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The geometry of rotation invariant Kähler metrics

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ABSTRACT

The present thesis consists of three results related to the geometry of rotation invariant Kähler metrics. In the first one, we prove that a 3-codimensional Kähler-Einstein submanifold of the complex projective space with rotation invariant metric is forced to be the product of complex projective spaces. In the second one, we prove that the only stable-projectively induced Ricci-flat Kähler metrics are flat. Finally, we prove as third result that given a Ricci-flat radial Kähler metric defined on a complex surface such that the third coefficient of its Tian-Yau-Zelditch expansion vanishes, then it is flat.

Declaration

I declare that to the best of my knowledge the contents of this thesis are original and my work except where indicated otherwise.

To my parents and my sister

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CONTENTS

INTRODUCTION

An interesting open question in complex geometry is concerned with the characterization of Kähler-Einstein metrics Kähler immersed (namely holomorphically and isometrically immersed) in a finite or infinite dimensional complex projective space endowed with the Fubini-Study metric. In particular, the only known examples of complete Kähler-Einstein manifolds Kähler immersed in $(\mathbb{C}\mathrm{P}^n, g_{FS})$, with $n < \infty$, are compact, simply connected and homogeneous with positive scalar curvature, as shown by the statement of the following theorem (see also [36] and [67]).

Theorem 1 (S. S. Chern [23], K. Tsukada [73]). *Let (M, g) be a complete n-dimensional Kähler–Einstein manifold ($n \geqslant 2$). If (M, g) admits a Kähler immersion in $\mathbb{C}\mathrm{P}^{n+2}$, then M is either totally geodesic or the complex quadric Q_n in $\mathbb{C}\mathrm{P}^{n+1}$.*

Consequently, we have the following conjecture (see also [1] and [3]).

Conjecture 1. *A complete Kähler-Einstein manifold Kähler immersed in $(\mathbb{C}\mathrm{P}^n, g_{FS})$ is homogeneous, compact, simply connected and it has positive scalar curvature.*

Observe that A. Loi and M. Zedda provide in [52] explicit examples of non-homogeneous Kähler-Einstein manifolds which admit a Kähler immersion in $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$. Therefore the finiteness hypothesis of the target complex projective space's dimension cannot be dropped.

Our conjecture is strengthened by the following results.

Theorem 2 (M. Takeuchi [68], A. Di Scala, H. Hishi, A. Loi [25])**.** *Every homogeneous manifold Kähler immersed in* $(\mathbb{C}\mathrm{P}^n, g_{FS})$ *is compact and every homogeneous manifold Kähler immersed in* $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$ *is simply connected.*

Theorem 3 (D. Hulin [38])**.** *If a compact Kähler-Einstein manifold is Kähler immersed in* $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$*, then its Einstein constant is positive.*

Moreover, one can also prove (see e.g. [25]) that, given a compact simply connected homogeneous Kähler (Einstein) manifold with integral Kähler form, then the Kodaira map suitably normalized is a Kähler immersion.

For the particular case of Ricci-flat Kähler metrics, we also believe the validity of the following.

Conjecture 2. *If a Ricci-flat Kähler metric can be Kähler immersed in* $(\mathbb{C}\mathrm{P}^N, g_{FS})$ *then it is flat and* $N = \infty$*.*

At the same time, we propose to study Ricci-flat Kähler metrics from a different point of view, namely that of Tian-Yau-Zelditch coefficients. The requirement that a Kähler metric is projectively induced is a somehow strong assumption. Thus one could try to approximate a Kähler metric g on a complex manifold M with projectively induced ones. Roughly speaking, under suitable conditions, we can construct for every sufficiently large positive integer m a holomorphic map $\varphi_m : M \to \mathbb{C}\mathrm{P}^{N_m}$ into an N_m-dimensional ($N_m \leqslant \infty$) complex projective space such that

$$\lim_{m\to\infty} \frac{1}{m}\varphi_m^* g_{FS} = g.$$

This approximation can be used to construct the so-called Tian-Yau-Zelditch coefficients, which are smooth functions globally defined on M that actually depend only on the metric g. Due to Donaldson's work (cf. [27], [28] and [2]) in the compact case and respectively to the theory of quantization in the noncompact case (see e.g. [8], [14] and [15]), it is natural to study metrics with the Tian-Yau-Zelditch coefficients being prescribed. For instance, the vanishing of this coefficients for large enough indexes turns out to be related to some important problems in the theory of psedoconvex manifolds (cf. [57], [33], [4]). In the last two decades a lot of work has been done both in the noncompact and compact cases. In the noncompact case, one can find, for instance, in [55] a characterization of the flat metric as a Taub-NUT metric with $a_3 = 0$, while Z. Feng and Z. Tu [35] solve a conjecture formulated in [80] by showing that the complex hyperbolic space is the only Cartan-Hartogs domain where the coefficient a_2 is constant. In [53] A. Loi and M. Zedda prove that a locally hermitian symmetric space with vanishing a_1 and a_2 is flat.

In this direction, we address the last conjecture.

Conjecture 3. *A Ricci-flat metric on an n-dimensional complex manifold such that $a_{n+1} = 0$ is flat.*

In the present thesis we are going to study the previous conjectures in the case of Kähler metrics that admit a local potential with rotational symmetries. This choice follows a kind of approach to the problem that has been often used in literature when, as in our case, the faced problems can be described in terms of PDEs too complicated to be solved directly. Hence, it is necessary to proceed with a restriction of the set of the solutions in order to be able to use analytical arguments. The restriction to rotationally symmetric solutions is the most common, since allows frequently to describe issues coming from Riemannian geometry through ODEs. The results achieved in this way have been in several cases particularly interesting as shown for instance by [19], [29] and [46].

The thesis is divided into four chapters organized as follows. The first chapter recalls preliminary notions and known results about Kähler manifolds, while the other three chapters contain our results (cf. [51] and [65]). Precisely, we are going to study Conjecture 1 in the second chapter, where we prove the following two theorems.

Theorem 4 (cf. Theor. 2.1)**.** *Let λ be the Einstein constant of a Kähler-Einstein rotation invariant n-dimensional manifold M (locally) Kähler immersed in a finite dimensional complex projective space. Then λ is a positive rational number less than or equal to $2(n+1)$. Hence, if M is complete, then it is compact and simply connected.*

Theorem 5 (cf. Theor. 2.2)**.** *Let (M, g) be an n-dimensional Kähler-Einstein manifold whose metric is rotation invariant with respect to Bochner's coordinates. Then (M, g) admits a (local) Kähler immersion in $\mathbb{C}\mathrm{P}^{n+k}$ for k less than or equal to 3, if and only if (M, g) is an open subset of $(\mathbb{C}\mathrm{P}^n, g_{FS})$, $(\mathbb{C}\mathrm{P}^2, 2g_{FS})$ or $(\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1, g_{FS} \oplus g_{FS})$.*

Notice that by D. Hulin [39] if a n-dimensional Kähler-Einstein manifold can be Kähler immersed in $\mathbb{C}\mathrm{P}^N$ then it can always be extended to a n-dimensional complete Kähler-Einstein manifold Kähler immersed in the same projective space and by Calabi [18] a local Kähler immersion of a simply connected manifold into a complex projective space can be extended to the whole manifold.

The Conjecture 2 is the topic of the third chapter. There, we prove the subsequent theorem.

Theorem 6 (cf. Theor. 3.1)**.** *The only Ricci-flat, stable-projectively induced and radial Kähler metric is the flat one.*

Finally, the following theorem summarizes our results concerning Conjecture 3, which is going to be studied in the fourth chapter.

Theorem 7 (cf. Theor. 4.2 and Theor. 4.5)**.** *Let us assume that one of the following two conditions holds true for a Ricci-flat Kähler metric defined on a complex surface such that the third Tian-Yau-Zelditch coefficient vanishes:*

1. *complete and ALE;*
2. *radial.*

Then the metric is flat.

CHAPTER 1

PRELIMINARIES

The present chapter's objective is just to recall some definitions and basic properties dealing with Kähler manifolds that are going to be used later. For this reason we suggest by now [61] as reference for a more detailed introduction to Kähler geometry, [9] and [69] as further references for Kähler-Einstein metrics.

The first section is purely introductory and it focuses mainly on the definition of Kähler manifold. The second section illustrates some techniques developed by E. Calabi for studying immersions in complex projective spaces, that will be used in some proofs of our results. Instead, the last two chapters deal with the problem of searching a canonical metric in a given cohomology class of Kähler metrics (precisely they deal with Kähler-Einstein and constant scalar curvature Kähler metrics respectively), in order to justify our interest in this kind of metrics.

1.1 Kähler metrics

Let M be a complex manifold with complex structure J. A Riemannian metric g on M is called *Hermitian metric* if J is an orthogonal transformation on each tangent space.

Definition 1.1. A Hermitian metric g is a *Kähler metric* if and only if the associated 2-form ω (*Kähler form*) defined as

$$\omega(X, Y) = g(JX, Y),$$

for any tangent vectors X, Y, is closed.

Unlike general Riemannian metrics, Kähler metrics can be locally described by a single real-valued function.

Lemma 1.1 ($\partial\bar{\partial}$-lemma)**.** *If ω is a Kähler form on a complex manifold M, then for any point $p \in M$ there exists an open subset $U \subset M$ containing p and a smooth real-valued function Φ defined on U (called* local Kähler potential*) such that*

$$\omega_{|_U} = \frac{i}{2}\partial\bar{\partial}\Phi. \tag{1.1}$$

Throughout this thesis we are interested in Kähler metrics with particular symmetries. Therefore, we need to introduce the following.

Definition 1.2. A Kähler metric g on a n-dimensional complex manifold M is *rotation invariant* (resp. *radial*) with respect to a coordinate system $(z_1, \ldots, z_n)$ centered in p, if there exists a Kähler potential Φ for g on a neighborhood of p such that Φ is rotation invariant (resp. radial) in $(z_1, \ldots, z_n)$, i.e. it only depends on $|z_1|^2, \ldots, |z_n|^2$ (resp. on $|z_1|^2 + \cdots + |z_n|^2$).

Finally, we give some remarkable examples of *Kähler manifolds* (with finite or infinite dimension n), i.e. complex manifolds endowed with a Kähler metric. Precisely, these are the unique Kähler manifolds which have constant holomorphic sectional curvature (i.e. sectional curvature restricted to complex lines in the tangent space). For this reason they are called *complex space forms* (cf. [11]).

Example 1.1 (zero holomorphic sectional curvature). The complex space $\mathbb{C}^n$ with the *Euclidean metric* g_0, where $\mathbb{C}^\infty$ denotes the Hilbert space $\ell^2(\mathbb{C})$ consisting of sequences of complex numbers z_i such that $\sum_{i=1}^{+\infty} |z_i|^2 < +\infty$. For this metric there exists a global Kähler potential defined as

$$\Phi(z_1, \ldots, z_n) = \sum_{i=1}^{n} |z_i|^2.$$

Example 1.2 (negative holomorphic sectional curvature). The unit ball

$$\mathcal{B}^n = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} |z_i|^2 < 1\right\},$$

endowed with a constant multiple of the *hyperbolic metric* g_{hyp} for which there exists a global Kähler potential given by

$$\Phi(z_1, \ldots, z_n) = -\log\left(1 - \sum_{i=1}^{n} |z_i|^2\right).$$

Example 1.3 (positive holomorphic sectional curvature). The complex projective space $\mathbb{C}\mathrm{P}^n$ endowed with a constant multiple of the *Fubini-Study metric* g_{FS}. In the affine chart $U_0 = \{Z_0 \neq 0\}$, where $[Z_0 : \ldots : Z_n]$ are homogeneous coordinates, we can relate to this metric a local Kähler potential given, with respect to affine coordinates $z_i = \frac{Z_i}{Z_0}$ for $i = 1, \ldots, n$, by

$$\Phi(z_1, \ldots, z_n) = \log\left(1 + \sum_{i=1}^{n} |z_i|^2\right).$$

1.2 Kähler immersions into $(\mathbb{C}\mathrm{P}^N, g_{FS})$

Riemannian manifolds are characterized by Nash theorem [58], which states that any Riemannian manifold can be smoothly and isometrically embedded into the Euclidean space $\mathbb{R}^n$ for a sufficiently large dimension n. In this sense, Hermitian manifolds are deeply different from Riemannian ones, since an analogous theorem in the complex setting does not hold true (even if we allow the dimension of the ambient space to be infinite). In fact many obstructions occur in the complex case, for example it does not exist any holomorphic immersion from a compact complex manifold into $\mathbb{C}^N$ as a consequence of the maximum principle for holomorphic functions.

The problem of the existence and uniqueness of holomorphic and isometric immersions (from now on *Kähler immersions*) from Kähler manifolds into the Euclidean space $\mathbb{C}^N$ was theoretically solved by Calabi in [18], where he develops a criterium to establish when such immersions exist. Calabi also applies the same techniques to solve the more general problem of the existence and uniqueness of Kähler immersions into (finite or infinite dimensional) complex space forms. Since we are interested in projectively induced metrics, we decided to summarize in this section Calabi's results dealing with these particular Kähler immersions.

Definition 1.3. A Kähler metric g on a connected complex manifold M will be called *projectively induced* if there exist a point $p \in M$, a neighbourhood V of p and a Kähler immersion $f : (V, g_{|V}) \to (CP^N, g_{FS})$, where $N \leqslant \infty$. We distinguish projectively induced metrics that admit a Kähler immersion into a complex projective space with finite dimension from the other, by calling the first ones *finitely projectively induced.*

Let (M, g) be a Kähler manifold with a projectively induced metric g and a local Kähler potential Φ. Since Φ is assumed to be real analytic by definition 1.3, by duplicating the variables z and $\bar{z}$, it can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood U of the diagonal containing $(p, \bar{p}) \in M \times \bar{M}$ (here $\bar{M}$ denotes the manifold conjugated to M). Thus one can consider the power expansion of Φ around the origin with respect to z and $\bar{z}$ and write it as

$$\Phi(z,\bar{z}) = \sum_{j,l=0}^{\infty} a_{jl} z^{m_j} \bar{z}^{m_l}. \tag{1.2}$$

Here, z^{m_j} denotes the monomial in n variables $\prod_{i=1}^{n} z_i^{m_j^i}$ and we arrange for practical reasons every n-tuple of nonnegative integers as a sequence $m_j = (m_j^1, \ldots, m_j^n)$ such that $m_0 = (0, \ldots, 0)$, $|m_j| \leqslant |m_{j+1}|$ for all positive integer j and all the m_j's with the same $|m_j|$ using reverse lexicographic order.

By $\partial\bar{\partial}$-lemma, a Kähler potential is defined up to an addition with the real part of a holomorphic function. Therefore there exists a unique Kähler potential (called *diastasis function*) defined as

$$\mathrm{D}_p(q) = \tilde{\Phi}(p,\bar{p}) + \tilde{\Phi}(q,\bar{q}) - \tilde{\Phi}(p,\bar{q}) - \tilde{\Phi}(q,\bar{p}),$$

where q are points in the neighborhood U of p. Observe that it is the unique potential such that the matrix (a_{jk}) defined according to eq. (1.2) with respect to coordinates system centered in p satisfies $a_{j0} = a_{0j} = 0$ for every nonnegative integer j.

Example 1.4 (Complex space forms). We can easily check that complex space forms' Kähler potentials described in examples 1.1, 1.2 and 1.3 are their diastasis functions around the origin (with respect to affine coordinates in case of projective spaces).

A fundamental consequence of diastasis' uniqueness consists in the subsequent theorem.

Theorem 1.2 (Hereditary property, cf. [18] Prop. 6)**.** *If f is a local Kähler immersion of a real analytic manifold S into $\mathbb{C}\mathrm{P}^N$, then the diastasis $\mathrm{D}_p^S(z)$ of S around a point $p \in f^{-1}([1:0:\ldots:0])$ is defined on $S \backslash f^{-1}(\mathbb{C}\mathrm{P}^N \backslash U_0)$, where U_0 is the affine chart $\{Z_0 \neq 0\}$, and it is equal to*

$$\mathrm{D}_{[1:0:\ldots:0]}^{\mathbb{C}\mathrm{P}^N} \circ f = \log(1 + |f(z)|^2)$$

The diastasis function plays a key role in the achievement of Calabi's results, as shown by the following.

Theorem 1.3 (Calabi's criterion, cf. [18] Theor. 8)**.** *There exists a local Kähler immersion around a point p of a real analytic Kähler manifold with diastasis* $\mathrm{D}_p(z)$ *into* $\mathbb{C}\mathrm{P}^N$ *if and only if the matrix of coefficients in the power expansion around p of*

$$e^{\mathrm{D}_p(z)} - 1 = \sum_{i,j} b_{ij} z^{m_i} \bar{z}^{m_j}$$

is positive semidefinite of rank at most N (we also say that the metric is 1-resolvable *of rank N).*

Theorem 1.4 (Rigidity, cf. [18] Theor. 9)**.** *A full Kähler immersion is unique up to unitary transformations of the complex projective space.*

Definition 1.4. A holomorphic immersion $f\colon U \to \mathbb{C}\mathrm{P}^N$ is *full* provided $f(U)$ is not contained in any $\mathbb{C}\mathrm{P}^h$ for $h < N$.

Moreover, projectively induced metrics are deeply characterized by the following properties.

Theorem 1.5 (Global character of projectively induced metrics, cf. [18] Theor. 10)**.** *In a connected Kähler manifold endowed with a projectively induced metric, each point admits a neighborhood where is defined a local Kähler immersion into* $\mathbb{C}\mathrm{P}^N$*.*

Theorem 1.6 (Immersion's extension, cf. [18] Theor. 11)**.** *If a Kähler metric is defined on a simply connected manifold M then a local Kähler immersion $f\colon V \subset M \to \mathbb{C}\mathrm{P}^N$ can be extended to a global one. This immersion is also injective if and only if* $\mathrm{D}(p,q) = 0$ *only for $p = q$.*

To conclude we recall that Bochner proved in [11] the existence of a holomorphic coordinate system particularly useful from the computational point of view.

Definition 1.5. For any real analytic Kähler manifold, there exists a coordinates system in a neighbourhood of each point, such that

$$\mathrm{D}_0(z) = \sum_{\alpha=1}^{n} |z_\alpha|^2 + \psi_{2,2}, \tag{1.3}$$

where $\psi_{2,2}$ is a power series with degree greater than or equal to 2 in both z and $\bar{z}$. They are called *Bochner's coordinates.*

These coordinates are uniquely determined up to unitary transformation (cf. [18] Prop. 7) and they also have an interesting behavior in relation to complex analytic submanifolds.

Theorem 1.7 (cf. [18] Theor. 7)**.** *Let S be a k dimensional (finite) analytic submanifold of a Kähler manifold M (possibly infinite dimensional) with a real-analytic metric. A Bochner's coordinate system $(z_1, \dots, z_k)$ in S can be extended to a Bochner's coordinate system $(x_1, \dots, x_{\dim(M)})$ in M such that the immersion equation are*

$$\begin{cases} x_i = z_i & \text{for } i = 1, \dots, k \\ x_i = f_i(z) & \text{for } i = k+1, \dots, \dim(M) \end{cases} \tag{1.4}$$

where the power expansion of immersion's components f_i with respect to z does not contain any term of degree less than 2.

1.3 Kähler-Einstein metrics

A *Kähler-Einstein metric* g is a Kähler metric such that for a real constant λ is satisfied

$$\mathrm{Ric}(g) = \lambda\omega, \tag{1.5}$$

where $\mathrm{Ric}(g)$ and ω are respectively the Ricci form and the Kähler form associated to g. We recall that the Ricci form is defined as

$$\big(\mathrm{Ric}(g)\big)(X, Y) = \mathrm{Ric}(JX, Y),$$

for any tangent vector X and Y, where J is the complex structure and Ric is the Ricci tensor. Moreover, the following expression holds true for any Kähler metric

$$\mathrm{Ric}(g) = -i\partial\bar{\partial} \log \det(g_{i\bar{j}}). \tag{1.6}$$

We clarify that throughout this thesis, if g is a Kähler metric, $g_{i\bar{j}}$ stands for $g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$ and a similar notation will be also used for Ricci and curvature's tensor. By formula (1.6), one can easily prove that the difference of two Ricci forms related to any two Kähler forms defined on the same manifold is an exact form globally defined on this manifold. Thus the cohomology class $[\mathrm{Ric}(g)]$ is independent of the choice of the Kähler metric and we call

$$c_1(M) = \frac{1}{2\pi}[\mathrm{Ric}(g)] \in H^2(M, \mathbb{R})$$

first Chern class of the complex manifold M. An algebraic approach to first Chern class (and the others) can be found in [79, Chap. 3], where it is proved, among various properties, that $c_1(M)$ actually belongs to $H^2(M, \mathbb{Z})$.

In [20] and [21] Calabi asked whether it is possible to find a Kähler form in each Kähler class (i.e. cohomology classes of type $(1,1)$ containing a positive definite form) of a compact complex manifold M such that its Ricci form is equal to any $(1,1)$-form arbitrary chosen in $c_1(M)$. Calabi proved in [20] the uniqueness of this metric, while the proof of the existence (well-known as Calabi's conjecture) was provided by Yau (cf. [75] and [77]). This problem is closely related to that of the existence of Kähler-Einstein metric on compact complex manifolds. In fact, if we suppose $\lambda = 0$ in eq. (1.5), the existence of a Ricci-flat metric in each Kähler class is an immediate consequence of Calabi's conjecture. If $\lambda < 0$, Aubin [5] and Yau [77] proved independently the existence of a unique (up to homotheties) Kähler-Einstein metric. Instead, the case of $\lambda > 0$ is still an open problem, although a great deal of progress has been made (see for instance the results achieved by Tian in [70]). Furthermore, we have to point out that the interest in Kähler-Einstein metrics is not only justified by their relevance in differential geometry, but also for the important role they play in algebraic geometry. In fact, in the mentioned Tian's paper [70] the author refined considerably a previous idea of Yau, who conjectured the existence of Kähler-Einstein metrics on manifolds with positive first Chern class (also called *Fano manifolds*) related to some sort of algebro-geometric notion of *stability* in sense of geometric invariant theory. Currently, most of the work focuses in this direction, since it is universally considered the key to solve this problem.

We conclude the present section about Kähler-Einstein metrics with two lemmas, that will be very useful in the proof of our results.

Lemma 1.8 (cf. [6] and [24]). *Every Kähler-Einstein metric is real analytic.*

Proof. The equation 1.5 can be locally rewritten as a fourth-order nonlinear and overdetermined elliptic PDE for the Kähler potential. Hence, by using regularity results for elliptic equations we get the claim (see e.g. [6, Prop. 3.56]). □

Among the various applications, the previous lemma ensures the possibility of studying Kähler-Einstein metric via Bochner's coordinates and diastasis' function.

Lemma 1.9 (cf. [1] and [38]). *A Kähler manifold (M, g) is Einstein if and only if by choosing Bochner's coordinates on a neighbourhood U of any point $p \in M$, the diastasis function $\mathrm{D}_0(z)$ satisfies the Monge–Ampère equation*

$$\det\left(\frac{\partial^2 \mathrm{D}_0}{\partial z_i \partial \bar{z}_j}\right) = e^{-\frac{\lambda}{2}\mathrm{D}_0(z)}. \tag{1.7}$$

Proof. By formula (1.5), by Ricci and Kähler forms' definition and by the $\partial\bar\partial$-lemma 1.1, it follows that there exists a holomorphic function f such that

$$\det\left(\frac{\partial^2 \mathrm{D}_0}{\partial z_i \partial \bar z_j}\right) = e^{-\frac{\lambda}{2}(\mathrm{D}_0 + f + \bar f)}.$$

Therefore, it is easy to check, once Bochner's coordinates are set, that the expansion of $\det\left(\frac{\partial^2 \mathrm{D}_0}{\partial z_i \partial \bar z_j}\right)$ in the $(z, \bar z)$-coordinates around the origin reads

$$\det\left(\frac{\partial^2 \mathrm{D}_0}{\partial z_i \partial \bar z_j}\right) = 1 + h(z, \bar z),$$

where $h(z, \bar z)$ is a power series in z, $\bar z$ which contains only mixed terms (i.e. of the form $z^{m_j} \bar z^{m_k}$, $j \neq 0$, $k \neq 0$). Further, also the expansion of $\mathrm{D}_0(z)$, given in equation (1.3), contains only mixed terms, forcing $f + \bar f$ to be zero. □

Example 1.5 (Complex space forms). We can easily check that complex space forms (see ex. 1.1, 1.2 and 1.3) are Kähler-Einstein manifolds and the Einstein constants of g_0, g_{hyp} and g_{FS} are respectively equal to 0, $-2(n+1)$ and $2(n+1)$, where n is the dimension of the manifold.

Example 1.6 (Taub-NUT metric). In [46] C. Lebrun construct a family of Kähler metrics on $\mathbb{C}^2$, whose Kähler forms are equal to $\omega_m = \frac{i}{2}\partial\bar\partial\Phi_m$, where

$$\Phi_m(u, v) = u^2 + v^2 + m(u^4 + v^4), \text{ for } m \geqslant 0,$$

and u and v are implicitly defined by

$$|z_1| = e^{m(u^2 - v^2)} u, \ |z_2| = e^{m(v^2 - u^2)} v.$$

For $m = 0$ one gets the flat metric, while for $m > 0$ each of the metrics of this family represents the first example of complete

		Compact	Noncompact
classical	A III	$\frac{\mathrm{SU}(p+q)}{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))}$	$\frac{\mathrm{SU}(p,q)}{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))}$
	D III	$\frac{\mathrm{SO}(2n)}{\mathrm{U}(n)}$	$\frac{\mathrm{SO}^*(2n)}{\mathrm{U}(n)}$
	C I	$\frac{\mathrm{Sp}(n)}{\mathrm{U}(n)}$	$\frac{\mathrm{Sp}(n,\mathbb{R})}{\mathrm{U}(n)}$
	BD I	$\frac{\mathrm{SO}(n+2)}{\mathrm{SO}(n)\times\mathrm{SO}(2)}$	$\frac{\mathrm{SO}_0(n,2)}{\mathrm{SO}(n)\times\mathrm{SO}(2)}$
exceptional	E III	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}_{10}+\mathbb{R})$	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}_{10}+\mathbb{R})$
	E VII	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6+\mathbb{R})$	$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_6+\mathbb{R})$

Table 1.1: Classification of irreducible compact and noncompact Hermitian symmetric spaces.

Ricci–flat (non-flat) metric on $\mathbb{C}^2$ having the same volume form of the flat metric ω_0, i.e. $\omega_m \wedge \omega_m = \omega_0 \wedge \omega_0$. Moreover, for $m > 0$, these metrics are isometric (up to dilation and rescaling) to the *Taub-NUT metric*.

Example 1.7 (Hermitian symmetric spaces). A *Hermitian symmetric space* is a Hermitian connected manifold M such that each point of M is an isolated fixed point of an involutive holomorphic isometry. Every Hermitian symmetric space is a Kähler manifold that can be decomposed into the Cartesian product of irreducible ones (cf. [37], which represents the main reference for the present example). We distinguish three different types of irreducible Hermitian symmetric spaces: Euclidean, noncompact and compact, which are an example of Einstein manifolds with respectively zero, negative and positive Einstein constant. The Hermitian symmetric space of Euclidean type are biholomorphically isometric to $(\mathbb{C}^n, g_0)$. The irreducible compact and noncompact Hermitian symmetric spaces have been classified by É. Cartan according to Table 1.1. An irreducible Hermitian symmetric space of noncompact type can be regarded as *bounded symmetric domain*, namely they are biholomorphically isometric to an open, bounded and connected subset of $\mathbb{C}^n$ endowed with the Bergman metric (we just recall that the Bergman metric is the Kähler metric related to the global potential

Type	Description
I_{pq}	$\{Z \in \mathcal{M}_{p\times q}(\mathbb{C}) \mid I_p - Z\bar{Z}^T > 0\}$,, where $p \leqslant q$
II_n	$\{Z \in \mathcal{M}_n(\mathbb{C}) \mid Z = -Z^T,\ I_p - Z\bar{Z}^T > 0\}$, where $n > 4$
III_n	$\{Z \in \mathcal{M}_n(\mathbb{C}) \mid Z = Z^T,\ I_p - Z\bar{Z}^T > 0\}$, where $n > 1$
IV_n	$\{Z \in \mathbb{C}^n \mid \lvert Z\rvert^2 < \frac{1}{2}(1 + \lvert ZZ^T\rvert^2) < 1\}$, where $n > 4$

Table 1.2: Classification of classical irreducible noncompact Hermitian symmetric spaces. I_p stands for the identity matrix of order p and $I_p - Z\bar{Z}^T > 0$ means that the matrix $I_p - Z\bar{Z}^T$ is positive definite.

$\log K(z, z)$, where K is the Bergman kernel of the complex domain, for details and examples about Bergman kernel and Bergman metrics we suggest to see the first chapter of [44]). A description of classical cases can be found in Table 1.2. The exceptional cases can be instead described in terms of matrices with coefficients in the algebra of complex octonions $\mathbb{O}_{\mathbb{C}}$, i.e. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{O}$ (see e.g. [78] for details). Furthermore, each of noncompact Hermitian spaces admits local chart defined on open subset whose complementary has zero measure and the restriction of the metric is given by a constant multiple of the potential $-\log K(z, -z)$, where K is the Bergman kernel of a bounded symmetric domain, called dual (cf. [30]).

Example 1.8 (Cartan-Hartogs domains). Let Ω be an irreducible bounded symmetric domain (cf. ex. 1.7) of complex dimension d and genus γ, called *base* of the Cartan-Hartogs domain. On this base we can construct a family of complex domains depending on a positive real constant μ

$$M_\Omega(\mu) = \{(z, w) \in \Omega \times \mathbb{C} \mid |w|^2 < N_\Omega(z, z)^\mu\},$$

where the *generic norm* N_Ω is defined from the volume of Ω and its Bergman kernel K_Ω in the following way

$$N_\Omega = (V(\Omega)K_\Omega(z, z))^{-\frac{1}{\gamma}}.$$

A *Cartan-Hartogs domains* can be endowed with the Kähler metric

$g(\mu)$ related to the Kähler potential

$$-\log(N_\Omega(z,z)^\mu - |w|^2).$$

They are the example of non-homogeneous complete Kähler-Einstein manifolds Kähler immersed in $\mathbb{C}\mathrm{P}^\infty$ cited in the Introduction. Indeed, in [78] the authors proved that complex hyperbolic spaces are the unique homogeneous Cartan-Hartogs domain and also that for every base Ω the Cartan-Hartogs domain $\left(M_\Omega(\frac{\gamma}{d+1}), g(\frac{\gamma}{d+1})\right)$ is Kähler-Einstein and complete, furthermore we can find in [52] a proof of the existence a Kähler immersion into $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$ of each $\left(M_\Omega(\frac{\gamma}{d+1}), \lambda g(\frac{\gamma}{d+1})\right)$ for a suitable choice of the real constant λ.

1.4 Constant scalar curvature Kähler metrics

A generalization of the problem of the existence of Kähler-Einstein metrics on compact manifolds described in the previous section consists in finding a canonical metric in each Kähler class. The best candidate to represent the whole class was proposed by E. Calabi in [17], where he defined an *extremal metric* on a n-dimensional compact manifold M in the Kähler class $[\omega]$ as a critical point of the functional (known as *Calabi's functional*)

$$\mathrm{Cal}(\eta) = \int_{\mathrm{M}} \rho_\eta^2 \eta^{\mathrm{n}},$$

where $\eta \in [\omega]$ and ρ_η is the scalar curvature.

Kähler metrics with constant scalar curvature (from now on *cscK metrics*) are the most important example of extremal metrics. Although there exist extremal metrics with non-constant scalar curvature (see e.g. [72], where we can find a construction of a family of extremal metrics on a particular type of minimal ruled surfaces), a Kähler metric is extremal if and only if the projection of the gradient of its scalar curvature on $T^{1,0}M$ is a holomorphic vector field. But most of compact Kähler manifolds does not admit non-zero vector vector field, hence in these cases an extremal metric (if it exists) needs to have constant scalar curvature.

What has been said so far justifies the interest in cscK metrics in the contemporary research, considering also that many problems are still open. For further details about mentioned properties and known results, we suggest to see [10].

CHAPTER 2

FINITELY PROJECTIVELY INDUCED KÄHLER-EINSTEIN METRICS

Our first result consists in proving Conjecture 1 in case of rotation invariant metrics with respect to Bochner's coordinates and lower codimensions, i.e. when the codimension with respect to the target projective space is less than or equal to 3. This is shown in the second section via the following two theorems (cf. [65]).

Theorem 2.1. *The Einstein constant λ of a Kähler-Einstein rotation invariant and finitely projectively induced n-dimensional manifold M is a positive rational number less than or equal to $2(n+1)$. Moreover, M is an open subset of a compact and simply connected manifold.*

Theorem 2.2. *Let (M, g) be an n-dimensional Kähler-Einstein manifold whose metric is rotation invariant with respect to Bochner's coordinates. Then (M, g) admits a Kähler immersion into $\mathbb{C}\mathrm{P}^{n+k}$ for k less than or equal to 3, if and only if (M, g) is an open subset of $(\mathbb{C}\mathrm{P}^n, g_{FS})$, $(\mathbb{C}\mathrm{P}^2, 2g_{FS})$ or $(\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1, g_{FS} \oplus g_{FS})$.*

Instead, in the next section we make some comments on the content of the above mentioned conjecture.

2.1 Some remarks

Our results make us suppose the following subconjecture of Conjecture 1 to be valid.

Conjecture. *If a complete Kähler-Einstein manifold (M, g) is finitely projectively induced and it has a rotation invariant metric then*

$$(M, g) = (\mathbb{C}\mathrm{P}^{n_1} \times \cdots \times \mathbb{C}\mathrm{P}^{n_k}, c_1 g_{FS} \oplus \cdots \oplus c_k g_{FS}),$$

where $c_j = 2a(n_j + 1)$ for $j = 1, \ldots, k$, and $a \in \mathbb{Z}^+$.

First of all, we specify that an explicit Kähler immersion of

$$(\mathbb{C}\mathrm{P}^{n_1} \times \cdots \times \mathbb{C}\mathrm{P}^{n_k}, c_1 g_{FS} \oplus \cdots \oplus c_k g_{FS}) \to \mathbb{C}\mathrm{P}^N,$$

with $N = \binom{n_1+c_1}{c_1} \cdots \binom{n_k+c_k}{c_k} - 1$, can be constructed by a suitable normalization of the Veronese and Segre embeddings. Indeed we can easily prove by using the techniques described in the previous chapter that the *normalized Veronese maps*

$$\begin{aligned} (\mathbb{C}\mathrm{P}^n, c g_{FS}) &\to (\mathbb{C}\mathrm{P}^{\binom{n+c}{c}-1}, g_{FS}) \\ [z_i]_{0 \leqslant i \leqslant n} &\mapsto \sqrt{\frac{(c-1)!}{c^{c-2}}} \left[\frac{z_0^{c_0} \ldots z_n^{c_n}}{\sqrt{c_0! \ldots c_n!}} \right]_{c_0 + \ldots + c_n = c} \end{aligned}$$

are Kähler immersions (cf. [18, Theor. 13]). Then we compose them with the *Segre embeddings*

$$\begin{aligned} (\mathbb{C}\mathrm{P}^{n_1} \times \mathbb{C}\mathrm{P}^{n_2}, g_{FS} \oplus g_{FS}) &\to (\mathbb{C}\mathrm{P}^{(n_1+1)(n_2+1)-1}, g_{FS}) \\ ([z_i]_{0 \leqslant i \leqslant n_1}, [w_j]_{0 \leqslant j \leqslant n_2}) &\mapsto [z_i w_j]_{(i,j) \in \{0, \ldots, n_1\} \times \{0, \ldots, n_2\}} \end{aligned}$$

to obtain the desired immersion.

Moreover, the following examples prove the necessity of the assumptions for the conjecture to be true both in the compact and noncompact case and when the Einstein constant is negative, zero or positive.

Example 2.1 (finitely projectively induced, rotation invariant and not Kähler-Einstein). There exist many examples of such metrics. For instance consider $\mathbb{C}\mathrm{P}^1$ endowed with the metric $\omega = \frac{i}{2}\partial\bar{\partial}\log(1+|z|^2+|z|^4)$ or any products of complex projective spaces which are not Kähler-Einstein, such as $(\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1, c_1 g_{FS} \oplus c_2 g_{FS})$, with $c_1 \neq c_2$, c_1, $c_2 \in \mathbb{Z}^+$.

Example 2.2 (noncompact Kähler-Einstein, rotation invariant and not finitely projectively induced). The complex hyperbolic space $(\mathbb{C}\mathrm{H}^n, g_{hyp})$ and the complex Euclidean space $\mathbb{C}^n$ are two examples of Kähler-Einstein and rotation invariant, but not finitely projectively induced manifolds (cf. [18] Theor. 13). A more interesting example is the Taub-NUT metric (see ex. 1.6). This metric is not finitely projectively induced as it follows by our Theorem 2.1.

Example 2.3 (compact Kähler-Einstein, rotation invariant and not finitely projectively induced). A simple example of compact Kähler-Einstein, rotation invariant manifolds which are not projectively induced is given by $(\mathbb{C}\mathrm{P}^n, c g_{FS})$, with $c > 0$, $c \in \mathbb{R}\backslash\mathbb{Z}$. Less trivial examples (as it follows by ex. 2.2) are the complex 1-dimensional torus $\mathrm{T}^1 = \mathbb{C}/\mathbb{Z}^2$ with the flat metric induced by the metric g_0 of $\mathbb{C}$ and the Riemann surfaces Σ_γ of genus $\gamma \geqslant 2$ with the hyperbolic metric induced by the metric g_{hyp} of $\mathbb{C}\mathrm{H}^1$. Indeed Σ_γ can be realized as a quotient $\mathcal{B}/\Gamma$, where Γ is a Fuchsian subgroup of $\mathrm{SU}(1,1)$, which acts on $\mathcal{B}$ via Möbius transforms. We can easily check that the metric g_{hyp} defined on $\mathcal{B}$ (cf. ex. 1.2) is invariant under such action.

Example 2.4 (Kähler-Einstein, finitely projectively induced and not rotation invariant). It is well-known since the work of Borel and Weil (see [48] or [68] for a proof) that a Hermitian symmetric space of compact type (cf. ex. 1.7) admits a Kähler immersion into $\mathbb{C}\mathrm{P}^N$ (for further results see also [26]). Moreover, it can be easily verified by analyzing the reproducing kernel functions of these domains (see [48] and [26]), that the only irreducible Hermitian symmetric space of compact type which is rotation invariant is biholomorphically isometric to the complex projective space.

2.2 Lower codimensions

In order to prove Theorem 2.1, we need the following lemma dealing with projectively induced and rotation invariant with respect to Bochner's coordinates metrics.

Lemma 2.3. *Let g be a projectively induced Kähler metric on a complex manifold (M, g) and let $f : V \to \mathbb{C}P^N$ be the holomorphic immersion such that $f^* g_{FS} = g$. Assume that f is full and g admits a diastasis D_0 on a neighbourhood U of a point $p \in M$, which is rotation invariant with respect to Bochner's coordinates $z_1, \ldots, z_n$ around p. Then there exists an open neighbourhood W of p such that $\mathrm{D}_0(z)$ can be written on W as*

$$\mathrm{D}_0(z) = \log\left(1 + \sum_{j=1}^{n} |z_j|^2 + \sum_{j=n+1}^{N} a_j |z^{m_{h_j}}|^2\right), \qquad (2.1)$$

where $a_j > 0$ and $h_j \neq h_l$ for $j \neq l$.

Proof. Up to a unitary transformation of $\mathbb{C}P^N$ and by shrinking V if necessary, we can assume $f(p) = [1, 0 \ldots, 0]$ and $f(V) \subset U_0 = \{Z_0 \neq 0\}$. Hence by Theor. 1.7 and Theor. 1.2, one gets

$$\mathrm{D}_0(z) = \log\left(1 + \sum_{j=1}^{n} |z_j|^2 + \sum_{j=n+1}^{N} |f_j(z)|^2\right),$$

where

$$f_j(z) = \sum_{l=n+1}^{\infty} \alpha_{jl} z^{m_l}, \; j = n+1, \ldots, N.$$

The rotation invariance of $\mathrm{D}_0(z)$ and the fact that f is full imply that the f_j's are monomials of z of different degree and formula (2.1) follows. □

To simplify the notation, from now on we write P_{z_i} for $\partial P/\partial z_i$, $P_{\bar z_j}$ for $\partial P/\partial \bar z_j$, $P_{z_i\bar z_j}$ for $\partial^2 P/\partial z_i \partial \bar z_j$, and so on.

Proof of Theorem 2.1. By Lemma 2.3 we can assume that there exists $p \in M$ such that the diastasis around p can be written as $\mathrm{D}_0(z) = \log(P)$ where P is a polynomial in the variables $|z_1|^2, \ldots, |z_n|^2$. By the equality

$$\frac{\partial^2 \mathrm{D}_0(z)}{\partial z_i \partial \bar z_j} = \frac{P P_{z_i\bar z_j} - P_{z_i} P_{\bar z_j}}{P^2},$$

we have

$$\begin{aligned} \det\left(\frac{\partial^2 \mathrm{D}_0(z)}{\partial z_i \partial \bar z_j}\right) &= \det\left(\frac{P P_{z_i\bar z_j} - P_{z_i} P_{\bar z_j}}{P^2}\right) \\ &= \frac{1}{P^{2n}} \det\left(P P_{z_i\bar z_j} - P_{z_i} P_{\bar z_j}\right). \end{aligned}$$

Given a polynomial Q in the variables $z_1, \ldots, z_n, \bar z_1, \ldots, \bar z_n$, we denote by $\deg Q$ the total degree of Q with respect to all the variables $z_1, \ldots, z_n, \bar z_1, \ldots, \bar z_n$. Then

$$\deg \det\left(P P_{z_i\bar z_j} - P_{z_i} P_{\bar z_j}\right) \leqslant 2n \deg P - 2n.$$

On the other hand, from Monge–Ampère equation (1.7) we get

$$\deg \det\left(P P_{z_i\bar z_j} - P_{z_i} P_{\bar z_j}\right) - 2n \deg P = -\frac{\lambda}{2} \deg P,$$

which forces $\frac{\lambda}{2} \geqslant \frac{2n}{degP} > 0$. Thus, if M is complete, by Myers' Theorem, M is also compact. Then M is simply connected by a well-known theorem of Kobayashi [43]. The final upper bound $\lambda \leqslant 2(n+1)$ and the extension of M to a complete manifold are consequences of the following result due to D. Hulin. □

Lemma 2.4 (D. Hulin [39]). *Let (V, h) be a Kähler-Einstein manifold which admits a Kähler immersion into $\mathbb{C}\mathrm{P}^N$. Then it can be extended to a complete n-dimensional Kähler-Einstein* *manifold* (M, g) *and the Einstein constant is a rational number. Further, let this immersion be full and let the Einstein constant* $\lambda = 2p/q$ *be positive, where* p/q *is irreducible, then* $p \leqslant n+1$ *and if* $p = n+1$ *(resp.* $n = p = 2$*), then* $(M, g) = (\mathbb{C}\mathrm{P}^n, qg_{FS})$ *(rep.* $(M, g) = (\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1, g_{FS} \oplus g_{FS})$*).*

In order to prove Theorem 2.2 we also need:

Lemma 2.5. *Let* (M, g) *be a complete* n*-dimensional rotation invariant Kähler-Einstein manifold. If* $n > 2k$*,* (M, g) *admits a (local) Kähler immersion into* $\mathbb{C}\mathrm{P}^{n+k}$ *if and only if* $(M, g) = (\mathbb{C}\mathrm{P}^n, g_{FS})$*.*

Proof. By Lemma 2.3 there exist a point $p \in M$ and local coordinates $z_1, \ldots, z_n$ around it such that the diastasis function $\mathrm{D}_0(z)$ for g centered at p can be written as

$$\mathrm{D}_0(z) = \log\left(1 + \sum_{j=1}^{n} |z_j|^2 + \sum_{j=1}^{n} a_j |z_j|^4 \right.$$
$$\left. + \sum_{1 \leqslant j < l \leqslant n} b_{jl} |z_j|^2 |z_l|^2 + \psi_{3,3}\right),$$

where $\psi_{3,3}$ is a (rotation invariant) polynomial of degree not less than three both in z and in $\bar{z}$. For $h = 1, \ldots, n$, differentiating with respect to z_h and $\bar{z}_h$ both sides of the Monge–Ampère equation (1.7) and evaluating at 0, i.e. by considering the n equalities for $h = 1, \ldots, n$

$$\frac{\partial^2}{\partial z_h \partial \bar{z}_h}\left(\det\left(\frac{\partial^2 \mathrm{D}_0(z)}{\partial z_j \partial \bar{z}_l}\right)\right)\Big|_0 = \frac{\partial^2}{\partial z_h \partial \bar{z}_h}\left(e^{-\frac{\lambda}{2}\mathrm{D}_0(z)}\right)\Big|_0,$$

we get n equations of the form

$$4a_h + \sum_{\substack{l=1 \\ l \neq h}}^{n} b_{hl} - (n+1) = -\frac{\lambda}{2}, \quad (h = 1, \ldots, n), \tag{2.2}$$

in which $b_{ij} = b_{ji}$. Thus, if the codimension of M into $\mathbb{C}\mathrm{P}^N$ is $k < n/2$, at least one of these equations is of the form $\lambda = 2(n+1)$, because the polynomial P consists of only $n + k + 1$ monomials, and then at most k of the variables $\{a_i, b_{ij}\}_{\substack{1 \leqslant i \leqslant n \\ i<j\leqslant n}}$ can be different from 0. Therefore the claim follows by Lemma 2.4. $\square$

We are now in the position of proving Theorem 2.2.

Proof of Theorem 2.2. By virtue of Lemma 2.4, we can consider directly the case where (M, g) is a complete Kähler-Einstein projectively induced and rotation invariant manifold.
By Lemma 2.3 there exist a point $p \in M$ and local coordinates $z_1, \ldots, z_n$ such that the diastasis function $\mathrm{D}_0(z)$ around p can be written as

$$\mathrm{D}_0(z) = \log\left(1 + \sum_{j=1}^{n} |z_j|^2 + \sum_{j=1}^{n} a_j |z_j|^4 + \sum_{1\leqslant j<l\leqslant n} b_{jl}|z_j|^2|z_l|^2 \right.$$
$$\left. + \sum_{j=1}^{n} c_j|z_j|^6 + \sum_{j,l=1 (j\neq l)}^{n} d_{jl}|z_j|^2|z_l|^4 + \psi_{4,4}\right),$$

where $\psi_{4,4}$ is a rotation invariant polynomial of degree not less than four both in z and in $\bar{z}$. If $k = 0$, we have $\mathrm{D}_0(z) = \log(1 + \sum_{j=1}^{n} |z_j|^2)$ and $(M, g) = (\mathbb{C}\mathrm{P}^n, g_{FS})$. The statement holds by Theorem 1 in the Introduction for those cases where the codimension is equal to 1 or 2. Thus let $k = 3$. By Lemma 2.5, the statement is true for $n > 6$. Here we need to analyze the cases $n \leqslant 6$ separately.

Consider the case $n = 2$. As in the proof of Lemma 2.5, for $h = 1, 2$, by differentiating with respect to z_h, $\bar{z}_h$ both sides of the Monge–Ampère equation (1.7), and evaluating at 0 we get the 2 equations

$$\begin{cases} 4a_1 + b_{12} = 3 - \frac{\lambda}{2}, \\ 4a_2 + b_{12} = 3 - \frac{\lambda}{2}, \end{cases}$$

from which follows immediately $a_1 = a_2$ and $b_{12} = 3 - \frac{\lambda}{2} - 4a_1$. If $a_1 = a_2 = b_{12} = 0$, we get $\lambda = 6$, and by Lemma 2.4 $(M, g) = (\mathbb{C}\mathrm{P}^2, g_{FS})$. If $a_1 \neq 0$, $b_{12} \neq 0$, since the codimension is 3, all the other coefficients must vanish, thus by differentiating both sides of the Monge–Ampère equation (1.7) by z_1, $\bar{z}_1$, z_2, $\bar{z}_2$, and twice by z_1 and $\bar{z}_1$, evaluating at zero, we get a system whose unique acceptable solution is $b_{12} = 1/2$, $a_1 = 1/4$ and $\lambda = 3$, that is $(M, g) = (\mathbb{C}\mathrm{P}^2, 2g_{FS})$. If $a_1 \neq 0$ but $b_{12} = 0$, by differentiating again both sides of the Monge–Ampère equation (1.7) by z_1, $\bar{z}_1$, z_2, $\bar{z}_2$ and evaluating at zero and at $\lambda = 2(3 - 4a_1)$, we have $4a_1 + 4d_{12} + 4d_{21} = 0$, which is impossible, since $a_1 \neq 0$ and all the coefficients must be non-negative. It remains to consider the case $b_{12} \neq 0$, $a_1 = a_2 = 0$. By differentiating the Monge–Ampère equation twice by z_j and twice by $\bar{z}_j$ (for $j = 1, 2$), evaluating at zero and at $a_1 = a_2 = 0$, $b_{12} = 3 - \frac{1}{2}\lambda$, we get $d_{12} = 9c_1$ and $d_{21} = 9c_2$. Since the codimension is 3, only two of them can be different from zero. If they are all zero, we get $\lambda = 6$, and by Lemma 2.4 $(M, g) = (\mathbb{C}\mathrm{P}^2, g_{FS})$ or $\lambda = 4$, and again by Lemma 2.4 $(M, g) = (\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1, g_{FS} \oplus g_{FS})$. Let us suppose that two of them are different from zero, say $d_{12} \neq 0$. Then all the terms of higher order vanish, and taking the third order derivative we get again $\lambda = 6$ or $\lambda = 4$.

The case $n = 3$ is very similar to that one. The system given in the proof of Lemma 2.5 reads

$$\begin{cases} 4a_1 + b_{12} + b_{13} = 4 - \frac{\lambda}{2}, \\ 4a_2 + b_{12} + b_{23} = 4 - \frac{\lambda}{2}, \\ 4a_3 + b_{13} + b_{23} = 4 - \frac{\lambda}{2}. \end{cases}$$

It is easy to see that only three cases do not reduce immediately to $(M, g) = (\mathbb{C}\mathrm{P}^3, g_{FS})$, that is, $a_1 = a_2 = a_3 \neq 0$, or $a_1 = b_{23} \neq 0$ (and all the symmetric to them), or $b_{12} = b_{23} = b_{13} \neq 0$. By taking the second order derivative of the Monge–Ampère equation and evaluating at zero, it follows that these cases are incompatible.

If $n = 4$, it easy to see from the system of linear equation

$$\begin{cases} 4a_1 + b_{12} + b_{13} + b_{14} = 5 - \frac{\lambda}{2}, \\ 4a_2 + b_{12} + b_{23} + b_{24} = 5 - \frac{\lambda}{2}, \\ 4a_3 + b_{13} + b_{23} + b_{34} = 5 - \frac{\lambda}{2}, \\ 4a_4 + b_{14} + b_{24} + b_{34} = 5 - \frac{\lambda}{2}, \end{cases}$$

that, up to symmetries, only the following cases may occur: all the coefficients are equal to zero, $a_1 = a_2 = b_{34} \neq 0$ or $b_{12} = b_{34} \neq 0$. In the third case, without loss of generality, we suppose that $d_{23} = d_{32} = 0$. Therefore if the second or third case holds, by differentiating both sides of the Monge–Ampère equation with respect to z_2, z_3, $\bar{z}_2$, $\bar{z}_3$, evaluating at zero, and considering the relations above, we get $b_{34} = 0$ and the conclusion follows.

The cases $n = 5$ and $n = 6$ are very similar to that one. For $n = 5$, by the system of linear equation either the coefficients of the system are zero or up to symmetries $4a_1 = b_{23} = b_{45} \neq 0$. Differentiating with respect to z_2, z_4, $\bar{z}_2$, $\bar{z}_4$ the Monge–Ampère equation and evaluating at zero, one gets $b_{23} = 0$. For $n = 6$, from the system of linear equation, one gets that either the coefficients are all zero or $b_{12} = b_{34} = b_{56} \neq 0$. By differentiating with respect to z_2, z_4, $\bar{z}_2$, $\bar{z}_4$ and evaluating at zero, one gets $b_{34} = 0$, and we are done. □

CHAPTER 3

PROJECTIVELY INDUCED RICCI-FLAT METRICS

In the current chapter we prove the Conjecture 2 by assuming that the metric involved is stable-projectively induced and restricting ourselves to radial Kähler metrics. This is shown in the second section via the following (cf. [51]).

Theorem 3.1. *The only Ricci-flat, stable-projectively induced and radial Kähler metric is the flat one.*

In the next section we introduce the definition of stable-projectively induced metrics and we make some general comments about them.

3.1 Stable-projectively induced metrics

In the noncompact case the structure of the set of the positive real numbers $\lambda \in \mathbb{R}^+$ for which the multiple λg of a Kähler metric g is projectively induced is in general less trivial than in the compact case, where it is always discrete since the well-known Kodaira's embedding theorem state that a compact Kähler manifold is projectively induced if and only if its Kähler form belongs to a cohomology class in $H^2(M, \mathbb{Z})$ (cf. [79, Chap. 6]). In the noncompact symmetric case, one has for example the following (see also [50] for the more general case of bounded homogeneous domains).

Theorem 3.2 (cf. [52] Theor. 2). *Let Ω be an irreducible bounded symmetric domain endowed with its Bergman metric g_B. Then there exist a positive real number a and an integer k (both depending on Ω) such that $(\Omega, \lambda g_B)$ admits an equivariant Kähler immersion into $\mathbb{C}P^\infty$ if and only if λ belongs to the set*

$$\{a, 2a, \ldots, ka\} \cup (ka, +\infty).$$

From this theorem it follows that the only irreducible bounded symmetric domain where λg_B is projectively induced for all $\lambda > 0$ is the complex hyperbolic space. More generally, for a homogeneous bounded domain (Ω, g) we have that λg is projectively induced for all $\lambda > 0$ if and only if $(\Omega, g) = \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_r}_{\lambda_r}$, where $\mathbb{C}H^{n_j}_{\lambda_j} = (\mathbb{C}H^{n_j}, \lambda_j g_{hyp})$ (cf.[25] Theor. 4). Inspired by these results, we give the following.

Definition 3.1. A projectively induced Kähler metric g is said to be *stable-projectively induced* if there exists $\delta > 0$ such that λg is projectively induced for all $\lambda \in (1-\delta, 1+\delta)$. A projectively induced Kähler metric is said to be *unstable* if it is not stable-projectively induced.

Example 3.1. The flat metric g_0 on the complex Euclidean space $\mathbb{C}^n$ is stable-projectively induced by the map (cf. [18])

$$(\mathbb{C}^n, g_0) \to (\mathbb{C}\mathrm{P}^\infty, g_{FS})$$
$$z \mapsto \left(\ldots, \sqrt{\frac{1}{m_j!}} z^{m^j}, \ldots\right)$$

Consequently, many examples of stable-projectively induced metrics can be constructed on those complex manifolds M which admit a holomorphic immersion into $\mathbb{C}^n$ (e.g. Stein manifolds) by simply taking the restriction of the flat metric g_0 to M.

Obviously a Kähler metric on a compact complex manifold is always unstable and Theor. 3.2 shows that there exists metrics g which are unstable-projectively induced, but they become stable-projectively induced by multiplying them for a suitable constant.

3.2 Proof of Theorem 3.1

As already mentioned, we are going to study metrics which admit a Kähler potential $\Phi : U \to \mathbb{R}$ that depends only on the sum of the local coordinates' moduli. Namely, there exists $f : \tilde{U} \to \mathbb{R}$, $\tilde{U} \subset \mathbb{R}^+$, such that

$$f(r) = \Phi(z),\ z = (z_1, \ldots, z_n), \tag{3.1}$$

where

$$\tilde{U} = \{r = |z|^2 = |z_1|^2 + \cdots + |z_n|^2 \mid z \in U\}.$$

Therefore, one can easily classify all the Ricci-flat metrics which also satisfy this further condition as shown by the following well-known result (cf. [16]).

Lemma 3.3. *Let U be a complex domain of $\mathbb{C}^n$ equipped with a radial Kähler Ricci-flat metric g. Then there esist $\lambda \in \mathbb{R}^+$ and $\varepsilon = -1, 0, 1$ such that the function $f : \tilde{U} \to \mathbb{R}$ defined by (3.1) has the following expression*

$$f(r) = \lambda \int (\varepsilon r^{-n} + 1)^{\frac{1}{n}} dr. \tag{3.2}$$

Proof. Being g Ricci-flat, its Ricci form vanishes, namely

$$\partial\bar{\partial} \log \det \left(\frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} \right) = 0 \tag{3.3}$$

where

$$\left(\frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j} \right) = \begin{pmatrix} f' + f''|z_1|^2 & f''\bar{z}_1 z_2 & \ldots & f''\bar{z}_1 z_n \\ f''\bar{z}_2 z_1 & f' + f''|z_2|^2 & \ldots & f''\bar{z}_2 z_n \\ \vdots & \vdots & \ldots & \vdots \\ f''\bar{z}_n z_1 & f''\bar{z}_n z_2 & \ldots & f' + f''|z_n|^2 \end{pmatrix}.$$

Thus, one easily sees that

$$\det\left(\frac{\partial^2\Phi}{\partial z_i \partial \bar{z}_j}\right) = (f')^{n-1}(f' + f''r).$$

If we denote $\Psi(r) = \log\det\left(\frac{\partial^2\Phi}{\partial z_i\partial \bar{z}_j}\right)$, equation (3.3) is equivalent to the following equations

$$\frac{\partial^2\Psi}{\partial z_i\partial \bar{z}_j} = \Psi'' \bar{z}_i z_j = 0 \ (i \neq j),$$

$$\frac{\partial^2\Psi}{\partial z_i\partial \bar{z}_i} = \Psi' + \Psi''|z_i|^2 = 0, \ (i = 1, \ldots, n).$$

This yields $\Psi' = 0$, i.e.

$$\log\big((f')^{n-1}(f' + f''r)\big) = c,$$

for some constant c.

Setting $\xi = (f')^n$ and $\tilde{c} = e^c > 0$, we get the following linear ODE in ξ

$$\xi' = -\frac{n}{r}\xi + \tilde{c}\frac{n}{r}.$$

Therefore, one finds

$$\xi = Cr^{-n} + \tilde{c}$$

that is

$$f' = (Cr^{-n} + \tilde{c})^{\frac{1}{n}}$$

and then the general solution is

$$f(r) = \int (Cr^{-n} + \tilde{c})^{\frac{1}{n}} dr, \ C \in \mathbb{R}, \tilde{c} > 0, \tag{3.4}$$

which is equivalent to (3.2) after a change of variables. □

Remark 3.2. It is known that the metrics corresponding to the Kähler potentials (3.2) are non-complete and non-flat except in the case of the Euclidean metric ($\varepsilon = 0$).

Unlike the case in which the origin is contained in the domain of definition of a radial diastasis, it is more difficult to apply Calabi's criterion (cf. Theor. 1.3), because the matrix b_{ij} coming from the power expansion of $e^{\mathrm{D}_p(z)} - 1$ around p, is not diagonal (see e.g. [54] for the case on which the origin is contained). Hence the following lemma is the key ingredient for the proof of our results.

Lemma 3.4. *Let $n \geqslant 2$ and $p = (u, 0, \ldots, 0)$, with $u \in \mathbb{R}$, $u \neq 0$, be a point of the complex domain $U \subset \mathbb{C}^n \backslash \{0\}$ on which is defined a metric g with radial Kähler potential $\Phi : U \to \mathbb{R}$ and corresponding diastasis $D_p : U \to \mathbb{R}$. Let $f : \tilde{U} \to \mathbb{R}$ defined by (3.1) and, for $h \in \mathbb{N}$, let $g_h : \tilde{U} \to \mathbb{R}$ be given by:*

$$g_h(r) = \frac{d^h e^{f(r)}}{dr^h} e^{-f(r)}. \tag{3.5}$$

Assume that the entries of the following infinite times infinite matrix

$$\left(det\left(\frac{\partial^{i+j}(e^{D_p(z)} g_h(|z|^2))}{\partial z_1^i \partial \bar{z}_1^j} \right)_{0 \leqslant i,j \leqslant l} \right)_{l,h \in \mathbb{N}} \tag{3.6}$$

are positive when evaluated at p. Then the metric g is 1-resolvable at p of infinite rank.

Proof. Let $z = (z_1, z_2, \ldots, z_n) = (z_1, z_*)$, let $m_i = (m_i^1, m_i^*) \in \mathbb{N} \times \mathbb{N}^{n-1}$ and let $D_p(z)$ be the diastasis function. We observe that if $m_i^* \neq m_j^*$ then

$$\frac{\partial^{|m_i|+|m_j|}}{\partial z^{m_i} \partial \bar{z}^{m_j}} \left(e^{D_p(z)} - 1 \right) \Big|_p = 0. \tag{3.7}$$

In fact, by definition of diastasis, $D_p(z)$ is the the sum of the Kähler potential $f(|z|^2)$, the constant $f(u^2)$ and the real part of a holomorphic function which depends only on z_1 and which is equal to $-2f(u^2)$ if evaluated at u. Therefore

$$\frac{\partial^{|m_i|+|m_j|}(e^{D_p(z)} - 1)}{\partial z_1^{m_i^1} \partial \bar{z}_1^{m_j^1} \partial z_*^{m_i^*} \partial \bar{z}_*^{m_j^*}} \Big|_p = \frac{\partial^{|m_i|+m_j^1}}{\partial z_1^{m_i^1} \partial \bar{z}_1^{m_j^1} \partial z_*^{m_i^*}} \left(z_*^{m_j^*} e^{D_p(z) - f} \frac{d^{|m_j|}}{dx^{|m_j|}} e^f \right) \Big|_p . \tag{3.8}$$

From which we can deduce obviously (3.7) and also

$$\frac{\partial^{|m_j^*|+|m_j^*|}}{\partial z_*^{m_j^*}\partial \bar{z}_*^{m_j^*}}\left(e^{D_p(z)}-1\right)\Big|_p = m_j^*! g_{|m_j^*|}(u^2). \tag{3.9}$$

With the order of multi-indices m_i's set at page 11, namely $m_0 = (0,\ldots,0)$, $|m_j| \leqslant |m_{j+1}|$ for all positive integers j, if $|m_i| = |m_j|$ and $m_i^1 > m_j^1$ then $i < j$, the square submatrix E_h of

$$b_{ij} = \frac{1}{m_i! m_j!}\frac{\partial^{|m_i|+|m_j|}}{\partial z^{m_i}\partial \bar{z}^{m_j}}\left(e^{D_p(z)}-1\right)\Big|_p$$

relative to multi-indices m_i such that $|m_i| \leqslant h$ assumes the following form

$$\begin{pmatrix} A_h & 0 \\ 0 & D_h \end{pmatrix} \tag{3.10}$$

where A_h is the square matrix relative to multi-indices m_i such that $|m_i| < h$ or $|m_i| = h$ and $m_i^1 \neq 0$, while D_h is the matrix relative to multi-indices m_i such that $|m_i| = h$ and $m_i^1 = 0$. Therefore, eq. (3.7) explains the null blocks in (3.10). Moreover, since if $m_i \neq m_j$, $|m_i| = |m_j| = h$ and $m_i^1 = m_j^1 = 0$ then $m_i^* \neq m_j^*$, it follows again by eq. (3.7) that D_h is diagonal (and the entries on the diagonal are described by (3.9)). Now, if every matrix E_h is positive definite, namely if for every positive integer h the matrix A_h is positive definite and the entries of D_h are positive, the metric examined is 1-resolvable at p of infinite rank.

Since we obtain the entries of D_h by multiplying $g_h(u^2)$ for a positive constant, these are positive for every integer h if and only if the entries of the first row ($l = 0$) of the matrix (3.6), given by $e^{D_p(z)} g_h,\ h = 0, 1, \ldots,$ are positive.

Now we consider the matrix A_h. We are free to change the above arrangement of the multi-indices m_i's, because this has just the effect to apply the same permutation to both rows and columns of the matrix, and then yields a similar matrix, which is positive definite if and only if the original one is. Thus we change their order as follows: $|m^*_j| < |m^*_{j+1}|$ for all positive integers j, if $|m^*_i| = |m^*_j|$ and m^*_i precedes m^*_j with respect to the lexicographical order or if $m^*_i = m^*_j$ and $m^1_i < m^1_j$ then $m_i < m_j$. Then, after the corresponding rows and columns exchanges on A_h and by using (3.7) we obtain a block matrix of the following form:

$$\begin{pmatrix} M^h_0 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & M^h_{|m^*_j|} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & M^h_{h-1} \end{pmatrix}$$

where $M^h_{|m^*_j|}$ are the square matrices related to fixed norm multi-indices. By virtue of (3.7) $M^h_{|m^*_j|}$ are themselves block matrices and, by (3.8), each block is equal to a suitable positive constant multiple of

$$\left(\frac{1}{i!j!}\frac{\partial^{i+j}}{\partial z_1^i \partial \bar{z}_1^j}\left(e^{D_p(z)} g_{|m^*_j|}\right)\right)_{0 \leqslant i,j \leqslant h-|m^*_j|}.$$

Therefore, by using Sylvester's criterion, if the entries from the second row onwards of the matrix (3.6) are positive, A_h is positive definite for every integer h. $\square$

Corollary 3.5. *Under the same assumptions of Lemma 1.3, if there exists $r \in \tilde{U}$ and $h \in \mathbb{N}$ such that the function given by (3.5) is negative, namely $g_h(r) < 0$, then the metric g is not projectively induced.*

Now we are in the position to prove our result.

Proof of Theorem 3.1. Let us denote by ω_ε the Kähler form corresponding to the potential (3.2) with $\lambda = 1$, namely

$$\omega_\varepsilon = \frac{i}{2}\partial\bar{\partial} f_\varepsilon, \tag{3.11}$$

where

$$f_\varepsilon(r) = \int (\varepsilon r^{-n} + 1)^{\frac{1}{n}} dr, \ \varepsilon = -1, 0, 1. \tag{3.12}$$

Notice that ω_ε is flat either for $n = 1$ or $\varepsilon = 0$. We will show that for $n \geqslant 2$ we have the following:

(a) $\lambda\omega_{-1}$ is not projectively induced for any $\lambda \in \mathbb{R}^+$;

(b) $\lambda\omega_1$ is not projectively induced for any $\lambda \in \mathbb{R}^+ \backslash \mathbb{Z}$.

Then the proof of Theorem 3.1 will follow by the very definition of stable-projectively induced metric.

A simple computation shows that the function $g_3(r)$ (namely (3.5) for $h = 3$) for the potential $f = \lambda f_{-1}$ is given by:

$$\lambda \frac{(r^n - 1)^{\frac{1-2n}{n}}}{r^3} \left(\lambda^2 (r^n - 1)^{\frac{2+2n}{n}} + 3\lambda (r^n - 1)^{\frac{1+n}{n}} - \Big(r^n (n+1) - 2 \Big) \right).$$

Hence, one has $\lim_{r \to 1^+} g_3(r) = -\infty$ and the proof of (a) follows by Corollary 3.5.

In order to prove (b) we first show by induction that the function $g_h(r)$ for the potential $f = \lambda f_1$ is given by:

$$\frac{\lambda}{r^h} \left((r^n + 1)^{1/n} \prod_{j=1}^{h-1} \Big(\lambda (r^n + 1)^{1/n} - j \Big) + \varphi_h(r) r \right), \tag{3.13}$$

where $\varphi_h \in C^\infty([0, +\infty))$. This statement is trivially true for g_1, because it is equal to $\frac{\lambda}{r}(r^n + 1)^{1/n}$. The functions g_h can be defined recursively as

$$g_{h+1} = g_h' + g_1 g_h,$$

where $g_1 = f'$. Hence

$$g_{h+1} = \frac{\lambda}{r^{h+1}} \left((r^n + 1)^{1/n} \prod_{j=1}^{h} \Big(\lambda (r^n + 1)^{1/n} - j \Big) + \varphi_{h+1} r \right),$$

with φ_{h+1} equal to

$$\frac{d}{dr}\Big((r^n+1)^{1/n}\prod_{j=1}^{h-1}\big(\lambda(r^n+1)^{1/n}-j\big)\Big) \\ +\big(1-h+\lambda(r^n+1)^{1/n}\big)\varphi_h+\varphi_h' r$$

and (3.13) is proved. Therefore, if $\lambda \in \mathbb{R}^+\backslash\mathbb{Z}$

$$\lim_{r\to 0^+} g_{\lfloor\lambda\rfloor+2}(r) = -\infty,$$

where $\lfloor\lambda\rfloor$ denotes the integral part of λ. Thus, Corollary 3.5 implies (b) and this concludes the proof of the theorem. $\square$

Remark 3.3. We are able to extend the proof of (b) also for some fixed integer values of λ with a case by case analysis. For example when $\lambda = 1$ one obtains the following table which expresses $g_h(r)$, for suitable values of r, depending on the dimension n of the domain, for $n = 2, 3, 4, 5$:

r	h	n	$g_h(r)$
$3/4$	7	2	$-\frac{12294367331}{2373046875}$
$3/4$	5	3	≈ -2.81
$3/4$	5	4	≈ -10.3
$6/5$	4	5	≈ -0.14

Moreover, for any n, we have:

$$g_4(1) = 2^{\frac{1-3n}{n}}(8(2^{\frac{1}{n}})^3 - 24(2^{\frac{1}{n}})^2 + 30(2^{\frac{1}{n}}) - 15 + 8n2^{\frac{1}{n}} - 9n)$$

which is seen to be negative for $n \geqslant 6$. We believe (in accordance with Conjecture 1) that $\lambda\omega_1$ is not projectively induced for all integer values of λ even if we are not able to provide a general proof.

Remark 3.4. For $n = 2$ and $\varepsilon = 1$ one can explicitely express a Kähler potential for the Kähler metric ω_1 on $\mathbb{C}^2\backslash\{0\}$, namely

$$f_1(r) = \sqrt{r^2+1} + \log r - \log(1+\sqrt{r^2+1}),\ r = |z_1|^2+|z_2|^2 \quad (3.14)$$

If M denotes the blow-up of $\mathbb{C}^2$ at the origin and E denotes the exceptional divisor, one can prove (see [29]) that there exists a complete Ricci-flat and ALE (Asymptotically Locally Euclidean, see page 57 and references therein) Kähler metric on M whose restriction to $\mathbb{C}^2\backslash\{0\}$ has Kähler potential given by (3.14). This metric is known in the literature as the Eguchi–Hanson metric and denoted here by g_{EH}.

Therefore as a byproduct of our analysis one gets that the Eguchi–Hanson metric g_{EH} is not projectively induced.

Notice that if one will be able to prove that λg_{EH} is not projectively induced for all $\lambda > 0$ (in accordance with our conjecture), this will provide an example of Ricci-flat and complete Kähler metric which does not admit a Kähler immersion into any finite or infinite dimensional complex space form (the reader is referred to [54] for details related to this issue).

CHAPTER 4

THE TIAN-YAU-ZELDITCH COEFFICIENTS

After giving Tian-Yau-Zelditch coefficients' definition in the first section, in second one we prove our last main result:

Theorem 4.1. *The third Tian-Yau-Zelditch coefficient of a radial cscK metric is constant if and only if the second one is constant.*

This theorem allows us to prove the validity of the Conjecture 3 in the case of radial Kähler metrics on complex surfaces.

Theorem 4.2. *A Ricci-flat radial Kähler metric defined on a complex surface such that the third Tian-Yau-Zelditch coefficient vanishes is flat.*

4.1 The Tian-Yau-Zelditch expansion

Let L be a Hermitian line bundle over a n-dimensional Kähler manifold M whose Kähler form is ω. We recall that a *Hermitian bundle* is a holomorphic vector bundle $\pi : E \to M$ endowed with a family of Hermitian inner products h smoothly varying on its fibers (namely $h \in \Gamma(E^* \otimes \bar{E}^*)$). In particular, a *Hermitian line bundle* is a Hermitian bundle of rank 1.

Let us assume that ω is *polarized* with respect to L, namely the Ricci form $\mathrm{Ric}(h)$ related to the Hermitian metric h satisfies

$$\mathrm{Ric}(h) = \omega. \tag{4.1}$$

Cohomological argumentations allow us to prove that such a Hermitian line bundle exists (and is unique up to isometries) if and only if $\frac{1}{2\pi}\omega$ is integral, i.e. $\frac{1}{2\pi}[\omega] \in H^2(M, \mathbb{Z})$ (cf. [79, Chap. 3]). Moreover, notice that a formula similar to (1.6) holds true also for the case of Hermitian line bundles and it can be easily proved via a direct computation by considering that in this case $\mathrm{Ric}(h)$ coincides with the $(1,1)$-component of the curvature of the Chern connection. We recall that the *Chern connection* on a holomorphic bundles is the unique connection compatible with both the hermitian metric and the complex structure (see e.g. [61, Chap. 4]).

In analogy with the quantum mechanics terminology, if m is a positive integer, $L^m := L^{\otimes m}$ is called the *quantum line bundle*, the pairs (L^m, h_m) is called a *geometric quantization* of the Kähler manifold $(M, m\omega)$, where the Hermitian metric h_m is defined as

$$h_m(\otimes^m \sigma, \otimes^m \sigma) = h(\sigma, \sigma)^m$$

if $\sigma : U \to L \backslash \{0\}$ is a trivializing local section for L. Consider the separable complex Hilbert space $\mathcal{H}_m$ consisting of global holomorphic sections of L^m whose norm is bounded with respect to the Hermitian product

$$\langle s, t \rangle_m = \int_M h_m(s, t) \frac{\omega^n}{n!}.$$

Notice that, if M is a compact manifold $\mathcal{H}_m$ coincides with the space of global holomorphic sections, instead if M is noncompact $\mathcal{H}_m$ may even contain just the zero section. If $\mathcal{H}_m \neq \{0\}$, we choose an orthonormal basis $(s_0, \ldots, s_{N_m})$, where $\dim(\mathcal{H}_m) = N_m + 1$

may even be infinite if M is noncompact, and we globally define the following smooth real-valued function

$$\varepsilon_{mg} = \sum_{j=0}^{N_m} h_m(s_j, s_j). \tag{4.2}$$

We call it *distortion function* (cf. [42], [40], [82]), but we specify that it is also known in literature as η-function (see [63]) or θ-function (see [12]).

Lemma 4.3. *The distortion function does not depend on the choice of the orthonormal basis of $\mathcal{H}_m$.*

Proof. Let $q \in L^m \backslash \{0\}$ be a fixed point of the fiber over $x \in M$. If one evaluates $s \in \mathcal{H}_m$ at x, one gets a multiple $\delta_q(s)$ of q. We can prove that the functional $\delta_q : \mathcal{H}_m \to \mathbb{C}$ is continuous and linear (see e.g. [12]). Hence, from Riesz's theorem there exists a unique $e_q^m \in \mathcal{H}_m$ such that

$$\delta_q(s) = \langle s, e_q^m \rangle_m$$

for every $s \in \mathcal{H}_m$. Thus, every element s_j of an orthonormal basis of $\mathcal{H}_m$ satisfies

$$s_j(x) = \langle s_j, e_q^m \rangle_m q. \tag{4.3}$$

By substituting (4.3) in (4.2), we get

$$\varepsilon_{mg}(x) = h_m(q, q) \langle e_q^m, e_q^m \rangle_m.$$

Hence, we have the independence of the choice of the basis. □

The name of "distortion function" finds justification in the following quantum-geometric interpretation. Let us suppose that there exists a sufficiently large integer m such that for every point of M we can find an element of $\mathcal{H}_m$ which does not vanishes at this point. For example, in the compact case Kodaira's theory guarantees the existence of such an m (see e.g. [79, Chap. 6] for a complete description of Kodaira's result). Therefore, we can use the basis $(s_j)_{0\leqslant j\leqslant N_m}$ of $\mathcal{H}_m$ to construct a holomphic map (called *coherent state map*) from M into $\mathbb{C}\mathrm{P}^{N_m}$. Indeed, if σ is a trivializing local section of L, for every j we can find a holomorphic function f_j such that a suitable restriction of s_j is equal to $f_j \otimes^m \sigma$ and we can easily prove that

$$\begin{aligned}\varphi_m : M &\to \mathbb{C}\mathrm{P}^{N_m}\\ x &\mapsto [f_0(x), \dots, f_{N_m}(x)]\end{aligned}$$

is globally well-defined. By considering that $\varphi_m^*\omega_{FS}$ reads locally as $\frac{i}{2}\partial\bar{\partial}\log\left(\frac{\sum_{j=0}^{N_m} h_m(s_j,s_j)}{h_m(\otimes^m\sigma,\otimes^m\sigma)}\right)$, we get

$$\varphi_m^*\omega_{FS} = m\omega + \frac{i}{2}\partial\bar{\partial}\log\varepsilon_{mg}. \tag{4.4}$$

Therefore, *balanced metrics*, namely polarized metrics whose distortion function is constant, play a particularly important role since in the compact case they are the unique projectively induced metrics via coherent state map. Nevertheless, any polarized metric on a compact complex manifold is the C^∞-limit of normalized projectively induced metrics $\frac{\varphi_m^* g_{FS}}{m}$, as conjectured by Yau and proved by Tian ([71]) and Ruan ([64]). Then, Catlin ([22]) and Zelditch ([81]) independently generalized the Tian-Ruan theorem by proving the existence of a complete asymptotic expansion for the distortion function related to any polarized metric defined on compact complex manifold. Unfortunately, we have not an analogous general theorem for polarized metric on a noncompact manifold. As a partial result in this direction we can cite M. Engliš [31], where he proved the expansion's existence for the case of strongly pseudoconvex bounded domains in $\mathbb{C}^n$ with real analytic boundary and for bounded symmetric domains endowed with Bergman metric. Furthermore, X. Ma and G. Marinescu proved its existence under some boundedness assumptions of the curvature of the bundles involved (see [59, Theor. 6.1.1]).

This asymptotic expansion

$$\varepsilon_{mg}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}, \tag{4.5}$$

is called *Tian–Yau–Zelditch expansion* and it means that for every compact subset H of M there exists a positive constant $c_{l,r}^H$ depending on the subset H and on two positive integer constants l and r such that

$$\left\| \varepsilon_{mg}(x) - \sum_{j=0}^{l} a_j(x) m^{n-j} \right\|_{C^r} \leqslant \frac{c_{l,r}^H}{m^{l+1}},$$

where $a_0(x) = 1$ and $a_j(x)$, $j = 1, \ldots$ are smooth functions on M depending on the curvature and its covariant derivatives at x of the metric g. In particular, Z. Lu [56] for the compact case and M. Engliš [32] for the noncompact case computed the first three coefficients:

$$\begin{cases} a_1(x) = \frac{1}{2}\rho \\ a_2(x) = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|\mathrm{Ric}|^2 + 3\rho^2) \\ a_3(x) = \frac{1}{8}\Delta\Delta\rho + \frac{1}{24}\mathrm{divdiv}(R, \mathrm{Ric}) - \frac{1}{6}\mathrm{divdiv}(\rho\mathrm{Ric}) \\ \qquad\quad +\frac{1}{48}\Delta(|R|^2 - 4|\mathrm{Ric}|^2 + 8\rho^2) + \frac{1}{48}\rho(\rho^2 - 4|\mathrm{Ric}|^2 + |R|^2) \\ \qquad\quad +\frac{1}{24}(\sigma_3(\mathrm{Ric}) - \mathrm{Ric}(R, R) - R(\mathrm{Ric}, \mathrm{Ric})), \end{cases} \tag{4.6}$$

where ρ, R, Ric denote respectively the scalar curvature, the curvature tensor and the Ricci tensor of (M, g), and we are using the following notations (in local coordinates $z_1, \ldots, z_n$):

$$\begin{aligned} &|D'\rho|^2 = g^{j\bar{i}} \frac{\partial\rho}{\partial z_i} \frac{\partial\rho}{\partial \bar{z}_j}, \\ &|D'\mathrm{Ric}|^2 = g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} \mathrm{Ric}_{i\bar{j},k} \overline{\mathrm{Ric}_{\alpha\bar{\beta},\gamma}}, \\ &|D'R|^2 = g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{l\bar{\delta}} g^{\varepsilon\bar{p}} R_{i\bar{j}k\bar{l},p} \overline{R_{\alpha\bar{\beta}\gamma\bar{\delta},\varepsilon}}, \\ &\mathrm{div}\,\mathrm{div}(\rho\mathrm{Ric}) = 2|D'\rho|^2 + g^{\beta\bar{i}} g^{j\bar{\alpha}} \mathrm{Ric}_{i\bar{j}} \frac{\partial^2\rho}{\partial z_\alpha \partial \bar{z}_\beta} + \rho\Delta\rho, \\ &\mathrm{div}\,\mathrm{div}(R, \mathrm{Ric}) = -g^{\beta\bar{i}} g^{j\bar{\alpha}} \mathrm{Ric}_{i\bar{j}} \frac{\partial^2\rho}{\partial z_\alpha \partial \bar{z}_\beta} - 2|D'\mathrm{Ric}|^2 \\ &\qquad\qquad + g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{l\bar{\delta}} R_{i\bar{j},k\bar{l}} R_{\beta\bar{\alpha}\delta\bar{\gamma}} - R(\mathrm{Ric}, \mathrm{Ric}) - \sigma_3(\mathrm{Ric}), \\ &R(\mathrm{Ric}, \mathrm{Ric}) = g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{l\bar{\delta}} R_{i\bar{j}k\bar{l}} \mathrm{Ric}_{\beta\bar{\alpha}} \mathrm{Ric}_{\delta\bar{\gamma}}, \end{aligned}$$

$$\begin{aligned} \mathrm{Ric}(R,R) &= g^{\alpha\bar{i}} g^{j\bar{\beta}} g^{\gamma\bar{k}} g^{\delta\bar{p}} g^{q\bar{\varepsilon}} \mathrm{Ric}_{i\bar{j}} R_{\beta\bar{\gamma}p\bar{q}} R_{k\bar{\alpha}\varepsilon\bar{\delta}}, \\ \sigma_3(\mathrm{Ric}) &= g^{\delta\bar{i}} g^{j\bar{\alpha}} g^{\beta\bar{\gamma}} \mathrm{Ric}_{i\bar{j}} \mathrm{Ric}_{\alpha\bar{\beta}} \mathrm{Ric}_{\gamma\bar{\delta}}, \end{aligned} \tag{4.7}$$

where the $g^{j\bar{i}}$'s denote the entries of the inverse matrix of the metric (i.e. $g_{k\bar{i}} g^{j\bar{i}} = \delta_{kj}$) and " ,p" represents the covariant derivative in the direction $\frac{\partial}{\partial z_p}$ and we are using the summation convention for repeated indices. The reader is also referred to [49] and [47] for a recursive formula for the coefficients a_j's and an alternative computation of a_j for $j \leqslant 3$ using Calabi's diastasis function (see also [74] for a graph-theoretic interpretation of this recursive formula).

Therefore, the study of elliptic PDEs $a_j = f$ (in [57] Z. Lu and G. Tian proved their ellipticity), where $j \geqslant 2$ and f is a smooth function on M, makes sense regardless of the existence of Tian-Yau-Zelditch expansion and so given any Kähler manifold (M, g) it makes sense to call the a_j's the *TYZ coefficients associated to metric g.*

4.2 Radial metrics

Before proving our result, we summarize for the reader's convenience the content of a recent paper written by Z. Feng, because some techniques developed there will be resumed later.

Theorem 4.4 (Z. Feng [34]). *The only radial Kähler potentials defined on a complex domain of dimension n which have constant first and second Tian-Yau-Zelditch coefficient are*

1. *the Euclidean metric,*

2. *constant multiple of the hyperbolic metric defined on $\mathcal{B}^n_{\frac{1}{\lambda}}$,*

3. *constant multiple of the Fubini-Study metric.*

Moreover, if $n = 2$

4. $|z|^2 + \lambda \log |z|^2$ *on* $\mathbb{C}^2 \backslash \{0\}$,

5. $\mu|z|^2 - \lambda \log |z|^2$ *on* $\mathbb{C}^2 \backslash \overline{\mathcal{B}}^2_{\frac{\lambda}{\mu}}$,

6. $-\mu\big(\log(1 - \xi|z|^{2\zeta}) + \frac{1-\zeta}{2} \log |z|^2\big)$ *on* $\mathcal{B}^2_{(\frac{1}{\xi})^{\frac{1}{\zeta}}} \backslash \overline{\mathcal{B}}^2_{((\frac{1}{\xi})(\frac{1-\zeta}{1+\zeta}))^{\frac{1}{\zeta}}}$,

7. $-\mu\big(\log(1 - \xi|z|^{2(\lambda+1)}) - \frac{\lambda}{2} \log |z|^2\big)$ *on* $\mathcal{B}^2_{(\frac{1}{\xi})^{\frac{1}{\lambda+1}}} \backslash \{0\}$,

8. $\mu\big(\log(1 + \xi|z|^{-2(\lambda+1)}) + \frac{2+\lambda}{2} \log |z|^2\big)$ *on* $\mathbb{C}^2 \backslash \mathcal{B}^2_{((\frac{1}{\xi})(\frac{\lambda}{2+\lambda}))^{\frac{1}{\lambda+1}}}$,

9. $\mu\big(\log(1 + |z|^{-2\zeta}) + \frac{1+\zeta}{2} \log |z|^2\big)$ *on* $\mathbb{C}^2 \backslash \{0\}$,

10a. $-\mu\big(\log(-\log |z|^2 + \kappa) + \frac{\log |z|^2}{2}\big)$ *on* $\mathcal{B}^2_{e^{\kappa}} \backslash \overline{\mathcal{B}}^2_{e^{\kappa-2}}$,

11a. $-\mu\big(\log |\cos(\lambda \log |z|^2 + \kappa)| + \frac{\log |z|^2}{2}\big)$ *on* $\mathcal{B}^2_{r_1(h,\kappa,\lambda)} \backslash \overline{\mathcal{B}}^2_{r_3(h,\kappa,\lambda)}$,

or if $n = 1$

10b. $-\mu \log |-\log |z|^2 + \kappa|$ *on* $\mathbb{C} \backslash (\partial \mathcal{B}_{e^{\kappa}} \cup \{0\})$,

11b. $-\mu \log |\cos(\lambda \log |z|^2 + \kappa)|$, *on* $\mathcal{B}_{r_1(h,\kappa,\lambda)} \backslash \overline{\mathcal{B}}_{r_2(h,\kappa,\lambda)}$.

Where of $\mathcal{B}^n_r$ *denotes the ball of radius* r *in* $\mathbb{C}^n$ *(cf. ex. 1.2),* $\partial \mathcal{B}^n_r$ *denotes the boundary of* $\mathcal{B}^n_r$, $\overline{\mathcal{B}}^n_r = \mathcal{B}^n_r \cup \partial \mathcal{B}^n_r$, $\mu, \lambda, \xi, \zeta \in \mathbb{R}^+$, $0 < \zeta < 1$, $\kappa \in \mathbb{R}$, $h \in \mathbb{Z}$, $r_1(h, \kappa) = e^{\frac{1}{\lambda}(\frac{2h+1}{2}\pi - \kappa)}$, $r_2(h, \kappa, \lambda) = e^{\frac{1}{\lambda}(\frac{2h-1}{2}\pi - \kappa)}$ *and* $r_3(h, \kappa, \lambda) = e^{\frac{1}{\lambda}(h\pi + \arctan(\frac{1}{2\lambda}) - \kappa)}$.

Proof. Let $2\Phi(\log r)$ be a Kähler potential defined on a radial complex domain of complex dimension n, where $r = |z_1|^2 + \ldots + |z_n|^2$.

Firstly, we classify Kähler metrics related to such kind of potentials which have constant scalar curvature (namely a_1 is constant). By definition, the metric tensor reads as

$$g_{i\bar{j}} = \frac{\partial^2 \Phi(\log r)}{\partial z_i \partial \bar{z}_j} = \frac{\Phi'' - \Phi'}{r^2} \bar{z}_i z_j + \frac{\Phi'}{r} \delta_{ij}, \tag{4.8}$$

where δ_{ij} is the Kronecker delta and Φ' represents the first derivative of Φ with respect to

$$t = \log r.$$

Since Riemannian metrics are positive definite, Φ'' needs to be a positive function. Hence we can consider the following substitutions

$$\begin{cases} y = \Phi'(t), \\ \psi(y) = \Phi''(t). \end{cases}$$

From eq. (4.8), we easily get

$$\det\left(g_{i\bar{j}}\right) = \frac{(\Phi')^{n-1}\Phi''}{r^n}.$$

By considering that

$$\frac{d}{dt}\Big((n-1)\log\Phi' + \log\Phi''\Big) = (n-1)\frac{\Phi''}{\Phi'} + \frac{\Phi'''}{\Phi''} = (n-1)\frac{\psi}{y} + \psi',$$

we introduce the further substitution

$$\sigma(y) = \frac{(y^{n-1}\psi)'}{y^{n-1}} = (n-1)\frac{\psi(y)}{y} + \psi'(y),$$

and we compute the Ricci tensor's components:

$$\mathrm{Ric}_{i\bar{j}} = -\frac{\partial^2 \log\det\left(g_{i\bar{j}}\right)}{\partial z_i \partial \bar{z}_j} = \frac{-\sigma'\psi + \sigma - n}{r^2} \bar{z}_i z_j + \frac{n-\sigma}{r}\delta_{ij}, \tag{4.9}$$

to be in the position to represent the scalar curvature as a function of y:

$$\rho = g^{i\bar{j}} Ric_{i\bar{j}} = \frac{n(n-1)}{y} - \frac{(n-1)\sigma + \sigma' y}{y}.$$

Thus we reach our initial objective by solving the ODE obtained by imposing ρ to be constant, namely

$$\frac{n(n-1)}{y} - \frac{(y^{n-1}\psi)''}{y^{n-1}} = -An(n+1),$$

whose solutions are

$$\psi(y) = Ay^2 + y + \frac{B}{y^{n-2}} + \frac{C}{y^{n-1}}, \tag{4.10}$$

where $A, B, C \in \mathbb{R}$.

Now, we determinate which conditions have to be satisfied so that the previous metrics also verify the PDE

$$a_2 = K, \tag{4.11}$$

where K is a real constant.

We are going to represent the first term in the previous PDE (see eq. 4.6) as a function of y, $\psi(y)$ and its derivatives, in order to convert it to an ODE.

Riemann tensor's components $R_{i\bar{j}k\bar{l}}$ are by definition equal to $\frac{\partial^2 g_{i\bar{l}}}{\partial z_k \partial \bar{z}_j} - g^{p\bar{q}} \frac{\partial g_{i\bar{p}}}{\partial z_k} \frac{\partial g_{q\bar{l}}}{\partial \bar{z}_j}$. Therefore, the following derivatives (cf. eq. 4.8)

$$\begin{aligned} \frac{\partial g_{i\bar{l}}}{\partial z_k} &= \frac{\Phi''-\Phi'}{r^2}(\delta_{kl}\bar{z}_i + \delta_{il}\bar{z}_k) + \frac{\Phi'''-3\Phi''+2\Phi'}{r^3}\bar{z}_i z_l \bar{z}_k \\ \frac{\partial^2 g_{i\bar{l}}}{\partial z_k \partial \bar{z}_j} &= \frac{\Phi''-\Phi'}{r^2}(\delta_{kl}\delta_{ij} + \delta_{il}\delta_{kj}) + \frac{\Phi''''-6\Phi'''+11\Phi''-6\Phi'}{r^4}\bar{z}_i z_j z_l \bar{z}_k \\ &\quad + \frac{\Phi'''-3\Phi''+2\Phi'}{r^3}(\delta_{kj}\bar{z}_i z_l + \delta_{kl} z_j \bar{z}_i + \delta_{il} z_j \bar{z}_k + \delta_{ij} z_l \bar{z}_k) \end{aligned} \tag{4.12}$$

allow us to state that the unique (up to consider tensor's symmetries) nonvanishing components in $(z_1, 0, \dots, 0)$ are equal to

$$\begin{aligned} R_{1\bar{1}1\bar{1}} &= \tfrac{\psi''\psi^2}{r^2}, \\ R_{1\bar{1}i\bar{i}} &= \tfrac{\psi' y - \psi}{y r^2}\psi, \\ R_{i\bar{i}i\bar{i}} &= 2R_{i\bar{i}j\bar{j}} = 2\tfrac{\psi - y}{r^2}, \end{aligned} \tag{4.13}$$

where $i \neq j$ and $i, j \neq 1$. By taking into account the invariance of $|R|^2$ under the action of the unitary group, we can use Riemann tensor's components of formula (4.13) and the metric tensor's components evaluated in $(z_1, 0, \dots, 0)$ (cf. eq. 4.8) to get the general formula

$$\begin{aligned} |R|^2 &= (g^{1\bar{1}})^4 (R_{1\bar{1}1\bar{1}})^2 + 4(n-1)(g^{1\bar{1}})^2 (g^{i\bar{i}})^2 (R_{1\bar{1}i\bar{i}})^2 \\ &\quad + (g^{i\bar{i}})^4 \big(4\tfrac{(n-2)(n-1)}{2}(R_{i\bar{i}j\bar{j}})^2 + (n-1)(R_{i\bar{i}i\bar{i}})^2\big) \\ &= (\psi'')^2 + 4(n-1)\big(\tfrac{\psi' y - \psi}{y^2}\big)^2 + 2n(n-1)\big(\tfrac{\psi - y}{y^2}\big)^2. \end{aligned} \tag{4.14}$$

Similarly, we compute

$$\begin{aligned} |Ric|^2 &= (g^{1\bar{1}})^2 (Ric_{1\bar{1}})^2 + (n-1)(g^{i\bar{i}})^2 (Ric_{i\bar{i}})^2 \\ &= (\sigma')^2 + (n-1)\big(\tfrac{\sigma - n}{y}\big)^2. \end{aligned} \tag{4.15}$$

Therefore the equation (4.11) is equivalent to

$$\begin{aligned} 4(n-1)\Big(\frac{\psi' y - \psi}{y^2}\Big)^2 &+ 2n(n-1)\Big(\frac{\psi - y}{y^2}\Big)^2 \\ &- 4\Big((\sigma')^2 + (n-1)\Big(\frac{\sigma - n}{y}\Big)\Big) + (\psi'')^2 \\ &\qquad = 24K - 3A^2 n^2 (n+1)^2. \end{aligned}$$

By imposing in the previous ODE, ψ to be equal to the solution (4.10), we get

$$\begin{aligned} (n-1)\,\Big(&2BCn^2 y^{-2n-1}(n+1) + C^2 n y^{-2(n+1)}(n+1)(n+2) \\ &+ B^2 n y^{-2n}(n+1)(n-2) + A^2 n(n+1)(3n+2)\Big) - 24K = 0. \end{aligned}$$

Hence, the previous equation holds true if and only if

$$\begin{cases} C = B = 0,\ K = A^2 n(n+1)(n-1)(3n+2)/24 & \text{if } n \geqslant 2 \\ C = 0,\ K = 2A^2 & \text{if } n = 2 \\ K = 0 & \text{if } n = 1 \end{cases}$$

namely, by definition of ψ and y and by renaming the constants for more convenience, one gets

$$\Phi'' = \begin{cases} a(\Phi')^2 + \Phi' & \text{if } n \geqslant 2, \\ a(\Phi')^2 + \Phi' + c & \text{if } n = 2, \\ a(\Phi')^2 + b\Phi' + c & \text{if } n = 1. \end{cases} \tag{4.16}$$

Let suppose that $a \neq 0$. Therefore

$$\Phi''(t) = a\left(\left(\Phi' + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right). \tag{4.17}$$

We distinguish now three different possibilities.

In the first case, we suppose that $\frac{c}{a} - \frac{b^2}{4a^2} = -D^2$, where $D \in \mathbb{R}^+$. By solving this ODE, we get

$$\left|\frac{\Phi'(t) + \frac{b}{2a} - D}{\Phi'(t) + \frac{b}{2a} + D}\right| = \xi e^{2aDt},$$

where $\xi \in \mathbb{R}^+$. Thus we have to distinguish two further different cases. The first one occurs when $\Phi' + \frac{b}{2a} - D < 0 < \Phi' + \frac{b}{2a} + D$, hence

$$\Phi'(t) = -\frac{\xi e^{2aDt}}{\xi e^{2aDt} + 1}\left(D + \frac{b}{2a}\right) + \frac{D - \frac{b}{2a}}{\xi e^{2aDt} + 1},$$

namely

$$\Phi(\log r) = -\frac{1}{a}\left(\log(1 + \xi r^{2aD}) + \left(\frac{b}{2} - Da\right)\log r\right).$$

Since $\Phi'' > 0$, then $a < 0$. If $n \geqslant 2$, also Φ' has to be positive by definition of Kähler potential. This condition is trivially satisfied if $n > 2$, while if $n = 2$ it is equivalent to $(2aD+1)\xi r^{2aD} > 2aD-1$. Moreover, if $n = 1$, we recall that $\partial\bar{\partial}\log r = 0$.
Otherwise we consider the cases where $\Phi' + \frac{b}{2a} - D > 0$ or $\Phi' + \frac{b}{2a} + D < 0$. Therefore

$$\Phi'(t) = -\frac{\xi e^{2aDt}}{\xi e^{2aDt} - 1}\left(D + \frac{b}{2a}\right) - \frac{D - \frac{b}{2a}}{\xi e^{2aDt} - 1},$$

namely

$$\Phi(\log r) = -\frac{1}{a}\left(\log|1 - \xi r^{2aD}| + \left(\frac{b}{2} - Da\right)\log r\right).$$

Since $\Phi'' > 0$, then $a > 0$. If $n \geqslant 2$, also Φ' has to be positive. This condition is satisfied if and only if $\xi r^{2aD} < 1$ and $\xi r^{2aD} > \frac{1-2aD}{2aD+1}$ if $n = 2$.

Now we consider $\frac{c}{a} - \frac{b^2}{4a^2} = 0$. This case may occur only if $n = 1$ or $n = 2$. By solving the initial ODE (4.17), we get

$$\Phi(\log r) = -\frac{1}{a}\log|\log r + \kappa| - \frac{b}{2a}\log r,$$

where $\kappa \in \mathbb{R}$. Since $\Phi'' > 0$, then $a > 0$. If $n = 2$, also Φ' has to be positive, hence $-2 - \kappa < \log r < -\kappa$ (in this case $b = 1$).

Finally we suppose $\frac{c}{a} - \frac{b^2}{4a^2} = D^2$. Also this case may occur only if $n = 1$ or $n = 2$. By solving the initial ODE (4.17), we get

$$\Phi(\log r) = -\frac{1}{a}\log|\cos(aD\log r + \kappa)| - \frac{b}{2a}\log r,$$

where $\kappa \in \mathbb{R}$. Since $\Phi'' > 0$, then $a > 0$. If $n = 2$, also Φ' has to be positive, hence $\frac{1}{aD}(\arctan(\frac{1}{2aD}) + h\pi - \kappa) < \log r < \frac{1}{aD}(\frac{\pi}{2} + h\pi - \kappa)$, where $h \in \mathbb{Z}$.

To conclude we have to consider that a may also be equal to 0, hence we have to solve

$$\Phi''(t) = b\Phi' + c.$$

If $b \neq 0$, we get

$$\Phi(\log r) = \frac{1}{b^2}\xi r^b - \frac{c}{b}\log r.$$

If $n \geqslant 2$, Φ' has to be positive, hence $\xi r^b > \frac{c}{b}$. This condition is always satisfied if $n > 2$, since $c = 0$. Moreover Φ'' is always positive.

Instead, if $b = 0$ (this case may occur only if $n = 1$), we get

$$\Phi(\log r) = \frac{c}{2}(\log r)^2 + \kappa \log r,$$

which is the Euclidean metric (already considered in the previous case). □

Remark 4.1 (Simanca's metric). Beyond the Euclidean metric, the unique scalar flat metric in the statement of the previous theorem are the ones related to the potentials (4) and (5), as it can be deduced from the proof. Among them, we recognize the remarkable example of the *Simanca's metric*. In [66] Simanca constructs a scalar flat (non Ricci-flat) Kähler complete metric on the blow-up $\tilde{\mathbb{C}}^2$ of $\mathbb{C}^2$ at the origin, whose Kähler potential on $\mathbb{C}^2 \setminus \{0\}$ is equal to the potential (4) for $\lambda = 1$.

Now, we propose the classification of the cscK radial metrics, whose third TYZ coefficient is constant.

Proof of Theorem 4.1. As in the proof of Theor. 4.4, we are going to represent the first term in the following PDE

$$a_3 = K, \tag{4.18}$$

where K is a real constant, as a function of y, $\psi(y)$ and its derivatives (see eq. 4.6), in order to convert it to an ODE and then to determinate which conditions have to satisfy a radial cscK metric (4.10) to verify it.

The same arguments about unitary invariance of terms of a_2 shown in the proof of Theor. 4.4 hold true also in this case for the term a_3. Therefore, we can easily compute by using eq. (4.13) and (4.8) evaluated at $(z_1, 0, \dots, 0)$ the following terms:

$$\begin{aligned}
&\mathrm{Ric}(R,R)\\
&\quad= (g^{1\bar1})^2 Ric_{1\bar1}\big((g^{1\bar1})^3(R_{1\bar11\bar1})^2 + 2(n-1)g^{1\bar1}(g^{i\bar i})^2(R_{1\bar1i\bar i})^2\big)\\
&\qquad\quad +(g^{i\bar i})^5 Ric_{i\bar i}\big((n-1)(R_{i\bar ii\bar i})^2 + 2(n-1)(n-2)(R_{i\bar ij\bar j})^2\big)\\
&\qquad\quad +2(n-1)(g^{1\bar1})^2(g^{i\bar i})^3 Ric_{i\bar i}(R_{1\bar1i\bar i})^2\\
&\quad= 2(n-1)\tfrac{n-\sigma}{y^5}\Big(n(\psi-y)^2 + (\psi' y-\psi)^2\Big)\\
&\qquad\quad -\sigma'\Big((\psi'')^2 + 2(n-1)\tfrac{(\psi' y-\psi)^2}{y^4}\Big);
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&R(\mathrm{Ric},\mathrm{Ric})\\
&\quad= (g^{1\bar1})^4(Ric_{1\bar1})^2 R_{1\bar11\bar1} + 2(n-1)(g^{1\bar1})^2(g^{i\bar i})^2 Ric_{1\bar1}Ric_{i\bar i}R_{1\bar1i\bar i}\\
&\qquad\quad +(g^{i\bar i})^4(Ric_{i\bar i})^2((n-1)R_{i\bar ii\bar i} + (n-1)(n-2)R_{i\bar ij\bar j})\\
&\quad= (\sigma')^2\psi'' - 2(n-1)\tfrac{(\psi' y-\psi)(n-\sigma)\sigma'}{y^3} + n(n-1)\tfrac{(\psi-y)(n-\sigma)^2}{y^4}.
\end{aligned} \tag{4.20}$$

By using formulas (4.14) and (4.15), we define

$$\chi(y) = |R|^2 - 4|\mathrm{Ric}|^2,$$

to get

$$\begin{aligned}
\Delta(|R|^2 - 4|\mathrm{Ric}|^2) &= g^{i\bar j}\frac{\partial^2\chi}{\partial z_j\partial\bar z_i}\\
&= g^{i\bar j}\big(\frac{\chi'\psi}{r}\delta_{ji} + \frac{\chi''\psi + \chi'\psi' - \chi'}{r^2}\psi\bar z_j z_i\big) = (\chi'\psi)' + \frac{n-1}{y}\chi'\psi.
\end{aligned} \tag{4.21}$$

Since Christoffel's symbols Γ^p_{ki} are equal to $g^{p\bar q}\frac{\partial g_{i\bar q}}{\partial z_k}$, we easily deduce from (4.12) that they are all equal to zero when evaluated at $(z_1, 0, \ldots, 0)$ except for:

$$\begin{aligned} \Gamma^1_{11} &= \left(\tfrac{\Phi'''-\Phi''}{r^2}\right)\left(\tfrac{r}{\Phi''}\right)\bar{z}_1 = \tfrac{\psi'-1}{r}\bar{z}_1, \\ \Gamma^i_{1i} &= \left(\tfrac{\Phi''-\Phi'}{r^2}\right)\left(\tfrac{r}{\Phi'}\right)\bar{z}_1 = \tfrac{\psi-y}{yr}\bar{z}_1. \end{aligned}$$

We also compute that the unique first derivatives of Ricci tensor's components different from zero in $(z_1, 0, \ldots, 0)$ (cf. (4.9)) are:

$$\begin{aligned} \tfrac{\partial}{\partial \bar{z}_1} Ric_{1\bar{1}} &= \tfrac{\sigma'\psi-(\sigma'\psi)'\psi}{r^2} z_1, \\ \tfrac{\partial}{\partial \bar{z}_1} Ric_{i\bar{i}} &= \tfrac{\partial}{\partial \bar{z}_i} Ric_{i\bar{1}} = \tfrac{\sigma-\sigma'\psi-n}{r^2} z_1, \end{aligned}$$

and we use them to evaluate Ricci tensor's first covariant derivatives at $(z_1, 0, \ldots, 0)$, which are defined as $Ric_{i\bar{j},k} = \frac{\partial}{\partial z_k} Ric_{i\bar{j}} - Ric_{p\bar{j}}\Gamma^p_{ki}$, and we get

$$\begin{aligned} |D^{'}\mathrm{Ric}|^2 &= (g^{1\bar{1}})^3 |Ric_{1\bar{1},1}|^2 + 2(n-1) g^{1\bar{1}} (g^{i\bar{i}})^2 |Ric_{i\bar{i},1}|^2 \\ &= (\sigma'')^2\psi + 2(n-1)\tfrac{\psi}{y^2}\left(\sigma' + \tfrac{n-\sigma}{y}\right)^2. \end{aligned} \tag{4.22}$$

Because some Riemann tensor's components in $(z_1, 0, \ldots, 0)$ are equal to zero (cf. eq. (4.13)), we need to compute just some Ricci tensor's second covariant derivatives, of which we recall the definition: $Ric_{i\bar{j},k\bar{l}} = \partial_{\bar{l}}\partial_k Ric_{i\bar{j}} + \Gamma^q_{ki}\Gamma^{\bar{p}}_{\bar{l}\bar{j}} Ric_{q\bar{p}} - \Gamma^p_{ki}\partial_{\bar{l}} Ric_{p\bar{j}} - \partial_{\bar{l}}\Gamma^p_{ki} Ric_{p\bar{j}} - \Gamma^{\bar{p}}_{\bar{l}\bar{j}}\partial_k Ric_{i\bar{p}}$. In particular, we preliminary evaluate at $(z_1, 0, \ldots, 0)$ the following derivatives ($\partial_{\bar{l}}\Gamma^p_{ki} = g^{p\bar{q}}\frac{\partial^2 g_{i\bar{q}}}{\partial \bar{z}_l \partial z_k} - g^{p\bar{\gamma}} g^{\alpha\bar{q}} \frac{\partial g_{\alpha\bar{\gamma}}}{\partial \bar{z}_l}\frac{\partial g_{i\bar{q}}}{\partial z_k}$)

$$\begin{aligned} \partial_{\bar{1}}\Gamma^1_{11} &= \tfrac{\psi''\psi}{r}, \\ \partial_{\bar{i}}\Gamma^1_{i1} &= \tfrac{\psi' y-\psi}{yr}, \\ \partial_{\bar{1}}\Gamma^i_{i1} &= \tfrac{\psi' y-\psi}{y^2 r}\psi, \\ \partial_{\bar{i}}\Gamma^i_{ii} &= 2\partial_{\bar{j}}\Gamma^i_{ji} = 2\tfrac{\psi-y}{yr} \end{aligned}$$

and

$$\begin{aligned} \tfrac{\partial^2}{\partial z_1 \partial \bar{z}_1} Ric_{1\bar{1}} &= \tfrac{-((\sigma'\psi)'\psi)'\psi+2(\sigma'\psi)'\psi-\sigma'\psi}{r^2}, \\ \tfrac{\partial^2}{\partial z_1 \partial \bar{z}_1} Ric_{i\bar{i}} &= \tfrac{\partial^2}{\partial z_i \partial \bar{z}_1} Ric_{1\bar{i}} = \tfrac{-(\sigma'\psi)'\psi+2\sigma'\psi-\sigma+n}{r^2}, \\ \tfrac{\partial^2}{\partial z_i \partial \bar{z}_i} Ric_{i\bar{i}} &= 2\tfrac{\partial^2}{\partial z_j \partial \bar{z}_j} Ric_{i\bar{i}} = 2\tfrac{\partial^2}{\partial z_j \partial \bar{z}_i} Ric_{i\bar{j}} = 2\tfrac{-\sigma'\psi+\sigma-n}{r^2}, \end{aligned}$$

to be in the position to express

$$\begin{aligned}
&(g^{1\bar 1})^4 R_{1\bar 1 1\bar 1} Ric_{1\bar 1,1\bar 1} + 2(n-1)(g^{1\bar 1})^2 (g^{i\bar i})^2 R_{1\bar 1 i\bar i}(Ric_{1\bar 1,i\bar i} + Ric_{i\bar i,1\bar 1})\\
&\quad + (g^{i\bar i})^4\big((n-1)R_{i\bar i i\bar i} Ric_{i\bar i,i\bar i} + 2(n-1)(n-2) R_{i\bar i j\bar j} Ric_{i\bar i,j\bar j}\big)\\
&= \tfrac{\psi''}{\psi}\Big(-((\sigma'\psi)'\psi)' - (\psi')^2\sigma' + 2\psi'(\sigma'\psi)' + \sigma'\psi''\psi\Big)\\
&\quad +2(n-1)\tfrac{\psi' y-\psi}{y^3}\left(-2(\sigma'\psi)' + 2\tfrac{\psi}{y^2}(n-\sigma) + 4\tfrac{\psi}{y}\sigma'\right)\\
&\quad +2(n-1)\tfrac{\psi' y-\psi}{y^3}\left(\psi' - \tfrac{\psi}{y}\right)\left(\sigma' + \tfrac{\sigma-n}{y}\right)\\
&\quad +2n\tfrac{\psi}{y^5}(n-1)(\psi-y)(\sigma-n-\sigma' y)
\end{aligned} \tag{4.23}$$

as a function of y, ψ and its derivatives.

Therefore, thanks to the formulas (4.14), (4.15), (4.20), (4.19), (4.21), (4.22) and (4.23) we convert the PDE (4.18) into an ODE and by imposing ψ to be equal to the solution (4.10), we get the following equality

$n(n-1)(n+1)\Big(A^3 n y^{3n+3}(n+1)(n-2) + AB^2 n y^{n+3}(2-n)(n+3) - 2Bny^{n+2}\big(AC(n+1)(n+4) + B(n^2-4)\big) - Cy^{n+1}(n+1)(n+2)\big(AC(n+6)+4Bn\big) - 2C^2y^n(n+1)(n+2)(n+3) + 2B^3ny^3(2-n)(2n+1) - 6B^2Cn^2y^2(2n+1) - 6BC^2y(n+1)(2n^2+3n+2) - 2C^3(n+1)(n+2)(2n+3)\Big) = -48Ky^{3n+3}.$

Therefore, if $n=1$ and $K=0$ the previous equation holds true independently from A, B and C. If $n \neq 1$, C needs to be equal to 0. After this simplification the equality is satisfied if $K=0$ and $n=2$ or $n>2$, $B=0$ and $K = -\frac{A^3}{48}n^2(n+1)^2(n-1)(n-2)$.

To sum up, we have just proved that a_3 is constant if and only if

$$\psi(y) = \begin{cases} Ay^2 + y & \text{if } n \geqslant 2,\\ Ay^2 + y + B & \text{if } n = 2,\\ Ay^2 + By + C & \text{if } n = 1.\end{cases}$$

Therefore the claim follows by comparing the previous equation with the (4.16). □

Proof of Theorem 4.2. Since we can easily check that the unique Ricci-flat metric in the statement of Theor. 4.4 (see also Rem. 4.1) is the Euclidean one, the claim follows from the previous theorem. □

Moreover, we can also prove the validity of Conjecture 3 for complete ALE metrics.

Theorem 4.5. *Let (M, g) be a Ricci-flat Kähler surface such that the third coefficient a_3 of the TYZ expansion vanishes. Assume that g is complete and ALE, then g is flat.*

Roughly speaking, an n-dimensional complete Riemannian manifold (M, g) is said to be ALE if there exists a compact subset $K \subset M$ such that $M \backslash K$ is diffeomorphic to the quotient of $\mathbb{R}^n \backslash B_R(0)$ (the ball of radius $R > 0$) by a finite group $G \subset O(n)$, and such that the metric g on this open subset tends to the flat euclidean metric at infinity. For the exact definition and construction of ALE Kähler metrics, which are interesting both from the mathematical and the physical point of view, the reader is referred to the foundational paper [45] (see also [7], [41], [62], [60]): in our theorem's proof we will need just the fact that the norm of the curvature tensor of such metrics vanishes at infinity.

Proof of Theorem 4.5. By (4.6) the assumption $a_3 = 0$ implies $\Delta|R|^2 = 0$. By a celebrated result of Yau [76] (being M complete) (M, g) does not admit a nonconstant positive harmonic function. Hence $|R|^2$ is constant. Being g an ALE metric $|R|^2 = 0$ and so the metric g is forced to be flat. □

Remark 4.2. The assumption of Ricci-flatness in Conjecture 2 and 3 cannot be weakened to scalar-flatness. Indeed Simanca's metric (see Rem. 4.1) is a radial scalar flat metric with vanishing a_3, as proved in Theor. 4.1. Furthermore, the holomorphic map

$$\varphi : \mathbb{C}^2 \backslash \{0\} \to \mathbb{C}\mathrm{P}^\infty$$

given by

$$(z_1, z_2) \mapsto \left(z_1, z_2, \ldots, \sqrt{\frac{j+k}{j!k!}} z_1^j z_2^k, \ldots \right), \; j+k \neq 0,$$

is a Kähler immersion from $(\mathbb{C}^2 \backslash \{0\}, g_S)$ into $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$, where g_S denotes the restriction of the Simanca metric g_S to $\mathbb{C}^2 \backslash \{0\}$. Indeed

$$\begin{aligned}\varphi^* \omega_{FS} &= \frac{i}{2} \partial \bar{\partial} \log \sum_{j,k \in \mathbb{N},\; j+k \neq 0} \frac{j+k}{j!k!} |z_1|^{2j} |z_2|^{2k} \\ &= \frac{i}{2} \partial \bar{\partial} \log(|z|^2 e^{|z|^2}) = \frac{i}{2} \partial \bar{\partial} \Phi_S = \omega_S.\end{aligned}$$

Since $\tilde{\mathbb{C}}^2$ is simply-connected it follows by Theor. 1.6 that φ extends to a Kähler immersion from $(\tilde{\mathbb{C}}^2, g_S)$ into $(\mathbb{C}\mathrm{P}^\infty, g_{FS})$.

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