroup of $A_0(S)$ at some point $o \in S$. Every simply connected Hermitian symmetric space is a product of irreducible elements

$$S = S_0 \times S_- \times S_+,$$

where all factors are simply connected Hermitian symmetric spaces, $S_0 = \mathbb{C} \times \cdots \times \mathbb{C}$, and S_-, S_+ are of the compact and noncompact type respectively (see [23] for details). The following table summarizes the classification of irreducible Hermitian symmetric spaces of compact type.

Table 1.1: Irreducible Hermitian symmetric spaces of compact type.

Type	HSSCT	Dimension
$A\ III$	$SU(n+m)/S(U_n \times U_m)$	nm
$C\ I$	$Sp(n)/U(n)$	$n(n+1)/2$
$D\ III$	$SO(2n)/U(n)$	$n(n-1)/2$
$BD\ I$	$SO_0(n+2)/SO(n) \times SO(2)$	n
$E\ III$	$(\mathfrak{e}_{-78}, \mathfrak{so}(10) + \mathbb{R})$	16
$E\ VII$	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6 + \mathbb{R})$	27

A Hermitian symmetric space of noncompact type of complex dimension d is biholomorphically isometric to (Ω, cg_B), where Ω is a bounded symmetric domain of $\mathbb{C}^d$ endowed with its Bergman metric g_B multyplied by a positive constant c. A globally defined potential for g_B is given by $\Phi(z) = \log \mathrm{K}$, where K is the Bergman kernel of Ω (cfr. Example 1.3.2). The domain Ω can be chosen to be circular (i.e. $z \in \Omega,\ \theta \in \mathbb{R} \Rightarrow\ e^{i\theta} z \in \Omega$) and convex. Every bounded symmetric domain is the product of irreducible factors, called Cartan domains. From E. Cartan classification, Cartan domains can be divided into two categories, classical and exceptional ones (see [28] for details). Classical domains correspond to the duals of $A\ III$, $C\ I$, $D\ III$ and $BD\ I$ in Table 1.1 and can be described in terms of complex matrices as follows (m and n are nonnegative integers, $n \geq m$):

$$\Omega_1[m,n] = \{Z \in M_{m,n}(\mathbb{C}),\ I_m - ZZ^* > 0\} \qquad (\dim(\Omega_1) = nm),$$

$$\Omega_2[n] = \{Z \in M_n(\mathbb{C}),\ Z = Z^T,\ I_n - ZZ^* > 0\} \qquad (\dim(\Omega_2) = \tfrac{n(n+1)}{2}),$$

$$\Omega_3[n] = \{Z \in M_n(\mathbb{C}),\ Z = -Z^T,\ I_n - ZZ^* > 0\} \qquad (\dim(\Omega_3) = \tfrac{n(n-1)}{2}),$$

$$\Omega_4[n] = \{Z = (z_1, \dots, z_n) \in \mathbb{C}^n,\ \sum_{j=1}^n |z_j|^2 < 1, 1 + |\sum_{j=1}^n z_j^2|^2 - 2\sum_{j=1}^n |z_j|^2 > 0\}$$

$$(\dim(\Omega_4) = n),\ n \neq 2,$$

where I_m (resp. I_n) denotes the $m \times m$ (resp $n \times n$) identity matrix, and $A > 0$ means that A is positive definite. In the last domain we are assuming $n \neq 2$ since $\Omega_4[2]$ is not irreducible (and hence it is not a Cartan domain). In fact, the biholomorphism

$$f\colon \Omega_4[2] \to \mathbb{C}\mathrm{H}^1 \times \mathbb{C}\mathrm{H}^1,\ (z_1, z_2) \mapsto (z_1 + iz_2, z_1 - iz_2)$$

satisfies

$$f^*(2(g_{hyp} \oplus g_{hyp})) = g_B.$$

The reproducing kernels of classical Cartan domains are given by

$$\mathrm{K}_{\Omega_1}(z,z) = \frac{1}{V(\Omega_1)}[\det(I_m - ZZ^*)]^{-(n+m)},$$

$$\mathrm{K}_{\Omega_2}(z,z) = \frac{1}{V(\Omega_2)}[\det(I_n - ZZ^*)]^{-(n+1)},$$

$$\mathrm{K}_{\Omega_3}(z,z) = \frac{1}{V(\Omega_3)}[\det(I_n - ZZ^*)]^{-(n-1)},$$

$$\mathrm{K}_{\Omega_4}(z,z)=\frac{1}{V(\Omega_4)}\left(1+|\sum_{j=1}^{n}z_j^2|^2-2\sum_{j=1}^{n}|z_j|^2\right)^{-n}, \tag{1.7}$$

where $V(\Omega_j)$, $j=1,\dots,4$, is the total volume of Ω_j with respect to the Euclidean measure of the ambient complex Euclidean space (see [19] for details).

Notice that for some values of m and n, up to multiply the metric by a positive constant, the domains coincide with the hyperbolic space $\mathbb{C}\mathrm{H}^n$, more precisely we have

$$(\Omega_1[1,n],g_B)=(\mathbb{C}\mathrm{H}^n,(n+1)g_{hyp}),$$

$$(\Omega_2[1],g_B)=(\Omega_3[2],g_B)=(\Omega_4[1],g_B)=(\mathbb{C}\mathrm{H}^1,2g_{hyp}),$$

$$(\Omega_3[3],g_B)=(\mathbb{C}\mathrm{H}^3,4g_{hyp}).$$

In general, $(\Omega,g_B)=(\mathbb{C}\mathrm{H}^n,cg_{hyp})$, for some $c>0$, if and only if the rank of Ω is equal to 1.

There are two kinds of exceptional domains $\Omega_5[16]$ of dimension 16 and $\Omega_6[27]$ of dimension 27, corresponding to the dual of $E\ III$ and $E\ VII$, that can be described in terms of 3×3 matrices with entries in the 8-dimensional algebra of complex octonions $\mathbb{O}_{\mathbb{C}}$. We refer the reader to [41] for a more complete description of these domains.

Remark 1.3.4. For future reference, observe that any irreducible bounded symmetric domain of rank greater or equal than 2, can be exhausted by totally geodesic submanifolds isomorphic to $\Omega_4[3]$ (cfr. [49]). Notice also that every homogeneous noncompact Kähler manifold different to $(\mathbb{C}\mathrm{H}^{n_1}\times\cdots\times\mathbb{C}\mathrm{H}^{n_s},c_1\,g_{hyp}\oplus\cdots\oplus c_s\,g_{hyp})$, for $c_1,\dots,c_s$ positive constant, admits $\Omega_4[3]$ as a Kähler submanifold (this last fact is due to a private conversation with A. J. Di Scala).

Example 1.3.5 (Calabi's complete and not locally homogeneous metric)**.** Consider the complex tubular domain $M_n=\frac{1}{2}D\oplus i\mathbb{R}^n\subset\mathbb{C}^n$, where D denotes any connected, open subset of $\mathbb{R}^n$. Let g_n be the metric on M_n whose associated Kähler form is given by

$$\omega_n=\frac{i}{2}\partial\bar{\partial}F(z),$$

with

$$F(z) = f(z_1 + \bar{z}_1, \dots, z_n + \bar{z}_n),$$

where $f : D \to \mathbb{R}$ is a radial function $f(x_1, \dots, x_n) = y(r)$, with $r = (\sum_{j=1}^n x_j^2)^{1/2}$, satisfying the differential equation

$$\left(\frac{y'}{r}\right)^{n-1} y'' = e^y,$$

with initial conditions

$$y'(0) = 0,\ y''(0) = e^{y(0)/n}.$$

This metric introduced by Calabi [13] is the first example of complete and not locally homogeneous Kähler–Einstein metric. In [56] J. A. Wolf gives a stronger more straightforward version of Calabi's result, namely if $n \geq 2$ and g is an $E(n)$-invariant Kähler metric on M_n, where $E(n) = \mathbb{R}^n \cdot \mathrm{SO}(n)$, then (M_n, g) cannot be both complete and locally homogeneous. Moreover, $E(n)$ is the largest connected group of holomorphic isometries of (M_n, g).

Example 1.3.6 (Taub-NUT metric)**.** In [30] C. Lebrun constructs the following family of Kähler forms on $\mathbb{C}^2$ defined by $\omega_m = \frac{i}{2}\partial\bar{\partial}\Phi_m$, where

$$\Phi_m(u, v) = u^2 + v^2 + m(u^4 + v^4),\ \text{for } m \geq 0,$$

and u and v are implicitly defined by

$$|z_1| = e^{m(u^2 - v^2)}u,\ |z_2| = e^{m(v^2 - u^2)}v.$$

For $m = 0$ one gets the flat metric, while for $m > 0$ each of the metrics of this family represents the first example of complete Ricci–flat (non-flat) metric on $\mathbb{C}^2$ having the same volume form of the flat metric ω_0, i.e. $\omega_m \wedge \omega_m = \omega_0 \wedge \omega_0$. Moreover, for $m > 0$, these metrics are isometric (up to dilation and rescaling) to the Taub-NUT metric.

Chapter 2

Kähler immersions into complex space forms

This chapter summarizes the work of E. Calabi [9] about the existence of a Kähler immersion of a complex manifold into a finite or infinite dimensional complex space form. In particular, Calabi provides an algebraic criterion to find out whether a complex manifold admits or not such an immersion. The basic tool he introduces is a particular Kähler potential called *diastasis*, to the description of which the first section is dedicated. The second section is devoted to illustrate Calabi's criterion.

2.1 The diastasis function

Let M be a n-dimensional complex manifold endowed with a real analytic Kähler metric g. The Kähler metric g is real analytic if fixed a local coordinate system $z = (z_1, \dots, z_n)$ on a neighbourhood U of a point $p \in M$, it can be described on U by a real analytic Kähler potential $\Phi : U \to \mathbb{R}$. In that case the potential $\Phi(z)$ can be analytically continued to an open neighbourhood $W \subset U \times U$ of the diagonal. Denote this extension by $\Phi(z, \bar{w})$.

Definition 2.1.1. *The* diastasis function $\mathrm{D}(z, w)$ *on* W *is defined by*

$$\mathrm{D}(z, w) = \Phi(z, \bar{z}) + \Phi(w, \bar{w}) - \Phi(z, \bar{w}) - \Phi(w, \bar{z}). \tag{2.1}$$

The following proposition due to Calabi (cfr. [9, pp. 3, 4]) describes the basic properties of the diastasis function.

Proposition 2.1.2 (E. Calabi)**.** *The diastasis function* $\mathrm{D}(z,w)$ *given by (2.1) satisfies the following properties:*

(i) it is uniquely determined by the Kähler metric and it does not depend on the choice of the local coordinate system;

(ii) it is real valued in its domain of (real) analyticity;

(iii) it is symmetric in z *and* w *and* $\mathrm{D}(z,z)=0$*;*

(iv) once fixed one of its two entries, it is a Kähler potential.

From now on, when in a coordinate neighbourhood we fix a point $p \in M$ with coordinates $w=(w_1,\dots,w_n)$ we write $\mathrm{D}_p(z)$ or $\mathrm{D}_w(z)$ for the diastasis centered at that point. In particular, if p is the origin of the coordinate system chosen, we write $\mathrm{D}_0(z)$.

The following proposition shows how the diastasis function is related to the geodesic distance explaining the name *diastasis*, from the Greek $\delta\iota\acute{\alpha}\sigma\tau\alpha\sigma\iota\varsigma$, that means *distance.*

Proposition 2.1.3. *If* $\rho(p,q)$ *is the geodesic distance between* p *and* q*, of coordinates respectively* $z=(z_1,\dots,z_n)$ *and* $w=(w_1,\dots,w_n)$ *then*

$$\mathrm{D}(z,w)=(\rho(p,q))^2+O((\rho(p,q))^4).$$

Example 2.1.4. The Kähler potentials given by (1.3), (1.4) and (1.5), are actually the diastasis functions for the complex space forms considered. In particular, given any point $p \in \mathbb{C}^n$, the globally defined diastasis $\mathrm{D}_0(z)=\sum_{j=1}^n |z_j|^2$ for the flat metric g_0, is exactly equal to the square of the geodesic distance.

Example 2.1.5. Let g_* be the Kähler metric on $\mathbb{C}^*$ whose associated Kähler form is given by $\omega_*=\frac{i}{2}\partial\bar\partial|z|$. The potential $\Phi(z)=|z|$ is globally defined, while the diastasis

$D_0(z)$, centered at any point $\alpha \in \mathbb{C}^*$, is not. In fact, by definition we have

$$D_\alpha(z) = |z| + |\alpha| - \sqrt{z\bar{\alpha}} - \sqrt{\bar{z}\alpha},$$

and as maximal domain of definition one can take $\mathbb{C}^* \setminus L$, where L is any half–line starting from the origin of $\mathbb{C}$ such that $\alpha \notin L$.

The importance of the diastasis function for our purposes, is expressed by the following proposition:

Proposition 2.1.6. *Let (M, g) be a Kähler manifold which admits a Kähler immersion $f\colon (M, g) \to (S, G)$ into a real analytic Kähler manifold (S, G). Then the metric g is real analytic. Let $D^M_p : U \to \mathbb{R}$ and $D^S_{f(p)} : V \to \mathbb{R}$ be the diastasis functions of (M, g) and (S, G) around p and $f(p)$ respectively. Then $D^S_{f(p)} \circ f = D^M_p$ on $f^{-1}(V) \cap U$.*

We now introduce the concept of Bochner's coordinates (cfr. [6], [9], [24], [25], [42], [45]). Given a real analytic Kähler metric g on M and a point $p \in M$, one can always find local (complex) coordinates in a neighborhood of p such that

$$D_p(z) = \sum_{\alpha=1}^{n} |z_\alpha|^2 + \psi_{2,2}, \tag{2.2}$$

where D_p is the diastasis of g relative to p and $\psi_{2,2}$ is a power series with degree ≥ 2 in both the variables z and $\bar{z}$. These coordinates, uniquely defined up to a unitary transformation (cfr. [6], [9]), are called the *normal* or *Bochner's coordinates* around the point p. Further we have the following:

Theorem 2.1.7 (E. Calabi)**.** *Let (M, g) be a n-dimensional Kähler manifold Kähler immersed into a N-dimensional real analytic Kähler manifold (S, G). Then if $z = (z_1, \dots, z_n)$ are Bochner's coordinates on M with respect to a point $p \in M$, then there exist Bochner's coordinates on S such that the immersion $i : M \to S$ is given in a neighbourhood of p by a graph*

$$(z_1, \dots, z_n) \mapsto (z_1, \dots, z_n, f_1(z), \dots, f_{N-n}(z)), \tag{2.3}$$

where for all $j = 1, \dots, N-n$, f_j is a holomorphic function with no terms of degree less then 2.

Remark 2.1.8. Observe that choosing Bochner's coordinates on a neighbourhood U of a point p of a Kähler manifold (M, g) whose diastasis on U is given by $\mathrm{D}_0(z)$, reduces the Monge–Ampère equation (Eq. (1.2)) to the form

$$\det(g_{\alpha\bar{\beta}}) = e^{-\frac{\lambda}{2}\mathrm{D}_0(z)}. \tag{2.4}$$

In fact, once set Bochner's coordinates, it is easy to check that the expansion of $\det(g_{\alpha\bar{\beta}})$ in the $(z, \bar{z})$-coordinates around the origin reads $\det(g_{\alpha\bar{\beta}}) = 1 + h(z, \bar{z})$, where $h(z, \bar{z})$ is a power series in z, $\bar{z}$ which contains only mixed terms (i.e. of the form $z^j \bar{z}^k$, $j \neq 0$, $k \neq 0$). Further, also the expansion of $\mathrm{D}_0(z)$, given by (2.2), contains only mixed terms, forcing $f + \bar{f}$ to be zero.

We conclude this section giving a very useful characterization of the diastasis, easily deducible by the definition, in terms of its power expansion. In order to semplify the notation, let us first fix the following multi-index convention that we are going to use through all the thesis. We arrange every n-tuple of nonnegative integers as the sequence $m_j = (m_{j,1}, \dots, m_{j,n})$ with not decreasing order, that is $m_0 = (0, \dots, 0)$ and if $|m_j| = \sum_{\alpha=1}^n m_{j,\alpha}$, we have $|m_j| \leq |m_{j+1}|$ for all positive integer j. Further z^{m_j} denotes the monomial in n variables $\prod_{\alpha=1}^n z_\alpha^{m_{j,\alpha}}$. For example, if $n = 2$ we can consider the ordering $m_0 = (0, 0)$, $m_1 = (1, 0)$, $m_2 = (0, 1)$, $m_3 = (1, 1)$, $m_4 = (2, 0)$, etc. and we would have $z^{m_0} = 1$, $z^{m_1} = z_1$, $z^{m_2} = z_2$, $z^{m_3} = z_1 z_2$, $z^{m_4} = z_1^2$, etc. Notice that the order is not uniquely determined by these rules, in fact, we are allowed to exchange terms of equal module $|m_j|$ (i.e. in the 2 dimensional case we may also take $m_1 = (0, 1)$, $m_2 = (1, 0)$, etc.).

Theorem 2.1.9 (Characterization of the diastasis)**.** *Among all the Kähler potentials the diastasis $\mathrm{D}_p(z)$ is characterized by the fact that in every coordinate system $(z_1, \dots, z_n)$ centered at p, the $\infty \times \infty$ matrix of coefficients (a_{jk}) in its power expansion around the origin*

$$\mathrm{D}_p(z) = \sum_{j,k=0}^{\infty} a_{jk} z^{m_j} \bar{z}^{m_k}, \tag{2.5}$$

satisfy $a_{j0} = a_{0j} = 0$ for every nonnegative integer j.

2.2 Calabi's criterion

Consider the indefinite Hilbert space E of sequences $(x_1, x_{-1}, x_2, x_{-2}, \dots, x_j, x_{-j}, \dots)$ such that $\sum_{j\in\mathbb{Z}^*} |x_j|^2 < \infty$, endowed with the indefinite Hermitian metric defined by the diastasis $\mathrm{D}_0^E(x) = \sum_{j\in\mathbb{Z}^*} (\mathrm{sgn} j)|x_j|^2$. The indefinite Hilbert space E constitutes a *universal* Kähler manifold, in the sense that it is a space in which every analytic Kähler manifold can be Kähler immersed. More precisely we have the following (cfr. [9, pp. 6-9]):

Theorem 2.2.1 (E. Calabi)**.** *A complex manifold M endowed with a metric g can be Kähler immersed into the indefinite Hilbert space E if and only if g is a real analytic Kähler metric.*

Let (S^N, g_b) be a N-dimensional complex space form of holomorphic sectional curvature $4b$ (see Example 1.3.1). As remarked in the previous section, if there exists a Kähler immersion of a complex manifold (M, g) into (S^N, g_b), then the metric g is forced to be a real analytic Kähler metric, being the pull–back via a holomorphic map of a real analytic Kähler metric. Thus consider a real analytic Kähler manifold (M, g) and fix a coordinate system $z = (z_1, \dots, z_n)$ with origin at $p \in M$. Let $\mathrm{D}_0(z)$ be the diastasis of g at p. Define the matrix (a_{jk}) to be the $\infty \times \infty$ matrix of coefficients given by (2.5).

Definition 2.2.2. *A real analytic Kähler manifold (M, g) is* resolvable *of rank N at $p \in M$ if (a_{jk}) is semipositive definite of rank N.*

For $b \neq 0$, consider the function $(e^{b\mathrm{D}_0(z)} - 1)/b$ and take its power expansion around the origin

$$\frac{e^{b\mathrm{D}_0(z)} - 1}{b} = \sum_{j,k=0}^{\infty} s_{jk}\, z^{m_j}\, \bar{z}^{m_k}.$$

Definition 2.2.3. *A real analytic Kähler manifold (M, g) is b-*resolvable *of rank N at $p \in M$ if the matrix (s_{jk}) is semipositive definite of rank N.*

In particular, (M, g) is 1-resolvable of rank N at p if the matrix of coefficients (b_{jk}) given by

$$e^{\mathrm{D}_0(z)} - 1 = \sum_{j,k=0}^{\infty} b_{jk}\, z^{m_j}\, \bar{z}^{m_k}, \tag{2.6}$$

is positive semidefinite of rank N. Similarly (M, g) is -1-resolvable of rank N at p if the matrix of coefficients (c_{jk}) given by

$$1 - e^{-\mathrm{D}_0(z)} = \sum_{j,k=0}^{\infty} c_{jk}\, z^{m_j}\, \bar{z}^{m_k}. \tag{2.7}$$

is positive semidefinite of rank N.

Calabi's criterion for local Kähler immersion can be stated as follows (cfr. [9, pp. 9, 18]):

Theorem 2.2.4 (local Calabi's criterion)**.** *Let (M, g) be a real analytic Kähler manifold. There exists a neighbourhood U of a point p that admits a Kähler immersion into $\mathbb{C}^N$ (resp. (S^N, g_b), for $b \neq 0$) if and only if (M, g) is resolvable (resp. b-resolvable) of rank at most N at $p \in M$. Furthermore if the rank is exactly N, the immersion is full.*

In particular, a neighbourhood $U \ni p$ of (M, g) admits a Kähler immersion into $\mathbb{C}\mathrm{P}^N$ (resp. $\mathbb{C}\mathrm{H}^N$), if and only if M is 1-resolvable (resp. -1-resolvable) of rank at most N at p.

Recall that a Kähler immersion $f : (M, g) \to (S, G)$ is *full* if the image $f(M) \subset S$ is not contained in any totally geodesic subspace of S.

In order to state the global version of Calabi's criterion, we need two further results (cfr. [9, pp. 8, 11, 18]):

Theorem 2.2.5 (rigidity)**.** *If a neighbourhood U of a point p admits a full Kähler immersion into (S^N, g_b), then N is univocally determined by the metric and the immersion is unique up to rigid motions of (S^N, g_b).*

Theorem 2.2.6 (global character of resolvability)**.** *If a Kähler manifold (M, g) is resolvable (resp. b-resolvable) of rank N at a point $p \in M$, then it also is at any other point.*

Last theorem states that if a local Kähler immersion around a point $p \in M$ exists, then the same is true for any other point. Due to this result, we can say that a manifold is resolvable (resp. b-resolvable) without specifying the point. In particular, if (M, g) is 1-resolvable, we say also that g is *projectively induced*. If M is chosen to be simply connected, then it is possible to extend the immersion to the whole manifold. More precisely, we have the following (cfr. [9, pp. 13, 20]):

Theorem 2.2.7 (global Calabi's criterion)**.** *A simply connected complex manifold* (M, g) *admits a Kähler immersion into* $\mathbb{C}^N$ *(resp.* (S^N, g_b)*, for* $b \neq 0$*) if and only if the following conditions are fulfilled:*

(i) the metric is a real analytic Kähler metric,

(ii) for each point $p \in M$ *the analytic extension of the diastasis* D_p *is single valued,*

(iii) the Kähler manifold (M, g) *is resolvable (resp. b-resolvable) of rank at most* N*.*

Further, the immersion is also injective if and only if

(iv) $\mathrm{D}_p(z) = 0$ *only for* $z = 0$*.*

We end this section with two examples of local Kähler immersions that cannot be extended to global ones.

Example 2.2.8. Consider the Kähler metric $\tilde{g}$ on $\mathbb{C}^*$ whose fundamental form is

$$\tilde{\omega} = \frac{i}{2}\frac{dz \wedge d\bar{z}}{|z|^2}.$$

Since $\mathbb{C}$ admits a Kähler immersion $f_0 \colon \mathbb{C} \to \mathbb{C}\mathrm{P}^\infty$ into $\mathbb{C}\mathrm{P}^\infty$ (cfr. Eq. (3.5) below) and it covers $\mathbb{C}^*$ throught the map $\exp \colon \mathbb{C} \to \mathbb{C}^*$, given by $\exp(z) = e^{2\pi i z}$, then a neighbourhood of each point of $\mathbb{C}^*$ can be Kähler immersed into $\mathbb{C}\mathrm{P}^\infty$. The immersion cannot be extended to a global one. In fact, since $\exp^*(\tilde{g}) = g_0$, such Kähler immersion f composed with exp, would be a Kähler immersion of $\mathbb{C}$ into $\mathbb{C}\mathrm{P}^\infty$. By Calabi's rigidity (Theorem 2.2.5), it would then exist a rigid motion T of $\mathbb{C}$ such that $T \circ f_0 = f \circ \exp$, that is impossible since f_0 is injective and exp is not.

Example 2.2.9. Consider the complex torus of complex dimension 1, $\mathrm{T}^1 = \mathbb{C}/\mathbb{Z}^2$, endowed with the flat metric g_0. The Kähler manifold (T^1, g_0) admits a local Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$. Since $\mathbb{C}$ is a covering for T^1, by applying the same arguments as in the previous example one can show that (T^1, g_0) does not admit a global Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$. Similar reasons show that a Riemann surface Σ_g admits a local but not global Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$, being covered by the hyperbolic disc of complex dimension 1. Observe that the same results follow by Hulin's work, since the surfaces considered have nonpositive constant scalar curvature (cfr. Section 3.2 below).

Chapter 3

Kähler immersions of Kähler–Einstein manifolds

Although Calabi's criterion solve theoretically the problem of characterizing Kähler manifolds admitting a Kähler immersion into a complex space form, it can be very hard or sometimes impossible to verify whether a particular manifold admits such a Kähler immersion or not. The problem of classifying such manifolds has been, and still is, largely studied by many mathematicians, in particular when the metric is Kähler–Einstein, that is the case we are interested in this thesis (see [17], [19], [24], [25], [30], [33], [36], [43], [50], [51] and [52]). The first three sections of this chapter give an outline of the problem in the case when the ambient space is finite dimensional, while the last one is devoted to the infinite dimensional case.

3.1 Umehara's work: Kähler immersions into $\mathbb{C}\mathrm{H}^N$ and $\mathbb{C}^N$

The following result due to M. Umehara [52] determines the nature of Kähler–Einstein manifolds admitting a Kähler immersion into $\mathbb{C}\mathrm{H}^N$ or $\mathbb{C}^N$, for finite N. Even if in this thesis we do not need directly the proof of Umehara's theorem, we report it for completeness and as example of beautiful application of the diastasis function.

Theorem 3.1.1 (M. Umehara)**.** *Every Kähler–Einstein manifold Kähler immersed into $\mathbb{C}^N$ or $\mathbb{C}\mathrm{H}^N$ is always totally geodesic.*

The proof of the theorem is based on results achieved by Umehara himself in [51] that can be summarized in the following lemma:

Lemma 3.1.2. *Let $f_1, \ldots, f_N$ be non-constant holomorphic functions on a complex manifold M such that $f_j(p) = 0$ for some $p \in M$. Then*

(1) $e^{\sum_{j=1}^N |f_j|^2} \notin \Lambda(M)$,

(2) $(1 - \sum_{j=1}^N |f_j|^2)^{-a} \notin \Lambda(M), \quad (a > 0)$.

Here $\Lambda(M)$ is a set of $\mathbb{R}$-linear combinations of real analytic functions of the form $h\bar{k} + \bar{h}k$ for $h,\, k \in \mathrm{Hol}(M)$. The set $\Lambda(M)$ is an associative algebra and coincides with the set of real analytic functions f on M whose associated form $\frac{i}{2}\partial\bar{\partial}f$ is of finite rank, i.e. the matrix of coefficients (a_{jk}) in the power expansion at a fixed point $p \in M$

$$f(z) = \sum_{j,k=0}^{\infty} a_{jk} z^{m_j} \bar{z}^{m_k},$$

has finite rank (see Section 2.2 for notation).

Let us prove first Umehara's result in the case when the ambient space is $\mathbb{C}^N$.

Proof of the first part of Theorem 3.1.1. Let (M, g) be a n-dimensional Kähler–Einstein manifold Kähler immersed into $\mathbb{C}^N$, ω the Kähler form associated to g and

$$\rho = -i\partial\bar{\partial}\log\det(g_{\alpha\bar{\beta}})$$

the Ricci form. Let $z = (z_1, \ldots, z_n)$ be a local coordinate system on $U \subset M$ such that $0 \in U$ and let

$$\omega_{|U} = \frac{i}{2}\partial\bar{\partial}\,\mathrm{D}_0^M,$$

where D_0^M is the diastasis for g on U centered at 0. The Gauss' Equation

$$\rho \leq 2b(n+1)\omega, \tag{3.1}$$

where b is the holomorphic sectional curvature of the ambient space (see for example [29, p. 177]), gives for $b = 0$ $\rho \leq 0$, where the equality holds if and only if M is totally geodesic. Hence, if M is not totally geodesic, ρ is negative definite and the Einstein's

Equation (1.1) implies $\lambda < 0$. Up to homothetic transformations of $\mathbb{C}^N$ we can suppose $\lambda = -1$.

Since M admits a Kähler immersion into $\mathbb{C}^N$, by Proposition 2.1.6 there exists $f_1, \dots, f_N$ holomorphic functions such that

$$\mathrm{D}_0^M(z) = \sum_{j=1}^{N} |f_j(z)|^2.$$

Thus, by previous lemma we have $e^{\mathrm{D}_0^M} \notin \Lambda(M)$. On the other hand, by the Monge-Ampère Equation (1.2) with $\lambda = -1$, the function $\log\det(g_{\alpha\bar\beta})$ is a Kähler potential for g, hence we have

$$\mathrm{D}_0^M(z) = h + \bar h + \log\det(g_{\alpha\bar\beta}),$$

for a holomorphic function h. Hence

$$e^{\mathrm{D}_0^M} = |e^h|^2 \det(g_{\alpha\bar\beta}).$$

Since $\det(g_{\alpha\bar\beta}) \in \Lambda(M)$, for it is a real valued function being the matrix $(g_{\alpha\bar\beta})$ Hermitian, we get the contradiction $e^{\mathrm{D}_0^M} \in \Lambda(M)$. □

Before proving the second part of Umehara's theorem we need the following lemma:

Lemma 3.1.3 (M. Umehara)**.** *Let M be a complex n-dimensional manifold and let $(z_1, \dots, z_n)$ be a local coordinate system on an open set $U \subset M$. If $f \in \Lambda(U)$ then*

$$f^{n+1} \det\left(\frac{\partial^2 \log f}{\partial z_\alpha \partial \bar z_\beta}\right) \in \Lambda(U).$$

Proof. Let us write f_α for $\partial f/\partial z_\alpha$, $f_{\bar\beta}$ for $\partial f/\partial \bar z_\beta$ and $f_{\alpha\bar\beta}$ for $\partial^2 f/\partial z_\alpha \partial \bar z_\beta$. We have

$$\frac{\partial^2 \log f}{\partial z_\alpha \partial \bar z_\beta} = \frac{f_{\alpha\bar\beta}}{f} - \frac{f_\alpha f_{\bar\beta}}{f^2},$$

thus we get

$$f^{n+1}\det\left(\frac{\partial^2\log f}{\partial z_\alpha\partial\bar z_\beta}\right)=f\det\left(f_{\alpha\bar\beta}-\frac{f_\alpha f_{\bar\beta}}{f}\right)=f\det\begin{pmatrix} f_{1\bar1}-f_1f_{\bar1}/f & \dots & f_{1\bar n}-f_1f_{\bar n}/f & 0\\ \vdots & & \vdots & \vdots\\ f_{n\bar1}-f_nf_{\bar1}/f & \dots & f_{n\bar n}-f_nf_{\bar n}/f & 0\\ f_{\bar1}/f & \dots & f_{\bar n}/f & 1\end{pmatrix}$$

$$=f\det\begin{pmatrix} f_{1\bar1} & \dots & f_{1\bar n} & f_1\\ \vdots & & \vdots & \vdots\\ f_{n\bar1} & \dots & f_{n\bar n} & f_n\\ f_{\bar1}/f & \dots & f_{\bar n}/f & 1\end{pmatrix}=f\det\begin{pmatrix} f_{1\bar1} & \dots & f_{1\bar n} & f_1\\ \vdots & & \vdots & \vdots\\ f_{n\bar1} & \dots & f_{n\bar n} & f_n\\ f_{\bar1} & \dots & f_{\bar n} & f\end{pmatrix}.$$

Hence

$$f^{n+1}\det\left(\frac{\partial^2\log f}{\partial z_\alpha\partial\bar z_\beta}\right)\in\Lambda(U),$$

for it is finitely generated by holomorphic and antiholomorphic functions on U and it is real valued, because the matrix $\left(\partial^2\log f/\partial z_\alpha\partial\bar z_\beta\right)$ is Hermitian. □

We can now prove the second part of Theorem 3.1.1.

Proof of the second part of Theorem 3.1.1. Let (M,g) be a n-dimensional Kähler–Einstein manifold Kähler immersed into $\mathbb{C}\mathrm{H}^N$. Comparing Gauss' Equation (3.1) with $b<0$ and Einstein's Equation (1.1), we get that the Einstein constant λ is negative. Let $(z_1,\dots,z_n)$ be local coordinates on an open set $U\subset M$ centered at $p\in U$. On U the Monge–Ampère Equation (1.2) for g reads

$$e^{-\frac{\lambda}{2}D_0^M(z)}=|e^h|^2\det(g_{\alpha\bar\beta}),$$

for some holomorphic function h. By Proposition 2.1.6, for some holomorphic functions $\varphi_1,\dots,\varphi_N$ that can be chosen to be zero at the origin, we have on U

$$\mathrm{D}_0^M(z)=-\log(1-\sum_{j=1}^N|\varphi_j(z)|^2).$$

Setting $f=1-\sum_{j=1}^N|\varphi_j|^2$ we have

$$\det(g_{\alpha\bar\beta})=(-1)^n\det\left(\frac{\partial^2\log f}{\partial z_\alpha\partial\bar z_\beta}\right).$$

Thus

$$f^{\frac{\lambda}{2}} = (-1)^n |e^h|^2 \det\left(\frac{\partial^2 \log f}{\partial z_\alpha \partial \bar{z}_\beta}\right),$$

and hence

$$f^{\frac{\lambda}{2}+n+1} = (-1)^n |e^h|^2 f^{n+1} \det\left(\frac{\partial^2 \log f}{\partial z_\alpha \partial \bar{z}_\beta}\right).$$

By previous lemma we obtain

$$f^{\frac{\lambda}{2}+n+1} = \left(1 - \sum_{j=1}^{N} |\varphi_j(z)|^2\right)^{\frac{\lambda}{2}+n+1} \in \Lambda(U),$$

and by (2) of Lemma 3.1.2 we get $\frac{\lambda}{2}+n+1 \geq 0$. On the other hand, the Gauss' Equation (3.1) implies $n+1+\frac{\lambda}{2} \leq 0$. Thus $\lambda = -2(n+1)$, and M is totally geodesic. $\square$

3.2 Kähler immersions into $\mathbb{C}\mathrm{P}^N$

The problem of classifying Kähler–Einstein manifolds that admit a Kähler immersion into the finite dimensional complex projective space $\mathbb{C}\mathrm{P}^N$ has been partially solved by S. S. Chern [17] and K. Tsukada [50], that determined all the projectively induced Kähler–Einstein manifolds in the case the codimension is respectively 1 or 2.

Theorem 3.2.1 (S. S. Chern, K. Tsukada)**.** *Let (M, g) be a n-dimensional Kähler–Einstein manifold ($n \geq 2$). If (M, g) admits a Kähler immersion into $\mathbb{C}\mathrm{P}^{n+2}$, then M is either totally geodesic or the quadric Q_n in $\mathbb{C}\mathrm{P}^{n+1}$ (which is totally geodesic in $\mathbb{C}\mathrm{P}^{n+2}$), with homogeneous equation $Z_0^2 + \cdots + Z_{n+1}^2 = 0$.*

In the case of Kähler immersions of a complex space form (S^N, g_b) into another, the problem has been solved by Calabi [9], which also gives the explicit expression of the Kähler immersion:

$$\begin{aligned} f\colon (S^N, g_b) &\hookrightarrow (S^{N'}, g_{b'}) \\ z &\mapsto \left(\sqrt{\frac{\prod_{k=1}^{|m_1|-1}(b'-kb)}{m_1!}}\, z^{m_1}, \ldots, \sqrt{\frac{\prod_{k=1}^{|m_{N'}|-1}(b'-kb)}{m_{N'}!}}\, z^{m_{N'}}\right), \end{aligned} \tag{3.2}$$

(for $b' = kb$, $N' \geq \binom{N+k}{k} - 1$, see Theorem 3.3.1 below for details) where we used the multi-index notation introduced in Section 2.1, with additionally $m_j! = m_{j,1}! \cdot \cdots \cdot m_{j,n}!$.

Observe that when $b = b' = 1$, we have an isometric version of the Veronese embedding. The following example describes another classical immersion, whose coefficients, differently from the Veronese embedding, do not need to be modified to be a Kähler immersion.

Example 3.2.2 (Segre embedding)**.** Consider the map $\sigma\colon \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^{(n+1)(m+1)-1}$ defined by

$$\sigma([Z_0, \ldots, Z_n], [W_0, \ldots, W_m]) \mapsto [Z_0 W_0, \ldots, Z_j W_k, \ldots, Z_n W_m],$$

where $[Z_0, \ldots, Z_n]$ and $[W_0, \ldots, W_m]$ are homogeneous coordinates for $\mathbb{C}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^m$ respectively and $0 \leq j \leq n$, $0 \leq k \leq m$, $(j,k) \neq (0,0)$, $(j,k) \neq (n,m)$. It is easy to verified that σ is a Kähler immersion.

In general, it is an open problem to classify projectively induced Kähler–Einstein manifolds. The only known examples of such manifolds are homogeneous and it is conjecturally true these are the only ones (see e.g. [3], [17], [43] and [50]).

Conjecture 3.2.3. *If a complete Kähler–Einstein manifold admits a Kähler immersion into* $\mathbb{C}\mathrm{P}^N$*, then it is homogeneous.*

The simplest examples of projectively induced homogeneous Kähler–Einstein manifolds are given by Hermitian symmetric space of compact type:

Example 3.2.4 (Hermitian symmetric spaces of compact type)**.** It is well-known since the work of Borel and Weil (see [31] or [43] for a proof) that a Hermitian symmetric space of compact type admits a full Kähler immersion into $\mathbb{C}\mathrm{P}^N$ (for further results see [19]).

Observe that a homogeneous projectively induced Kähler–Einstein manifold is forced to be compact (see [43, §2 p.178]). Thus we can state the following weaker conjecture:

Conjecture 3.2.5. *If a complete Kähler–Einstein manifold admits a local Kähler immersion into* $\mathbb{C}\mathrm{P}^N$*, then it is compact.*

Notice that the previous conjecture does not hold if the manifold considered is not equipped with a Kähler–Einstein metric. The following example due to A. J. Di Scala private conversation, represents a complete not Einstein noncompact manifold admitting a Kähler immersion into $\mathbb{C}\mathrm{P}^8$.

Example 3.2.6. Consider the complex torus of complex dimension 2, $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2 \times \mathbb{C}/\mathbb{Z}^2$, endowed with the flat metric ω_0, and consider the map $f\colon \mathbb{C} \to \mathbb{T}^2$, $f(z) = (z, \alpha z)$, for $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Notice that this map is a injective immersion, in fact if $(z, \alpha z) \sim (w, \alpha w)$ then on one hand $z = w + m + in$ and $\alpha z = \alpha w + p + iq$ for some integer m, n, p and q, and on the other hand $\alpha z = \alpha w + \alpha m + i\alpha n$. Since α is not rational this gives $m = n = p = q = 0$, and thus $z = w$. Notice that $f^*(\omega_0) = (|\alpha|^2 + 1)\omega_0$. Since $\mathbb{C}/\mathbb{Z}^2$ is algebraic, we have a holomorphic map $\mathbb{T}^2 \to \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2$. By the Segre embedding $\sigma\colon \mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^8$ (see Example 3.2.2 above) we can consider $\mathbb{C}\mathrm{P}^2 \times \mathbb{C}\mathrm{P}^2$ Kähler immersed into $\mathbb{C}\mathrm{P}^8$. Let us call ω_2 the metric on $\mathbb{T}^2$ pull–back of the Fubini-Study metric on $\mathbb{C}\mathrm{P}^8$, and let $\omega_1 = f^*(\omega_2)$. The metric ω_1 is complete. In fact, since $\mathbb{T}^2$ is compact, ω_2 is complete, and there exist two positive constants a, b such that $a\,\omega_0 < \omega_2 < b\,\omega_0$. Thus $a\,f^*(\omega_0) < f^*(\omega_2) < b\,f^*(\omega_0)$, that is

$$a\,(|\alpha|^2 + 1)\omega_0 < \omega_1 < b\,(|\alpha|^2 + 1)\omega_0,$$

from which the completeness of ω_1 is straightforward.

A different approach to the problem is considered by D. Hulin (cfr. [24], [25]) that studies Kähler–Einstein manifolds Kähler immersed into $\mathbb{C}\mathrm{P}^N$ in relation with the sign of the Einstein constant. By the Bonnet–Meyers Theorem it follows that if the Einstein constant of a Kähler–Einstein manifold M is positive then M is compact. Hulin proves that in the case when M is projectively induced the converse is also true (see [25] for a proof and [3] for further details):

Theorem 3.2.7 (D. Hulin)**.** *Let (M, g) be a (connected) Kähler–Einstein manifold Kähler immersed into $\mathbb{C}\mathrm{P}^N$. Then the Einstein constant is strictly positive.*

Corollary 3.2.8. *Any Calabi-Yau manifold is not projectively induced.*

We conclude this section with the following theorem that summarizes some further results given in [24]:

Theorem 3.2.9 (D. Hulin)**.** *Let (M, g) be a Kähler–Einstein manifold which admits a Kähler immersion $f\colon M \to \mathbb{C}\mathrm{P}^N$ into $\mathbb{C}\mathrm{P}^N$. Then the Einstein constant λ is rational. Further, if the immersion is full and we write $\lambda = 2p/q > 0$, where p/q is irreducible, then $p \leq n+1$ and if $p = n+1$ (resp. $p = n$), then $(M, g) = (\mathbb{C}\mathrm{P}^n, qG_{FS})$ (resp. $(M, g) = (Q_n, qG_{FS})$).*

3.3 The infinite dimensional case

In this section we consider Kähler immersions of Kähler–Einstein manifolds into infinite dimensional complex space forms, namely $\mathbb{C}\mathrm{H}^\infty$, $\ell^2(\mathbb{C})$ and $\mathbb{C}\mathrm{P}^\infty$. Obviously if a Kähler manifold admits a Kähler immersion into $\mathbb{C}\mathrm{H}^\infty$ or $\ell^2(\mathbb{C})$ then it is noncompact by the maximum principle. Also for a Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ we can assume M noncompact by the following reason: *if a compact Kähler manifold M admits a Kähler immersion $f\colon M \to \mathbb{C}\mathrm{P}^\infty$, then f cannot be full, i.e. there exists a positive integer N such that $f(M) \subset \mathbb{C}\mathrm{P}^N \subset \mathbb{C}\mathrm{P}^\infty$.* In fact, assume by contradiction that $f\colon M \to \mathbb{C}\mathrm{P}^\infty$ is a full Kähler immersion. Then we can write $f(p) = [s_0(p), \dots, s_j(p), \dots]$, where each s_j is a holomorphic section of some holomorphic line bundle L on M. Since the map is full, the $\{s_j\}$ are linearly independent in contradiction with the fact that the space of holomorphic sections $H^0(L)$ on a compact manifold is of finite dimension. Therefore troughout this section we can assume our manifold to be noncompact.

As application of its criterion Calabi studies the existence of Kähler immersion of a complex space form into another (cfr. [9, pp. 21-22]):

Theorem 3.3.1 (E. Calabi)**.** *A complex space form (S^N, g_b) admits a global Kähler immersion into $(S^{N'}, g_{b'})$ if an only if $b \leq b'$ and*

either $b \leq 0$ and $N' = \infty$,

or $b' = kb$ for some positive integer k, and $N' \geq \binom{N+k}{k} - 1$ for N finite, $N' = \infty$ for $N = \infty$.

In the proof of the theorem, Calabi gives the explicit expression of the full Kähler immersion, that is the map described in (3.2). In particular if we take the ambient space to be infinite dimensional and the holomorphic sectional curvature to be 0, 1 or -1 respectively, we have the following full Kähler immersions:

$$f : \mathbb{C}\mathrm{H}^n \hookrightarrow \ell^2(\mathbb{C}) : z \mapsto \left(\dots, \sqrt{\frac{(|m_j| - 1)!}{m_j!}} z^{m_j}, \dots \right), \tag{3.3}$$

$$f : \mathbb{C}\mathrm{H}^n \hookrightarrow \mathbb{C}\mathrm{P}^\infty : z \mapsto \left(\dots, \sqrt{\frac{|m_j|!}{m_j!}} z^{m_j}, \dots \right), \tag{3.4}$$

$$f : \mathbb{C}^n \hookrightarrow \mathbb{C}\mathrm{P}^\infty : z \mapsto \left(\dots, \sqrt{\frac{1}{m_j!}} z^{m_j}, \dots \right). \tag{3.5}$$

Notice that, since the immersions above are full, it follows from Calabi's rigidity (Theorem 2.2.5) that if b and b' have different sign, a complex space form (S^N, g_b) cannot be Kähler immersed into a finite dimensional complex space form $(S^{N'}, g_{b'})$ (in [31] this assertion has been generalized to Hermitian symmetric spaces, namely an Hermitian symmetric space cannot Kähler immersed into another of different type).

In the following theorem which will be used later, S. Bochner [6] consider the relations between Kähler manifolds admitting a Kähler immersion into $\ell^2(\mathbb{C})$ and the ones admitting a Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$.

Theorem 3.3.2 (S. Bochner)**.** *If a Kähler manifold (M, g) admits a Kähler immersion into the infinite dimensional flat space $\ell^2(\mathbb{C})$ then it also does into $\mathbb{C}\mathrm{P}^\infty$.*

Proof. Fix a local coordinate system $(z_1, \dots, z_n)$ on a neighbourhood U of $p \in M$. By Theorem 2.1.9 for some holomorphic functions $f_1, \dots, f_j, \dots$, the diastasis function for g reads

$$\mathrm{D}_0^M(z) = \sum_{j=1}^{\infty} |f_j|^2.$$

Let $\mathrm{D}_0^M(z) = \log\psi$ with $\psi = e^{\mathrm{D}_0^M(z)}$. Then for some suitable functions h_j, $j = 1, 2, \ldots$ we get

$$\psi = 1 + \sum_{j=1}^{\infty} |h_j|^2,$$

and the conclusion follows. □

Notice that the same assertion as in Theorem 3.3.2 does not hold for finite dimensional ambient spaces. In fact, in [53] Umehara proves that if a Kähler manifold admits a Kähler immersion into $\mathbb{C}^N$ then it cannot be Kähler immersed into any complex hyperbolic space or complex projective space, and if it can be Kähler immersed into $\mathbb{C}\mathrm{H}^N$ then it cannot be into any complex projective space (this last assertion has been recently generalized by A. J. Di Scala and A. Loi [20] by replacing $\mathbb{C}\mathrm{H}^N$ with any bounded domain endowed with its Bergman metric). Although, we can prove a statement similar to Bochner's one for Kähler immersion into $\mathbb{C}\mathrm{H}^\infty$. More precisely we have the following:

Theorem 3.3.3. *If a Kähler manifold (M, g) admits a Kähler immersion into the infinite dimensional hyperbolic space $\mathbb{C}\mathrm{H}^\infty$ then it also does into $\ell^2(\mathbb{C})$.*

Proof. Consider a local coordinate system $(z_1, \ldots, z_n)$ on M in a neighbourhood of $p \in M$ and let $\mathrm{D}_0^M(z)$ be the diastasis function for g at p. By Theorem 2.1.9, there exists $f_1, \ldots, f_j, \ldots$ holomorphic functions such that

$$\mathrm{D}_0^M(z) = -\log(1 - \sum_{j=1}^{\infty} |f_j|^2).$$

Hence

$$\mathrm{D}_0^M(z) = \sum_{j=1}^{\infty} |h_j|^2,$$

for some suitable holomorphic functions h_j, $j = 1, 2, \ldots$, and we are done. □

Regarding $\mathbb{C}\mathrm{H}^\infty$ and $\ell^2(\mathbb{C})$, notice that Umehara's result cannot be extended to that cases. In fact, it is enough to consider the Kähler immersion (3.3) given by Calabi of $\mathbb{C}\mathrm{H}^n$ into $\ell^2(\mathbb{C})$. Nevertheless, we conjecture that this is the only exception:

Conjecture 3.3.4. *If a Kähler–Einstein manifold (M, g) admits a Kähler immersion into $\mathbb{C}\mathrm{H}^\infty$ or $\ell^2(\mathbb{C})$, then either (M, g) is totally geodesic or $(M, g) = (\mathbb{C}\mathrm{H}^{n_1} \times \cdots \times \mathbb{C}\mathrm{H}^{n_r}, c_1 g_{hyp} \oplus \cdots \oplus c_r g_{hyp})$ for positive constants $c_1, \ldots, c_r$ and some $r \in \mathbb{N}$.*

In [19] A. J. Di Scala and A. Loi prove this conjecture holds true in the case when (M, g) is a bounded symmetric domain. The remaining part of this section is dedicated to a summary of their work. We report a proof of the main theorem since it is an example of direct application of Calabi's criterion. Observe that the second part of that proof is a simplified version of the original ones, based on Remark 1.3.4.

Remark 3.3.5. Notice that by circularity of Ω, the diastasis around the origin of (Ω, g_B) is given by

$$\mathrm{D}_0^\Omega(z) = \log(V(\Omega)\mathrm{K}(z, z)), \tag{3.6}$$

where $V(\Omega)$ is the Euclidean volume of Ω. For an explicit proof of equality (3.6) see Proposition 4.3.2.

Remark 3.3.6. Let g_B be the Bergman metric on a bounded domain D (see Example 1.3.2). Then (D, g_B) admits by definition a full Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ (see [31] and [27] for details). Further the immersion can be written as $f = (f_0, f_1, \ldots, f_j, \ldots)$, where $\{f_j\}$ is a orthonormal basis for the Hilbert space $L^2_{hol}(D)$.

If two Kähler manifolds (M_1, g_1) and (M_2, g_2) admit Kähler immersions, say f_1 and f_2, into $\ell^2(\mathbb{C})$, then the Kähler manifold $(M_1 \times M_2, g_1 \oplus g_2)$ admits a Kähler immersion into $\ell^2(\mathbb{C})$ obtained by mapping $(z_1, z_2) \in M_1 \times M_2$ to $(c_1 f_1(z_1), c_2 f_2(z_2)) \in \ell^2(\mathbb{C})$. The converse is also true, as expressed by the following lemma.

Lemma 3.3.7 (A. J. Di Scala, A. Loi)**.** *A Kähler map $f\colon M \times N \to \ell^2(\mathbb{C})$ from the product $M \times N$ of two Kähler manifolds is a product, i.e. up to unitary transformation of $\ell^2(\mathbb{C})$ $f(p, q) = (f_1(p), f_2(q))$, where $f_1\colon M \to \ell^2(\mathbb{C})$ and $f_2\colon N \to \ell^2(\mathbb{C})$ are Kähler maps.*

Theorem 3.3.8 (A. J. Di Scala, A. Loi)**.** *If an open subset U of a n-dimensional Hermitian symemtric space of noncompact type (Ω, cg_B) admits a Kähler immersion* *into* $\mathbb{C}\mathrm{H}^\infty$ *then* $(\Omega, cg_B) = \mathbb{C}\mathrm{H}^n$. *If* U *admits a Kähler immersion into* $\ell^2(\mathbb{C})$ *then* (Ω, g_B) *is either* $(\mathbb{C}\mathrm{H}^n, c_0 g_{hyp})$ *or* $(\mathbb{C}\mathrm{H}^{n_1} \times \cdots \times \mathbb{C}\mathrm{H}^{n_k}, c_1 g_{hyp} \oplus \cdots \oplus c_k g_{hyp})$ *for some* $c_0, c_1, \dots, c_k \in \mathbb{R}^+$. *Furthermore, if* $f : \mathbb{C}\mathrm{H}^{n_1} \times \cdots \times \mathbb{C}\mathrm{H}^{n_k} \to \ell^2(\mathbb{C})$ *is a Kähler immersion then, up to unitary transformation of* $\ell^2(\mathbb{C})$, f *is the product of* k *maps i.e.* $f = (f_1, \dots, f_k)$ *where each* $f_j \colon \mathbb{C}\mathrm{H}^{n_j} \to \ell^2(\mathbb{C})$ *is given by (3.3) with* $n = n_j$.

Proof. Let $f : U \to \mathbb{C}\mathrm{H}^\infty$ be a Kähler immersion of an open subset U of $(\Omega_4[3], g_B)$. We can assume without loss of generality that the origin belongs to U. By (3.6), the diastasis of g_B at 0 restricted to U is given by

$$\mathrm{D}_0^{\Omega_4}(z) = \log(V(\Omega_4[3])\mathrm{K}_4(z, \bar{z})),$$

where K_4 is the Bergman kernel of Ω_4. In order to study the -1-resolvability of g_B, we consider the following power expansion

$$1 - e^{-\mathrm{D}_0^{\Omega_4}(z)} = \sum_{jk} c_{jk} z^{m_j} \bar{z}^{m_k}. \tag{3.7}$$

By the expression of the Bergman kernel given by (1.7) the matrix (c_{jk}) is such that $c_{jk} = 0$ for j, k sufficiently large. Thus, by Theorem 2.2.4 there exists a nonnegative integer N such that $f(U) \subset \mathbb{C}\mathrm{H}^N \subset \mathbb{C}\mathrm{H}^\infty$ and by Umehara's Theorem 3.1.1 $\Omega = \mathbb{C}\mathrm{H}^n$ and $cg_B = g_{hyp}$.

In order to prove the second part of the theorem, it is enough to show that an open subset U of $(\Omega_4[3], g_B)$ does not admit a Kähler immersion into $\ell^2(\mathbb{C})$ (see Remark 1.3.4). By (1.7) and (3.6), the diastasis of g_B around the origin restricted to U is given by

$$\mathrm{D}_0^{\Omega_4}(z) = -3 \log\left(1 - 2(|z_1|^2 + |z_2|^2 + |z_3|^2) + |z_1^2 + z_2^2 + z_3^2|^2\right),$$

thus the matrix of coefficients (a_{jk}) in the power expansion (2.5) for $j, k = 0, \dots, 9$

reads

$$A = 3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

The matrix A is not semipositive definite, indeed it has a negative eigenvalue, namely -3. Hence the whole matrix (a_{jk}) cannot be semipositive definite and the diastasis of g_B is not resolvable of any rank. Finally, the proof of the last part of the theorem follows by Lemma 3.3.7 and Calabi's rigidity (Theorem 2.2.5). □

Chapter 4

Construction of Kähler immersions of complete nonhomogeneous Kähler–Einstein manifolds into $\mathbb{CP}^\infty$

This chapter deals with Kähler immersions of complete noncompact Kähler-Einstein manifolds into $\mathbb{CP}^\infty$. The first section presents our results with a brief outline of the problematic correlated. The second one is dedicated to some general facts about the constant multypling the metric in relation with Kähler immersions into $\mathbb{CP}^\infty$. The last two sections are devoted to the proof of our results.

4.1 Introduction

To the author's knowledge, the only known examples of Kähler-Einstein metrics admitting a full Kähler immersion into $\mathbb{CP}^\infty$ are the flat metric on $\mathbb{C}^n$, the hyperbolic metric on $\mathbb{CH}^n$ (see Section 3.3 or [9]) or the Bergman metric on a bounded homogeneous domain of $\mathbb{C}^n$ (cfr. [27]). Hence, it is natural to ask if there exists a projectively induced, complete, nonhomogeneous Kähler-Einstein manifold. The following theorem gives a positive answer to this question (see Section 4.4 for the definition of Cartan–Hartogs domain).

Theorem 4.1.1. *There exists a continuous family of homothetic, complete, nonhomogeneous and projectively induced Kähler-Einstein metrics on each Cartan–Hartogs domain based on an irreducible bounded symmetric domain of rank $r \neq 1$.*

Our result should be compared with the compact case. First, it is an open problem to classify the compact Kähler-Einstein manifolds which admit a Kähler immersion into a finite dimensional complex projective space (cfr. Section 3.2). Actually, the only known examples of such manifolds are homogeneous and it is conjecturally true these are the only ones (cfr. Conjecture 3.2.3). Moreover, a family as in the previous theorem cannot exist in the compact case. In fact, let cg, $c > 0$, be homothetic Kähler metrics on a compact complex manifold M such that (M, g) admits a Kähler immersion into $\mathbb{C}\mathrm{P}^N$. Then if (M, cg) admits a Kähler immersion into $\mathbb{C}\mathrm{P}^N$, c is forced to be a positive integer. Indeed, the Kähler form ω on M is integral (being the pull-back of the integral Fubini-Study form on $\mathbb{C}\mathrm{P}^N$), namely $\int_\Gamma \omega \in \mathbb{Z}$, for $\Gamma \in H_2(M, \mathbb{Z})$. Thus, if (M, cg) admits a Kähler immersion into $\mathbb{C}\mathrm{P}^N$, we have $c \int_\Gamma \omega \in \mathbb{Z}$, implying $c \in \mathbb{Z}$.

The proof of Theorem 4.1.1 is based on recent results (cfr. [41], [55]) about Einstein metrics on Cartan–Hartogs domains and on the following theorem (see Section 4.3 below or [2] for the definition of the Wallach set of the domain Ω).

Theorem 4.1.2. *Let Ω be an irreducible bounded symmetric domain endowed with its Bergman metric g_B. Then (Ω, cg_B) admits a equivariant Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ if and only if $c\gamma$ belongs to $W(\Omega) \setminus \{0\}$, where γ denotes the genus of Ω.*

4.2 The importance of the constant

Let (M, g) be a Kähler manifold and let c be a positive constant. The multiplication of the metric g by c is harmless when one studies Kähler immersions into the infinite dimensional complex Euclidean space $\ell^2(\mathbb{C})$ equipped with the flat metric g_0. In fact, if $f : M \to \ell^2(\mathbb{C})$ satisfies $f^*(g_0) = g$ then $(\sqrt{c}f)^*(g_0) = cg$. Although, the situation is much different when one deals with Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$. We prove in this section a general fact (Corollary 4.2.2 below) about projectively induced metrics in relation with the value of the constant.

Lemma 4.2.1. *If a Kähler manifold (M, g) does not admit a local Kähler immersion into $\ell^2(\mathbb{C})$ then there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, $(M, \epsilon g)$ does not admit a local Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$.*

Proof. Fix a coordinate system $(z_1, \ldots, z_n)$ on an open set $U \subset M$ and let $\mathrm{D}_0^M(z)$ be the diastasis function for g on U centered at the origin. Let (a_{jk}) be the matrix of coefficients in the power expansion of $\mathrm{D}_0^M(z)$ around the origin given by (2.5). If (M, g) does not admit a local Kähler immersion into $\ell^2(\mathbb{C})$ then there exists a submatrix $A = (a_{jk})_{s_1 \leq j,k \leq s_r}$ of (a_{jk}) such that $\det(A) < 0$. The power expansion around the origin of $e^{\epsilon \mathrm{D}_0^M} - 1$ can be written as

$$e^{\epsilon \mathrm{D}_0^M(z)} - 1 = \sum_{j,k=0}^{\infty} \epsilon \, (a_{jk} + \epsilon b_{jk}) \, z^{m_j} \bar{z}^{m_k}.$$

for suitable b_{jk}. Let $B = (b_{jk})_{s_1 \leq j,k \leq s_r}$. We have

$$\det(A + \epsilon B) = \det(A) + P(\epsilon), \tag{4.1}$$

where $P(\epsilon)$ is a polinomial in the variable ϵ with coefficients depending on A and B and such that $P(0) = 0$, resulting by expanding the left hand side of (4.1). Since $P(\epsilon)$ is a polinomial, there exists $\epsilon_0 < 1$ such that $P(\epsilon) < |\det(A)|$ for all $\epsilon < \epsilon_0$. It follows that for all $\epsilon < \epsilon_0$, $\det(A + \epsilon B) < 0$ and conclusion follows by Calabi's criterion (Theorem 2.2.4). $\square$

Observe that we may also have $P(\epsilon) < 0$ for some value of $\epsilon > 1$. In that case we could have $\det(A+\epsilon B) < 0$ for some discrete values of ϵ. Notice also that the procedure described does not determine the highest value of ϵ_0 or the value of any discrete point. In the next section we show how to establish exactly for which values of the constant the immersion exists in the case when M is a bounded symmetric domain.

Corollary 4.2.2. *A Kähler manifold* (M, cg) *admits a local Kähler immersion into* $\mathbb{C}\mathrm{P}^\infty$ *for all* $c > 0$ *if and only if* (M, g) *does into* $\ell^2(\mathbb{C})$.

Proof. If $(M, c_0 g)$ does not admit a Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ for some $c_0 > 0$, then by Bochner's Theorem 3.3.2 $(M, c_0 g)$ does not admit a Kähler immersion into $\ell^2(\mathbb{C})$ either. Further by the discussion at the beginning of this section, (M, cg) does not admit a Kähler immersion into $\ell^2(\mathbb{C})$ for any value of $c > 0$, in particular for $c = 1$. Conversely, if (M, g) does not admit a Kähler immersion into $\ell^2(\mathbb{C})$, by previous lemma there exists c_0 small enough such that $(M, c_0 g)$ cannot be Kähler immersed into $\mathbb{C}\mathrm{P}^\infty$, and we are done. $\square$

4.3 The Wallach set and Kähler immersions into $\mathbb{C}\mathrm{P}^\infty$

A bounded symmetric domain (Ω, cg_B) is uniquely determined by a triple of integers (r, a, b), where r represents the rank of Ω and a and b are positive integers. It remains defined by (r, a, b) the genus $\gamma = (r-1)a + b + 2$. Observe that $(\Omega, cg_B) = \mathbb{C}\mathrm{H}^n$ if and only if its rank is equal to 1. The table below summarizes the numerical invariants and the dimension of Ω according to its type (for a more detailed description of this invariants, which is not necessary in our approach, see e.g. [2], [58]).

Table 4.1: Bounded symmetric domains, invariants and dimension.

Type	r	a	b	γ	dimension
$\Omega_1[m, n]$	m	2	$n - m$	$n + m$	nm
$\Omega_2[n]$	n	1	0	$n + 1$	$n(n+1)/2$
$\Omega_3[n]$	$[n/2]$	4	0 (n even) 2 (n odd)	$n - 1$	$n(n-1)/2$
$\Omega_4[n]$	2	$n - 2$	0	n	n
$\Omega_V[16]$	2	6	4	12	16
$\Omega_{VI}[27]$	3	8	0	18	27

We give now the definition of the *Wallach set* of an irreducible bounded symmetric domain (Ω, cg_B) of genus γ and Bergman kernel K, referring the reader to [2], [21] and [54] for more details and results. This set, denoted by $W(\Omega)$, consists of all $\eta \in \mathbb{C}$ such that there exists a Hilbert space $\mathcal{H}_\eta$ whose reproducing kernel is $\mathrm{K}^{\frac{\eta}{\gamma}}$. This is equivalent to the requirement that $\mathrm{K}^{\frac{\eta}{\gamma}}$ is positive definite, i.e. for all n-tuples of points $x_1, \dots, x_n$ belonging to Ω the $n \times n$ matrix $(\mathrm{K}(x_\alpha, x_\beta)^{\frac{\eta}{\gamma}})$, is positive *semidefinite*. It turns out (cfr. [2, Cor. 4.4, p. 27] and references therein) that $W(\Omega)$ consists only of real numbers and depends on two of the domain's invariants, a and r. More precisely we have

$$W(\Omega) = \left\{0,\, \frac{a}{2},\, 2\frac{a}{2},\, \dots,\, (r-1)\frac{a}{2}\right\} \cup \left((r-1)\frac{a}{2},\, \infty\right). \tag{4.2}$$

The set $W_d = \left\{0,\, \frac{a}{2},\, 2\frac{a}{2},\, \dots,\, (r-1)\frac{a}{2}\right\}$ and the interval $W_c = \left((r-1)\frac{a}{2},\, \infty\right)$ are called respectively the *discrete* and *continuous* part of the Wallach set of the domain Ω.

Remark 4.3.1. If Ω has rank $r = 1$, namely Ω is the complex hyperbolic space $\mathbb{C}\mathrm{H}^d$, then $g_B = (d+1)g_{hyp}$. In this case (and only in this case) $W_d = \{0\}$ and $W_c = (0, \infty)$. If $d = 1$, the Hilbert space $\mathcal{H}$ associated to the kernel

$$\mathrm{K} = \frac{1}{(1-|z|^2)^\alpha}, \qquad \alpha > 0,$$

is the space

$$\mathcal{H} = \left\{ f \in \mathrm{Hol}(\mathbb{C}\mathrm{H}^1), f(z) = \sum_{j=0}^{\infty} a_j z^j \mid \sum_{j=0}^{\infty} \frac{\Gamma(\alpha)\Gamma(j+1)}{\Gamma(j+\alpha)} |a_j|^2 < \infty \right\},$$

endowed with the scalar product

$$< g, h >= \sum_{j=0}^{\infty} \frac{\Gamma(\alpha)\Gamma(j+1)}{\Gamma(j+\alpha)} b_j \bar{c}_j,$$

where $g(z) = \sum_{j=0}^{\infty} b_j z^j$, $h(z) = \sum_{j=0}^{\infty} c_j z^j$ and Γ is the Gamma function.
If $\alpha > 1$, $\mathcal{H}$ is the weighted Bergman space of Ω, namely the Hilbert space of analytic functions $f \in \mathrm{Hol}(\mathbb{C}\mathrm{H}^1)$ such that

$$\int_{\mathbb{C}\mathrm{H}^1} |f(z)|^2 d\mu_\alpha(z) < \infty,$$

where $\mu_\alpha(z)$ is the Lebesgue measure on $(\mathbb{C}\mathrm{H}^1, \alpha\omega_B)$.

Consider a bounded symmetric domain (Ω, cg_B) and let K be its Bergman kernel. The following proposition provides the expression of the diastasis function for (Ω, g_B) and proves a very useful property of the matrix of coefficients (b_{jk}) given by (2.6).

Proposition 4.3.2. *Let Ω be a bounded symmetric domain. Then the diastasis for its Bergman metric g_B around the origin is*

$$\mathrm{D}_0^\Omega(z) = \log(V(\Omega)\mathrm{K}(z, z)), \tag{4.3}$$

where $V(\Omega)$ denotes the total volume of Ω with respect to the Euclidean measure of the ambient complex Euclidean space. Moreover the matrix (b_{jk}) given by (2.6) for D_0^Ω satisfy $b_{jk} = 0$ whenever $|m_j| \neq |m_k|$.

Proof. The Kähler potential $\mathrm{D}_0^\Omega(z)$ is centered at the origin, in fact by the reproducing property of the kernel we have

$$\frac{1}{\mathrm{K}(0,0)} = \int_\Omega \frac{1}{\mathrm{K}(\zeta,0)}\mathrm{K}(\zeta, 0)d\mu,$$

hence $\mathrm{K}(0,0) = 1/V(\Omega)$, and substituting in (4.3) we obtain $\mathrm{D}_0^\Omega(0) = 0$. By the circularity of Ω (i.e. $z \in \Omega,\ \theta \in \mathbb{R}$ imply $e^{i\theta}z \in \Omega$), rotations around the origin are automorphisms and hence isometries, that leave D_0^Ω invariant. Thus we have $\mathrm{D}_0^\Omega(z) = \mathrm{D}_0^\Omega(e^{i\theta}z)$ for any $0 \leq \theta \leq 2\pi$, that is, each time we have a monomial $z^{m_j}\bar{z}^{m_k}$ in $\mathrm{D}_0^\Omega(z)$, we must have

$$z^{m_j}\bar{z}^{m_k} = e^{i|m_j|\theta}z^{m_j}e^{-i|m_k|\theta}\bar{z}^{m_k} = z^{m_j}\bar{z}^{m_k}e^{(|m_j|-|m_k|)i\theta},$$

implying $|m_j| = |m_k|$. This means that every monomial in the expansion of $\mathrm{D}_0^\Omega(z)$ has holomorphic and antiholomorphic part with the same degree. Hence, by Theorem 2.1.9, $\mathrm{D}_0^\Omega(z)$ is the diastasis for g_B. By the chain rule the same property holds true for $e^{\mathrm{D}_0^\Omega(z)} - 1$ and the second part of the proposition follows immediately. □

Proof of Theorem 4.1.2. Let $f : (\Omega, cg_B) \to \mathbb{C}\mathrm{P}^\infty$ be a Kähler immersion, we want to show that $c\gamma$ belongs to $W(\Omega)$, i.e. K^c is positive definite. Since Ω is contractible it is not hard to see that there exists a sequence $f_j, j = 0, 1 \ldots$ of holomorphic functions defined on Ω, not vanishing simultaneously, such that the immersion f is given by $f(z) = [\ldots, f_j(z), \ldots]$, $j = 0, 1 \ldots$, where $[\ldots, f_j(z), \ldots]$ denotes the equivalence class in $\ell^2(\mathbb{C})$ (two sequences are equivalent iff they differ by the multiplication by a nonzero complex number). Let $x_1, \ldots, x_n \in \Omega$. Without loss of generality (up to unitary transformation of $\mathbb{C}P^\infty$) we can assume that $f(0) = e_1$, where e_1 is the first vector of the canonical basis of $\ell^2(\mathbb{C})$, and $f(x_j) \notin H_0$, $\forall\, j = 1, \ldots, n$. Therefore, by Theorem 2.1.6 and Proposition 4.3.2, we have

$$c\,\mathrm{D}_0^\Omega(z) = \log[V(\Omega)^c\,\mathrm{K}^c(z,z)] = \log\left(1 + \sum_{j=1}^\infty \frac{|f_j(z)|^2}{|f_0(z)|^2}\right), \quad z \in \Omega \setminus f^{-1}(H_0).$$

$$V(\Omega)^c\,\mathrm{K}^c(x_\alpha, x_\beta) = 1 + \sum_{j=1}^\infty g_j(x_\alpha)\overline{g_j(x_\beta)},\ g_j = \frac{f_j}{f_0}.$$

Thus for every $(v_1, \ldots v_n) \in \mathbb{C}^n$ one has

$$\sum_{\alpha,\beta=1}^n v_\alpha \mathrm{K}^c(x_\alpha, x_\beta)\bar{v}_\beta = \frac{1}{V(\Omega)^c}\sum_{k=0}^\infty |v_1 g_k(x_1) + \cdots + v_n g_k(x_n)|^2 \geq 0, g_0 = 1,$$

and hence the matrix $(\mathrm{K}^c(x_\alpha, x_\beta))$ is positive semidefinite.

Conversely, assume that $c\gamma \in W(\Omega)$. Then, by the very definition of Wallach set, there exists a Hilbert space $\mathcal{H}_{c\gamma}$ whose reproducing kernel is $\mathrm{K}^c = \sum_{j=0}^\infty |f_j|^2$, where f_j is an orthonormal basis of $\mathcal{H}_{c\gamma}$. Then the holomorphic map $f : \Omega \to \ell^2(\mathbb{C}) \subset \mathbb{C}\mathrm{P}^\infty$ constructed by using this orthonormal basis satisfies $f^*(g_{FS}) = cg_B$. In order to prove that this map is equivariant write $\Omega = G/K$ where G is the simple Lie group acting holomorphically and isometrically on Ω and K is its isotropy group. For each $h \in G$ the map $f \circ h : (\Omega, cg_B) \to \mathbb{C}\mathrm{P}^\infty$ is a full Kähler immersion and therefore by Calabi's rigidity (Theorem 2.2.5) there exists a unitary transformation U_h of $\mathbb{C}\mathrm{P}^\infty$ such that $f \circ h = U_h \circ f$ and we are done. □

Remark 4.3.3. In [2] it is proven that if η belongs to $W(\Omega) \setminus \{0\}$ then G admits a representation in the Hilbert space $\mathcal{H}_\eta$. This is in accordance with our result. Indeed if $c\gamma$ belongs to $W(\Omega) \setminus \{0\}$ then the correspondence $h \mapsto U_h, h \in G$ defined in the last part of the proof of Theorem 4.1.2 is a representation of G.

Observe that by Corollary 4.2.2 it follows that a bounded symmetric domain (Ω, cg_B) admits a Kähler immersion into $\ell^2(\mathbb{C})$ if and only if $\Omega = \mathbb{C}\mathrm{H}^{n_1} \times \cdots \times \mathbb{C}\mathrm{H}^{n_k}$ and $cg_B = c_1 g_{hyp} \oplus \cdots \oplus c_k g_{hyp}$, for some suitable positive constant $c_1, \ldots, c_k$. In fact, by Theorem 4.1.2, the hyperbolic space is the only bounded symmetric domain admitting a Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ for any value of the constant multiplying the metric. This can be an alternative proof of Theorem 3.3.8 (see also [19]).
Furthermore, if (Ω, cg_B) admits a Kähler immersion into $\mathbb{C}\mathrm{H}^N$ (resp. $\mathbb{C}\mathrm{H}^\infty$) then $(\Omega, cg_B) = (\mathbb{C}\mathrm{H}^n, g_{hyp})$ and $c = 1$ (resp. $c \leq 1$). In fact, a Kähler manifold which admits a Kähler immersion into a complex hyperbolic space is locally irreducible (cfr. [1, Th. 17]), and by Theorem 3.3.3 any bounded symmetric domain different from the hyperbolic space cannot be Kähler immersed into $\mathbb{C}\mathrm{H}^\infty$. The values of c follow from Calabi's Theorem 3.3.1.

4.4 Proof of the main result

Let Ω be an irreducible bounded symmetric domain of complex dimension d and genus γ. For all positive real numbers μ consider the family of Cartan-Hartogs domains

$$\mathrm{M}_\Omega(\mu) = \left\{(z, w) \in \Omega \times \mathbb{C},\ |w|^2 < \mathrm{N}_\Omega^\mu(z, z)\right\}, \tag{4.4}$$

where $\mathrm{N}_\Omega(z, z)$ is the *generic norm* of Ω, i.e.

$$\mathrm{N}_\Omega(z, z) = (V(\Omega)\mathrm{K}(z, z))^{-\frac{1}{\gamma}},$$

with $V(\Omega)$ the total volume of Ω with respect to the Euclidean measure of the ambient complex Euclidean space and $\mathrm{K}(z, z)$ is its Bergman kernel.

The domain Ω is called the *base* of the Cartan–Hartogs domain $\mathrm{M}_\Omega(\mu)$ (one also says that $\mathrm{M}_\Omega(\mu)$ is based on Ω). Consider on $\mathrm{M}_\Omega(\mu)$ the metric $g(\mu)$ whose globally defined Kähler potential around the origin is given by

$$\mathrm{D}_0(z, w) = -\log(\mathrm{N}_\Omega^\mu(z, z) - |w|^2). \tag{4.5}$$

The following theorem summarizes what we need about these domains (see [41] and [55] for a proof).

Theorem 4.4.1 (G. Roos, A. Wang, W. Yin, L. Zhang, W. Zhang)**.** *Let* $\mu_0 = \gamma/(d+1)$. *Then* $(\mathrm{M}_\Omega(\mu_0), g(\mu_0))$ *is a complete Kähler–Einstein manifold which is homogeneous if and only if the rank of* Ω *equals* 1*, i.e.* $\Omega = \mathbb{C}\mathrm{H}^d$.

Remark 4.4.2. Observe that when $\Omega = \mathbb{C}\mathrm{H}^d$, we have $\mu_0 = 1$, $\mathrm{M}_\Omega(1) = \mathbb{C}\mathrm{H}^{d+1}$ and $g(1) = g_{hyp}$.

In the following proposition, interesting on its own sake, we describe the Kähler immersions of a Cartan–Hartogs domain into $\mathbb{C}\mathrm{P}^\infty$ in terms of its base.

Proposition 4.4.3. *The potential* $\mathrm{D}_0(z, w)$ *given by (4.5) is the diastasis around the origin of the metric* $g(\mu)$*. Moreover,* $cg(\mu)$ *is projectively induced if and only if* $(c+m)\frac{\mu}{\gamma} g_B$ *is projectively induced for every integer* $m \geq 0$.

Proof. The power expansion around the origin of $\mathrm{D}_0(z, w)$ can be written as

$$\mathrm{D}_0(z, w) = \sum_{j,k=0}^{\infty} A_{jk} (zw)^{m_j} (\bar{z}\bar{w})^{m_k} \tag{4.6}$$

where m_j are ordered $(d+1)$-uples of integer and

$$(zw)^{m_j} = z_1^{m_{j,1}} \cdots z_d^{m_{j,d}} w^{m_{j,d+1}}.$$

In order to prove that $\mathrm{D}_0(z, w)$ is the diastasis for $g(\mu)$ we need to verify that $A_{j0} = A_{0j} = 0$ (see Theorem 2.1.9). This is straightforward. Indeed if we take derivatives with respect either to z or $\bar{z}$ is the same as deriving the function $-\log(\mathrm{N}_\Omega^\mu(z, z)) = \frac{\mu}{\gamma}\mathrm{D}_0^\Omega(z)$ that is the diastasis of $(\Omega, \frac{\mu}{\gamma} g_B)$, thus we obtain 0. If we take derivatives with respect either to w or $\bar{w}$ we obtain zero no matter how many times we derive with respect to z or $\bar{z}$, since $\mathrm{D}_0(z, w)$ is radial in w.

In order to prove the second part of the proposition take the function

$$e^{c\mathrm{D}_0(z,w)} - 1 = \frac{1}{(\mathrm{N}_\Omega^\mu(z, z) - |w|^2)^c} - 1, \tag{4.7}$$

and using the same notations as in (4.6) write the power expansion around the origin as

$$e^{c\mathrm{D}_0(z,w)} - 1 = \sum_{j,k=0}^{\infty} B_{jk}(zw)^{m_j}(\bar{z}\bar{w})^{m_k}.$$

By Calabi's criterion (Theorem (2.2.4)), $cg(\mu)$ is projectively induced if and only if $B = (B_{jk})$ is positive semidefinite of infinite rank. The generic entry of B is given by

$$B_{jk} = \frac{1}{m_j! \cdot m_k!} \frac{\partial^{|m_j|+|m_k|}}{\partial(zw)^{m_j}\partial(\bar{z}\bar{w})^{m_k}} \left(\frac{1}{(\mathrm{N}^{\mu}_{\Omega}(z,z) - |w|^2)^c} - 1 \right)\Bigg|_0,$$

where $m_j! = m_{j,1}! \cdots m_{j,d+1}!$ and $\partial(zw)^{m_j} = \partial z_1^{m_{j,1}} \cdots \partial z_d^{m_{j,d}} \partial w^{m_{j,d+1}}$. By Proposition (4.3.2) we have

$$m_{j,1} + \cdots + m_{j,d} \neq m_{k,1} + \cdots + m_{k,d} \Rightarrow B_{jk} = 0, \tag{4.8}$$

and since (4.7) is radial in w we also have

$$m_{j,d+1} \neq m_{k,d+1} \Rightarrow B_{jk} = 0. \tag{4.9}$$

Thus, B is a $\infty \times \infty$ matrix of the form

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & E_2 & 0 & 0 & \dots \\ 0 & \vdots & 0 & E_3 & 0 & \dots \\ 0 & & \vdots & 0 & \ddots & \end{pmatrix},$$

where the generic block E_i contains derivatives $\partial(zw)^{m_j}\partial(\bar{z}\bar{w})^{m_k}$ of order $2i$, $i = 1, 2, \dots$ such that $|m_j| = |m_k| = i$. We can further write

$$E_i = \begin{pmatrix} F_{z(i)}(0) & 0 & 0 \\ 0 & F_{w(i)}(0) & 0 \\ 0 & 0 & F_{(z,w)(i)}(0) \end{pmatrix}, \tag{4.10}$$

where $F_{z(i)}(0)$ (resp. $F_{w(i)}(0)$, $F_{(z,w)(i)}(0)$) contains derivatives $\partial(zw)^{m_j}$ $\partial(\bar{z}\bar{w})^{m_k}$ (of order $2i$ with $|m_j| = |m_k| = i$) such that $m_{j,d+1} = m_{k,d+1} = 0$ (resp. $m_{j,d+1} = m_{k,d+1} = i$, $m_{j,d+1}, m_{k,d+1} \neq 0, i$). (Notice also that we have 0 in all the other entries because of

(4.8) and (4.9)). Since the derivatives are evaluated at the origin, deriving (4.7) with respect to $\partial(zw)^{m_j}\ \partial(\bar z\bar w)^{m_k}$ with $|m_j| = |m_k| = i$ and $m_{j,d+1} = m_{k,d+1} = 0$ is the same as deriving the function

$$\frac{1}{(\mathrm{N}^{\mu}_{\Omega}(z,z))^{c}} - 1 = e^{c\frac{\mu}{\gamma}\mathrm{D}^{\Omega}_{0}(z)} - 1. \tag{4.11}$$

Thus, by Calabi's criterion, all the blocks $F_{z(i)}(0)$ are positive semidefinite if and only if $c\frac{\mu}{\gamma}g_B$ is projectively induced. Observe that the blocks $F_{w(i)}(0)$ is semipositive definite without extras assumptions. Indeed if we consider derivatives $\partial(zw)^{m_j}\partial(\bar z\bar w)^{m_k}$ of (4.7) with $|m_j| = |m_k| = i$ and $m_{j,d+1} = m_{k,d+1} = i$, since $\mathrm{N}^{\mu}_{\Omega}(z,z)$ evaluated in 0 is equal to 1, it is the same as deriving the function $1/(1-|w|^2)^c - 1 = \left(\sum_{j=0}^{\infty}|w|^{2j}\right)^{c} - 1$ and the claim follows. Finally, consider the block $F_{(z,w)(i)}(0)$. It can be written as

$$F_{(z,w)(i)}(0) = \begin{pmatrix} H_{z(i-1),w(1)}(0) & 0 & 0 & 0 \\ 0 & H_{z(i-2),w(2)}(0) & 0 & 0 \\ \vdots & & \ddots & \\ 0 & 0 & 0 & H_{z(1),w(i-1)}(0) \end{pmatrix}$$

where the generic block $H_{z(i-m),w(m)}(0)$, $1 \le m \le i-1$, contains derivatives $\partial(zw)^{m_j}$ $\partial(\bar z\bar w)^{m_k}$ of order $2i$ such that $|m_j| = |m_k| = i$ and $m_{j,d+1} = m_{k,d+1} = m$ evaluated at zero (as before, by (4.8) and (4.9) all entries outside these blocks are 0). Now it is not hard to verify that these blocks can be obtained by taking derivatives $\partial(zw)^{m_j}\partial(\bar z\bar w)^{m_k}$ of order $2(i-m)$ such that $|m_j| = |m_k| = 2(i-m)$ and $m_{j,d+1} = m_{k,d+1} = 0$ of the function

$$\frac{(m+c-1)!}{(c-1)!\ m!\ \mathrm{N}^{\mu(c+m)}_{\Omega}(z,z)} - 1 = e^{(c+m)\frac{\mu}{\gamma}\mathrm{D}^{\Omega}_{0}(z)} - 1, \tag{4.12}$$

and evaluating at $z = \bar z = 0$. Thus, again by Calabi's criterion, $F_{(z,w)(i)}(0)$ is positive semidefinite iff $(c+m)\frac{\mu}{\gamma}g_B$, $m \ge 1$, is projectively induced and this ends the proof of the proposition. $\square$

Remark 4.4.4. Proposition 4.4.3 can be also proved for "general" Cartan-Hartogs domains with dimension $n = d + r$, namely

$$\mathrm{M}_{\Omega}(\mu) = \left\{(z,w) \in \Omega \times \mathbb{C}^r,\ ||w||^2 < \mathrm{N}^{\mu}_{\Omega}(z,z)\right\},$$

where $||w||^2 = |w_1|^2 + \cdots + |w_r|^2$. In that case Equation (4.12) can be obtained using the following formula

$$\frac{1}{m_1!^2 \cdots m_r!^2} \frac{\partial^{2m}}{\partial w_1^{m_1} \partial \bar{w}_1^{m_1} \cdots \partial w_r^{m_r} \partial \bar{w}_r^{m_r}} \left(\frac{1}{f(z,\bar{z}) - ||w||^2} \right)^c =$$
$$= \frac{1}{m_1!^2 \cdots m_r!^2} \sum_{k_1=1}^{m_1+1} \cdots \sum_{k_r=1}^{m_r+1} \left[\frac{(\sum_{j=1}^r (k_j) + m + c - r - 1)!}{(c-1)!} \cdot \right.$$
$$\left. \cdot \prod_{i=1}^r \left[\binom{m_i}{k_i - 1}^2 (m_i + 1 - k_i)! (w_i \bar{w}_i)^{k_i - 1} \right] \frac{1}{(f(z,\bar{z}) - ||w||^2)^{\sum_{j=1}^r (k_j) + m + c - r}} \right].$$

We are now in the position to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Take $\mu = \mu_0 = \gamma/(d+1)$ in (4.4) and $\Omega \neq \mathbb{C}\mathrm{H}^d$. By Theorem 4.4.1 $(\mathrm{M}_\Omega(\mu_0), cg(\mu_0))$ is Kähler-Einstein, complete and nonhomogeneous for all positive real number c. By Proposition 4.4.3 $cg(\mu_0)$ is projectively induced if and only if $\frac{c+m}{d+1} g_B$ is projectively induced, for all nonnegative integer m. By Theorem 4.1.2 this happens if $\frac{(c+m)}{d+1} \geq \frac{(r-1)a}{2\gamma}$. Hence $cg(\mu_0)$ with $c \geq \frac{(r-1)(d+1)a}{2\gamma}$ is the desired family of projectively induced Kähler-Einstein metrics. $\square$

Remark 4.4.5. Observe that the scalar curvature of Cartan–Hartogs domains is negative. It still is an open question whether there exists or not a nonhomogeneous Ricci–flat projectively induced manifold. Notice also that the Taub-NUT metric described in Example 1.3.6 is not projectively induced. More precisely, we can prove that for $m > 1/2$ it does not exist a Kähler immersion of $(\mathbb{C}^2, \omega_{TN})$ into $\mathbb{C}\mathrm{P}^\infty$. In fact, it is enough to show that the holomorphic submanifold defined by $z_2 = 0$, $z_1 = z$, endowed with the induced metric (still denoted ω_{TN}) having potential $\phi = u^2 + mu^4$, with u defined implicitly by $z\bar{z} = e^{2mu^2} u^2$, does not admit a Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$ for $m > 1/2$. Let us denote $u^2 = w$ and let $\tilde{w}$ (resp. $\tilde{\phi}$) be the real function associated to w (resp. to ϕ). In order to use Calabi's criterion, we need to calculate the coefficients of the power series development of $G_+ = e^\phi - 1 = e^{w + mw^2} - 1$. By direct computation we get

$$\frac{d^k \tilde{w}}{dx^k}(0) = (-2km)^{k-1},$$

and in particular we obtain

$$\frac{d^2\tilde{G}_+}{dx^2}(0) = 1 - 2m,$$

for $\tilde{G}_+ = e^{\tilde{\phi}} - 1$. Thus, for $m > 1/2$ this term is negative and by Calabi's criterion $(\{z_2 = 0\}, \omega_{TN})$ does not admit a Kähler immersion into $\mathbb{C}\mathrm{P}^\infty$. It is still an open question if the same holds for all $m > 0$. Although, one notices that going on with computation gets

$$\frac{d^3\tilde{G}_+}{dx^3}(0) = 1 - 6m + 12m^2,$$

$$\frac{d^4\tilde{G}_+}{dx^4}(0) = 1 - 12m + 60m^2 - 128m^3,$$

$$\frac{d^5\tilde{G}_+}{dx^5}(0) = 1 - 20m + 180m^2 - 880m^3 + 2000m^4,$$

and so on. Thus it seems to be true that for any $m > 0$ there exists $k \in \mathbb{N}$ such that $\frac{d^k\tilde{G}_+}{dx^k}(0)$ is negative. Observe also that, by Calabi's criterion follows easily that for any $m \neq 0$ it does not exist a Kähler immersion of $(\mathbb{C}^2, \omega_{TN})$ into $\ell^2(\mathbb{C})$ or $\mathbb{C}\mathrm{H}^\infty$.

By applying the same argument with $0 < c < \frac{a(d+1)}{2\gamma}$ (and $r \neq 1$) one also gets the following:

Corollary 4.4.6. *There exists a continuous family of nonhomogeneous, complete, Kähler-Einstein metrics which does not admit a local Kähler immersion into $\mathbb{C}\mathrm{P}^N$ for any $N \leq \infty$.*

Remark 4.4.7. As direct consequence of Corollary 4.4.6 together with Corollary 4.2.2, we get that a Cartan-Hartogs domain $(\mathrm{M}_\Omega(\mu_0), cg(\mu_0))$ does not admit a Kähler immersion into $\ell^2(\mathbb{C})$. Further by Theorem 3.3.3, it does not admit a Kähler immersion into $\mathbb{C}\mathrm{H}^\infty$ for any value of $c > 0$ either.

Chapter 5

A characterization of $\mathbb{C}\mathrm{H}^n$

This chapter deals with Kähler immersions of bounded symmetric domains into the indefinite complex Euclidean space. The first section describes what we need about indefinite Kähler metrics and introduces our third and last result, to the proof of which the second section is dedicated.

5.1 Indefinite Kähler metrics

In Section 3.3 we saw that the hyperbolic space $\mathbb{C}\mathrm{H}^n$ is the only bounded symmetric domain admitting a Kähler immersion into $\ell^2(\mathbb{C})$ (cfr. Theorem 3.3.8 or [19]). In this chapter we address the problem of extending this result when the ambient space is the indefinite complex Euclidean space $\mathbb{C}^{r,s} = (\mathbb{C}^{r+s}, g_{r,s})$, r, $s \in \mathbb{N}\cup\{\infty\}$, $\{r, s\} \neq \{\infty, \infty\}$. Here $g_{r,s}$ is the indefinite Kähler metric on $\mathbb{C}^{r+s}$ whose associated (indefinite) Kähler form is given by

$$\omega_{r,s} = \frac{i}{2}\partial\bar{\partial}(\sum_{j=1}^{r} |z_j|^2 - \sum_{k=r+1}^{r+s} |z_k|^2),$$

when $r \in \mathbb{N}$ and $s \in \mathbb{N} \cup \{\infty\}$, and by

$$\omega_{\infty,s} = \frac{i}{2}\partial\bar{\partial}(-\sum_{j=1}^{s} |z_j|^2 + \sum_{k=s+1}^{+\infty} |z_k|^2), \tag{5.1}$$

when $s \in \mathbb{N}$ and $r = \infty$. One calls s the *index* of $g_{r,s}$. Notice that we are excluding the case when both $s = \infty$ and $r = \infty$, since by Theorem 2.2.1 every real analytic Kähler manifold admits a local Kähler immersion into $\mathbb{C}^{\infty,\infty}$. Observe also that

$(\mathbb{C}\mathrm{H}^n, (n+1)g_{hyp})$ can be Kähler immersed into $\mathbb{C}^{\infty,s}$ via the map $i \circ f \colon \mathbb{C}\mathrm{H}^n \to \mathbb{C}^{\infty,s}$, where $i : \ell^2(\mathbb{C}) \to \mathbb{C}^{\infty,s}$ denotes the natural inclusion and f is the map described in Section 3.3 (Eq. (3.3)). It is worth pointing out that one can construct infinitely many noncongruent Kähler immersion of $(\mathbb{C}\mathrm{H}^n, (n+1)g_{hyp})$ into $\mathbb{C}^{\infty,s}$. For example for any holomorphic function ψ on $\mathbb{C}\mathrm{H}^n$ the map $z \mapsto (\psi(z), \psi(z), f(z))$ is a Kähler immersion of $\mathbb{C}\mathrm{H}^n$ into $\mathbb{C}^{\infty,1}$.

Behind the pure mathematical interest, indefinite Kähler geometry can be viewed in the case when $s = 1$, as a combination of the Lorentzian geometry of space-time and the symplectic geometry of phase space. Among the authors that have been studying the geometry of Kähler submanifolds of *finite* indefinite space forms we cite M. Barros, A. Romero, Y. J. Suh and M. Umehara (see [5], [40], [51]).

In the next section we show that $(\mathbb{C}\mathrm{H}^n, g_{hyp})$ can be characterized among irreducible bounded symmetric domains as the only one which admits a Kähler immersion into $\mathbb{C}^{\infty,s}$, $s < \infty$, more precisely we have (cfr. [32]):

Theorem 5.1.1 (A. Loi, M. Zedda)**.** *Let (Ω, g_B) be a Cartan domain. Assume that there exists a local Kähler immersion (Ω, g_B) into $\mathbb{C}^{r,s}$, then $r = \infty$, $s \in \mathbb{N}$ and $(\Omega, g_B) = (\mathbb{C}\mathrm{H}^n, (n+1)g_{hyp})$.*

5.2 Proof of the main result

The proof of Theorem 5.1.1 is based on the following lemma.

Lemma 5.2.1. *Let (M, g) be a Kähler manifold and let $A = (a_{jk})$ be the $\infty \times \infty$ Hermitian matrix given by equation (2.5) for the diastasis function D_0^M of g on a neighborhood U of a point $p \in M$. If $(U, g_{|U})$ admits a Kähler immersion into $\mathbb{C}^{\infty,s}$ (resp. $\mathbb{C}^{r,\infty}$) then the number of negative eigenvalues of A is less or equal than s (resp. r).*

Proof. Let $\{z_1, \dots, z_n\}$ be complex coordinates centered at the origin of U. In order to prove the lemma we introduce the following notations. Denote by $\{v_i\}_{i \in \mathbb{N}}$ a sequence of complex numbers and consider the complex vector space

$$V = \left\{ \{v_i\}_{i\in\mathbb{N}} \mid \sum_{i=0}^{+\infty} v_i z^{m_i} < \infty \right\}.$$

Every holomorphic function $\psi = \sum_{i=0}^{+\infty} a_i z^{m_i}$ induces a linear functional $\widehat{\psi} \in V^* = \mathrm{Hom}(V, \mathbb{C})$ by

$$\widehat{\psi}(v) = \sum_{i=0}^{+\infty} a_i v_i, \quad v = \{v_i\}_{i\in\mathbb{N}} \in V.$$

Define a sesquilinear form on V by

$$\widehat{\mathrm{D}}_0^M \colon V \times V \to \mathbb{R}, \quad \widehat{\mathrm{D}}_0^M(u, v) = uAv^*. \tag{5.2}$$

Assume now that $f\colon U \to \mathbb{C}^{\infty,s}$ is a Kähler immersion of a neighbourhood of a point $p \in M$ into $\mathbb{C}^{\infty,s}$. We can assume that $f(p) = 0$ (the case $\mathbb{C}^{r,\infty}$ is treated similarly). In local coordinates f is given by

$$f(z) = (f_1(z), \dots, f_s(z), f_{s+1}(z), \dots) \in \mathbb{C}^{\infty,s},$$

for suitable holomorphic functions f_j. Since f is a Kähler immersion it follows by (5.1) and by the very definition of the diastasis function that

$$\mathrm{D}_0^M(z) = -|f_1(z)|^2 - \dots - |f_s(z)|^2 + \sum_{k=s+1}^{+\infty} |f_k(z)|^2.$$

Hence in our notation

$$\widehat{\mathrm{D}}_0^M(u, v) = -\widehat{f}_1(u)\overline{\widehat{f}_1}(v) - \dots - \widehat{f}_s(u)\overline{\widehat{f}_s}(v) + \sum_{k=s+1}^{+\infty} \widehat{f}_k(u)\overline{\widehat{f}_k}(v). \tag{5.3}$$

Let $W \subset V$ be the complex subspace of V consisting of those $w \in V$ such that $\widehat{\mathrm{D}}_0^M(w, w) < 0$. By (5.2) each eigenvector of a negative eigenvalue of the matrix A belongs to W. Hence, in order to prove the lemma we need to show that $\dim W \leq s$. Assume, by a contradiction, that $\dim W > s$ and let $\widehat{\xi}_1, \dots, \widehat{\xi}_m$, $m \leq s$ be a basis for the subspace of V^* spanned by the linear functionals $\widehat{f}_1, \dots, \widehat{f}_s$. If $m = 0$ then $f(U) \subset \ell^2(\mathbb{C})$ and by Calabi's criterion the matrix A does not have negative eigenvalues. On the other hand, if $m \geq 1$, the $\mathbb{C}$-linear map $L\colon W \to \mathbb{C}^m$ defined by

$$L(w) = (\widehat{\xi}_1(w), \dots, \widehat{\xi}_m(w))$$

is surjective. Thus there exists $0 \neq w_0 \in W$ such that $\widehat{\xi}_1(w_0) = \cdots = \widehat{\xi}_m(w_0) = 0$ and hence $\widehat{f}_1(w_0) = \cdots = \widehat{f}_s(w_0) = 0$. By (5.3) $\widehat{\mathrm{D}}_0^M(w_0, w_0) \geq 0$ which contradicts the fact that $w_0 \in W$. $\square$

Recall that by Proposition 4.3.2 the diastasis function around the origin for the Bergman metric g_B of a Cartan domain Ω is globally defined and given by

$$\mathrm{D}_0^\Omega(z) = \log(V(\Omega)\mathrm{K}_\Omega(z, z)). \tag{5.4}$$

and furthermore, if (a_{jk}) is the matrix in (2.5) for D_0^Ω, we have $a_{jk} = 0$ whenever $|m_j| \neq |m_k|$.

In order to simplify the proof of our theorem we introduce the following definition. We say that a square submatrix C of a square matrix M is *central* if its diagonal lies on the diagonal of M. Furthermore we say that M is a *block* matrix if it is of the form

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & M_1 & 0 & 0 & \dots \\ \vdots & 0 & M_2 & 0 & \dots \\ & \vdots & 0 & \ddots & \end{pmatrix},$$

where each block M_i is a central submatrix of M.

Notice that Proposition 4.3.2 says that the matrix (a_{jk}) given by (2.5) for the diastasis D_0^Ω of a Cartan domain Ω, is a block matrix, where each block M_i contains the elements a_{jk} with $|m_j| = |m_k| = i$.

We can now prove the main theorem:

Proof of Theorem 5.1.1. Let (Ω, g_B) be a Cartan domain. Then it is easily seen that $(\mathbb{C}\mathrm{H}^1, \gamma\, g_{hyp})$ admits a Kähler immersion in (Ω, g_B) where γ is the genus of Ω. By equation (1.5) the matrix (a_{jk}) for γg_{hyp} is the diagonal matrix given by $a_{jk} = \delta_{jk}/j$, thus it has infinite positive eigenvalues (given by $\gamma/j,\ j = 1, 2, \dots$). By Lemma 5.2.1 it follows that $\mathbb{C}\mathrm{H}^1$, and hence (Ω, g_B), can not be Kähler immersed into $\mathbb{C}^{r,s}$ with $r \in \mathbb{N}$, $s \leq \infty$.

Thus it remains to prove that a Cartan domain of rank greater than 1 can not admit a Kähler immersion into $\mathbb{C}^{\infty,s}$ for $s \in \mathbb{N}$.

Assume, by contradiction, that there exists $f : \Omega \to \mathbb{C}^{\infty,s}$. Without loss of generality we can assume $f(0) = 0$. We are going to prove that the matrix (a_{jk}) in (2.5) for D_0^{Ω} has infinite negative eigenvalues. By Lemma 5.2.1 this will be the desired contradiction. Any irreducible bounded symmetric domain of rank at least two can be exhausted by totally geodesic submanifolds isomorphic to $\Omega_4[3]$ (cfr. Remark 1.3.4), hence we need only to prove the assertion for the case $\Omega = \Omega_4[3]$.

By Proposition 4.3.2 and equation (1.7) the diastasis of $(\Omega_4[3], g_B)$ is given by

$$\mathrm{D}_0^{\Omega_4}(z) = -3\log(1 - 2(|z_1|^2 + |z_2|^2 + |z_3|^2) + (z_1^2 + z_2^2 + z_3^2)(\bar{z}_1^2 + \bar{z}_2^2 + \bar{z}_3^2)).$$

We will show that every block (except the first one) has at least one negative eigenvalue. Consider the 3×3 matrix

$$B = \frac{12(2|z_3|^2 + 1)}{(1 - 2|z_3|^2 + |z_3|^4)^2} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Let B_0 be the matrix obtained by evaluating B at $z_3 = \bar{z}_3 = 0$. Then B_0 is the submatrix of (a_{jk}) with j, k corresponding to the triples $(2, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$. Further let B_n be the submatrix of (a_{jk}) with j, k corresponding to the triples $(2, 0, n)$, $(0, 2, n)$, $(0, 0, n + 2)$. The matrix B_n can be obtained from B by deriving each of its entries n times with respect to z_3, n times with respect to $\bar{z}_3$ and evaluating at $z_3 = \bar{z}_3 = 0$. From

$$\frac{\partial^{2n}}{\partial z_3^n \partial \bar{z}_3^n} \frac{12(2|z_3|^2 + 1)}{(1 - 2|z_3|^2 + |z_3|^4)^2}\Big|_{z_3=\bar{z}_3=0} > 0,$$

we have $\det(B_n) < 0$ for all $n \in \mathbb{N}$. Thus every B_n must have a negative eigenvalue. This implies that the $(n+2)$th block of (a_{jk}) which contains B_n as a central submatrix, has at least one negative eigenvalue and hence (a_{jk}) has infinite negative eigenvalues. $\square$

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