Università degli Studi di Cagliari

PHD DEGREE
in Mathematics and Computer Science

Cycle 33rd

Quantizations of Kähler metrics on blow-ups

Scientific Disciplinary Sector
MAT/03

PhD Student: **Francesco Cannas Aghedu**

Supervisor: **Professor Andrea Loi**

Final exam. Academic Year 2019–2020
Thesis defence: February 2021 Session

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To my parents and Andrea
who have always supported and
motivated me to never give up.
I couldn't have done it without all of you!

Contents

Acknowledgements

First of all, my sincere gratitude goes to my mentor Prof. Andrea Loi for accepting me as his student, for his guidance through each phase of my doctoral research. His deep knowledge helped me in all the time of the research and writing of this thesis.

I am also very grateful to several other people in Cagliari University: colleagues, professors, administrative staff have always been very helpful and kind during my studies. Thank you for this.

Besides Cagliari University, I also spent three months at King's College London: I would like to thank Prof. Simon Salamon for accepting me as visiting research assistant and for the interesting mathematical discussions. In addition I spent two months in the Laboratory of Geometry and Dynamical Systems of the Polytechnic University of Catalonia: I wish to extend my thanks to Prof. Eva Miranda for inviting me as visiting PhD Student and to all the students of the Lab for their hospitality. Finally I spent three months in the Department of Mathematics and Statistics at University of São Paulo: a special thanks to Prof. Roberto Mossa for inviting me, for his support and for his warm hospitality.

I would like also to thank Prof. Michela Zedda and Prof. Roberto Mossa for agreeing to referee my PhD thesis.

Declaration

To the best of my knowledge, I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification.

Abstract

The present thesis consists of three main results (contained in [14, 15]) related to Kähler metrics on blow-ups. In the first one, we prove that the blow-up $\tilde{\mathbb{C}}^2$ of $\mathbb{C}^2$ at the origin endowed with the Burns–Simanca metric g_{BS} admits a regular quantization. We use this fact to prove that all coefficients in the Tian-Yau-Catlin-Zelditch expansion for the Burns–Simanca metric vanish and that a dense subset of $(\tilde{\mathbb{C}}^2, g_{BS})$ admits a Berezin quantization. In the second one, we prove that the generalized Burns–Simanca metric on the blow-up $\tilde{\mathbb{C}}^n$ of $\mathbb{C}^n$ at the origin is projectively induced but not balanced for any integer $n \geq 3$. Finally, we prove as third result that any positive integer multiple of the Eguchi–Hanson metric, defined on a dense subset of $\tilde{\mathbb{C}}^2/\mathbb{Z}_2$, is not balanced.

Introduction

The modern theory of quantization was developed in the second half of the twentieth century and the term *quantization* mainly refers to a construction for passing from a classical mechanics system to the "corresponding" quantum system, which has the classical system as its limit. It is known however, that not all quantum systems have a classical counterpart and moreover, several quantum systems may reduce to the same classical theory. From the mathematical point of view, there are also obstacles of different kinds, namely no general theorem of existence of quantization which satisfies the physical interpretation. As a result, nowadays we are faced the existence of many different quantization theories, ranging from geometric quantization, deformation quantization, Berezin quantization, asymptotic quantization or stochastic quantization, to mention just a few. None of the existing approaches completely solve the quantization problem; on the other hand, on the mathematics side all these have evolved into rich theories of their own right, and with results of great depth and beauty.

The main theme of this thesis is the study of quantizations of Kähler metrics on complex blow-ups. In particular geometric quantization and Berezin quantization of the blow-up of $\mathbb{C}^2$ at the origin endowed with suitable Kähler metrics.

A *geometric quantization* of a Kähler manifold (M, ω) is a pair $(L, \mathfrak{h})$, where L is a holomorphic line bundle over M and $\mathfrak{h}$ is a Hermitian structure on L such that $\operatorname{curv}(L, \mathfrak{h}) = -2\pi i\omega$. The line bundle L is called *quantum line*

bundle of (M,ω). A Kähler manifold (M,ω) is said to be *quantizable* if it admits a geometric quantization (see Section 2.3).
Not all Kähler manifolds admit a geometric quantization. In terms of cohomology classes a Kähler manifold (M,ω) admits a geometric quantization $(L,\mathfrak{h})$ if and only if $c_1(L)=[\omega]$ (see [38]).
Consider the separable complex Hilbert space [12] $\mathcal{H}_{\mathfrak{h}}$ consisting of global holomorphic sections s of L which are L^2-bounded, namely

$$\langle s,s\rangle_{\mathfrak{h}} := \int_M h(s(x),s(x))\frac{\omega^n(x)}{n!} < \infty.$$

Under suitable conditions (see Subsection 2.4.1), one can define the so called *epsilon function* of the pair $(L,\mathfrak{h})$, that is a smooth real valued function on M defined, for any $x\in M$, by

$$\epsilon_{(L,\mathfrak{h})}(x)=\sum_{j=0}^{N} h(s_j(x),s_j(x)),$$

where $\{s_j\}_{j=0,\ldots,N}$, $(\dim\mathcal{H}_{\mathfrak{h}}=N+1\leq\infty)$ is an orthonormal basis for $\mathcal{H}_{\mathfrak{h}}$. The metric g on M is called *balanced* if $\epsilon_{(L,\mathfrak{h})}$ is a positive constant. The definition of balanced metric was originally given by Donaldson [20] in the case of compact quantizable Kähler manifold and generalized in [5] to the non compact case (see also [25, 30, 48]).
Consider now the quantum line bundle $(L^m,\mathfrak{h}_m)$ for $(M,m\omega)$, where L^m is the m-th tensor power of L and $\mathfrak{h}_m$ is the m-th tensor power of $\mathfrak{h}$. A geometric quantization $(L,\mathfrak{h})$ of a Kähler manifold (M,g) is called a *regular quantization* if mg is balanced for any (sufficiently large) natural number m (see Section 2.4) i.e. if $\epsilon_{(L^m,\mathfrak{h}_m)}$ is a positive constant for any (sufficiently large) natural number m..
Many authors (see, e.g. [3, 7] and references therein) have been trying to understand what kind of properties are enjoyed by those Kähler manifolds which admit a regular quantization. Here we recall two facts.

- A Kähler metric which admits a regular quantization is cscK (constant scalar curvature Kähler) metric (see [43]).

- A geometric quantization of a homogeneous and simply connected Kähler manifold is regular (see [3, 41]).

Therefore, the following question naturally arises:

Question 1. *Is it true that a complete Kähler manifold* (M, ω) *which admits a regular quantization is necessarily homogeneous (and simply-connected)?*

The assumption of completeness is necessary otherwise one can construct regular quantizations on non-homogeneous Kähler manifolds obtained by deleting a measure zero set from a homogeneous Kähler manifold (see [45] for more details). The connection request is in brackets since one can prove that every homogeneous and projectively induced Kähler manifold is simply connected (see [18]). In the compact case the previous question is still open and of great interest also because the Kähler manifolds involved are projectively algebraic.
In this thesis we give a negative answer to Question 1 in the non compact case by considering the Burns–Simanca metric g_{BS} [10, 39, 60] on the complex blow-up $\tilde{\mathbb{C}}^2$ of $\mathbb{C}^2$ at the origin. The Burns–Simanca metric is an important example (both from mathematical and physical point of view) of non-homogeneous complete, zero constant scalar curvature metric (see Section 3.1). The main result is then the following:

Theorem 1. *Let* $\tilde{\mathbb{C}}^2$ *be the blow-up of* $\mathbb{C}^2$ *at the origin endowed with the Burns–Simanca metric* g_{BS}*. Then* $(\tilde{\mathbb{C}}^2, g_{BS})$ *admits a regular quantization such that* $\epsilon_{mg_{BS}} = m^2$*.*

We believe Theorem 1 could be used to built regular quantizations of non-homogeneous compact Kähler manifolds.
We also prove a result on Berezin quantization on the dense subset $\mathbb{C}^2 \setminus \{0\}$ of $\tilde{\mathbb{C}}^2$ equipped with the restriction of the Burns–Simanca Kähler form ω_{BS} associated to the metric g_{BS} (see Section 3.3). This is expressed by the following corollary:

Corollary 1. $(\mathbb{C}^2 \setminus \{0\}, g_{BS})$ *admits a Berezin quantization.*

The construction in the proof of Theorem 1 stops to work when $\mathbb{C}^2$ in replaced by $\mathbb{C}^n$, $n \geq 3$ and the metric g_{BS} is replaced by its natural generalization $g_{S(n)}$ on the blow-up $\tilde{\mathbb{C}}^n$ of $\mathbb{C}^n$ at the origin (see Chapter 4). This is expressed by the following theorem:

Theorem 2. *Let $\tilde{\mathbb{C}}^n$ be the blow-up of $\mathbb{C}^n$ at the origin endowed with the generalized Burns–Simanca metric $g_{BS(n)}$. For any integer $m \geq 1$ the following statements hold*

1. *$(\tilde{\mathbb{C}}^n, mg_{BS(n)})$ is projectively induced for any $n \geq 2$,*
2. *$mg_{BS(n)}$ is not balanced for all $n \geq 3$.*

The theorem gives also an example of Kähler metric g on the blow-up of $\mathbb{C}^n$ at the origin such that mg is projectively induced but it is not balanced for any positive integer m. Here a Kähler metric g on a complex manifold M is said to be *projectively induced* if there exists a Kähler immersion (isometric and holomorphic) of (M, g) into the complex projective space $(\mathbb{C}P^N, g_{FS}), N \leq +\infty$, endowed with the Fubini-Study metric g_{FS}. It is important to stress the link between balanced metrics and projectively induced ones: it can be shown that a balanced metric is projectively induced via a suitable map (see subsection 2.4.3), but in general the converse is not true.
It is a classical and interesting open problem in Kähler geometry characterizing Kähler metrics which are projectively induced.
It is a well-known theorem by John Nash [56] that any Riemannian manifold admits an isometric smooth embedding into the real Euclidean space $\mathbb{R}^N$, for sufficiently large N. In contrast with the Riemannian case a Kähler manifold does not always admit a Kähler immersion into the complex Euclidean space $\mathbb{C}^N$ (not even if N is infinite).
For example, there are no holomorphic immersion of compact complex manifolds into $\mathbb{C}^N$ for any positive value of N [66], since global holomorphic functions on connected compact complex manifolds are necessarily constant [66] and the maximum principle for holomorphic functions on domains in $\mathbb{C}^N$ is also valid [32]. But even if we consider non compact manifolds, there are still many obstructions to the existence of such an immersion as can be seen in the seminal paper of Eugenio Calabi [13] (see also [49]).

Notice that when one considers $(\mathbb{C}P^n, g_{FS})$, with finite dimension n, the only known examples of complete Kähler-Einstein projectively induced metrics are compact and homogeneous and it is still an open problem to show that these are the only possibilities (see [4, 6, 18, 63, 64] for more details).

D. Hulin [34] proves that a compact Kähler-Einstein manifold Kähler immersed into $\mathbb{C}P^N$ has positive scalar curvature. This result implies for examples that a Calabi–Yau manifold, namely a Ricci-flat compact Kähler manifolds, does not admit a Kähler immersion into $\mathbb{C}P^N$. The first example of Ricci-flat (non-flat) Kähler metric constructed on non-compact complex manifolds is the celebrated *Taub-NUT metric* described by C. Le Brun in [40]. This is a 1-parameter family of complete Kähler metrics on $\mathbb{C}^2$ defined by the Kähler potential $\Phi_m(u, v) = u^2 + v^2 + m(u^4 + v^4)$, where $m \geq 0$ and u and v are implicitly given by $|z_1| = e^{m(u^2+v^2)}u, |z_2| = e^{m(v^2-u^2)}v$. Then one can prove [50, Lemma 5, p. 522] that for $m > 1/2$ the Taub-NUT metric is not projectively induced. Actually, with the same techniques in [50], one can prove the non existence of a Kähler immersion also for smaller values of the parameter. Although, it is hard to prove it in general for any $m > 0$. Observe that for $m = 0$ the Taub-NUT metric reduces to be the flat metric on $\mathbb{C}^2$. It is well known that the flat metric on $\mathbb{C}^n$ admits a Kähler immersion into the infinite dimensional complex projective space. In [47] the authors conjecture that this is the only possible case proposing the following:

Conjecture 1. *A Ricci-flat projectively induced metric is flat.*

They verify the conjecture under the assumptions that the metric involved is stable-projectively induced (see [47] for more details) and radial, i.e. the Kähler potential depends only on the sum $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$ of the local coordinates' moduli (see [51] and [67] for details). In Chapter 5 we provide a result in line with the conjecture by restricting it to the class of balanced metrics, in particular by considering the Eguchi–Hanson metric g_{EH} [21, 23] on $\tilde{\mathbb{C}}^2/\mathbb{Z}_2$. More exactly we prove the following theorem

Theorem 3. *The restriction of the metric mg_{EH} on $\mathbb{C}^2 \setminus \{0\}$ is not balanced for any positive integer m.*

The Eguchi–Hanson metric is an interesting example of complete Ricci-flat Kähler metric (non-flat). It is also important from mathematical physics point of view since it is the prototypical example of a gravitational instantons [22], which consists of a 4-real manifold endowed with a complete, non-singular, positive definite metric which satisfies the Einstein equation.
The previous theorem can be extended to the whole manifold $\tilde{\mathbb{C}}^2/\mathbb{Z}_2$. Indeed, more recently than [14], A. Loi, M. Zedda and F. Zuddas showed that mg_{EH} is not projectively induced for any positive integer m (see [51, Cor. 1]).

The thesis is divided into five chapters which are organized as follows.
Chapter 1 contains some preliminaries about complex and Kähler manifolds (Section 1.1 and 1.2, respectively), a briefly introduction to the immersions of a Kähler manifolds into the complex projective spaces (Section 1.3) and to the Ricci-flat Kähler metrics (Section 1.4).
The main theme of Chapter 2 is the interplay between the geometric quantization of a Kähler manifold and the *realization* of a Kähler manifold as a Kähler submanifold of some complex projective space endowed with the Fubini-Study metric. Sections 2.1 and 2.2 provide an introduction to holomorphic Hermitian line bundles and to the interaction between divisors and line bundles, respectively. Lines bundles are a key ingredient in definition of a geometric quantization: we will give it in Section 2.3. Section 2.4 touches on the main definitions of this thesis: that of regular quantization of Kähler manifold and that of balanced metric. Section 2.6 provides the computation of the epsilon function for the complex Euclidean space.
Chapter 3 is dedicated to the definition and to the properties of the Burns–Simanca metric. We prove that the Burns–Simanca metric admits a regular quantization (Section 3.2) and that the dense subset $\mathbb{C}^2 \setminus \{0\}$ of $\tilde{\mathbb{C}}^2$ equipped with the restriction of the Burns–Simanca Kähler form ω_{BS} associated to the metric g_{BS} admits a Berezin quantization (Section 3.3).
In Chapter 4 we investigate the generalization of the Burns–Simanca metric on the blow-up of $\mathbb{C}^n$ at the origin (Section 4.1) provinig Theorem 2 (Section 4.2).
Finally, Chapter 5 contains the construction of the Eguchi-Hanson metric on

$\tilde{\mathbb{C}}^2/\mathbb{Z}_2$ (Section 5.1) and the proof of Theorem 3 (Section 5.2).

Chapter 1

Preliminaries

Kähler manifolds may be considered as special Riemannian manifolds. Besides the Riemannian structure, they also have compatible symplectic and complex structures. Kähler structures were introduced by Erich Kähler in [36]. Section 1.1 and 1.2 are devoted to the definition of Complex and Kähler manifolds, respectively, and to the construction of a few concrete examples. We will mainly refer to [35, 55]. Section 1.3 contains a briefly introduction to the immersions of a Kähler manifolds into the complex projective spaces, based on Calabi's seminal paper [13]. Finally, Section 1.4 is an introduction to Ricci-flat Kähler metrics.

1.1 Complex manifolds

Definition 1.1.1. A *complex manifold* of complex dimension n is a topological manifold $(M, \mathcal{U})$ whose atlas $(\phi_U)_{U\in\mathcal{U}}$ satisfies the following compatibility condition: for every intersecting $U, V \in \mathcal{U}$, the map

$$\phi_{UV} := \phi_U \circ \phi_V^{-1}$$

between open sets of $\mathbb{C}^n$ is holomorphic. A pair (U, ϕ_U) is called a chart and the collection of all charts is called a *holomorphic structure*.

A function $F : M \to \mathbb{C}$ is called holomorphic if $F \circ \phi_U^{-1}$ is holomorphic for every $U \in \mathcal{U}$. By $\mathcal{O}(M)$ we denote the space of holomorphic functions on M.

Since every holomorphic map between open sets of $\mathbb{C}^n$ is in particular a smooth map between open sets of $\mathbb{R}^{2n}$, every complex manifold M of complex dimension n defines a real $2n$-dimensional smooth manifold, which is the same as M as topological space. The converse does not hold: it is strictly false that smooth manifolds admit complex structures in general, since, in particular, complex manifolds must have even topological dimension.

Note that there is an essential difference between smooth and complex manifolds. A smooth manifold can always be covered by open subsets diffeomorphic to $\mathbb{R}^n$. In contrast, a general complex manifold cannot be covered by open subsets biholomorphic to $\mathbb{C}^n$. This phenomenon is due to the fact that $\mathbb{C}$ in not biholomorphic to a bounded open disc.

Definition 1.1.2. A $(1,1)$-tensor $J : TM \to TM$ on a smooth (real) manifold M which satisfies $J^2 = -\mathrm{Id}$ is called an *almost complex structure*. The pair (M, J) is then referred to as an *almost complex manifold*.

A complex manifold M is thus in a canonical way an almost complex manifold: one can prove that M induces an almost complex structure on its underlying smooth manifold [66, Prop. 3.4, p. 30]. The converse is not true in general, but it holds under some integrability condition: it is a deep result due to Newlander and Nirember [57, Theor. 1.1, p. 393].

Let (M, J) be a complex manifold and let us fix the following notations: denote by $\Omega^{p,q}(M) \subset \Omega^{p+q}(M, \mathbb{C})$ the space of complex-valued differential forms of type (p, q) on M. Recall that the exterior derivate, denoted by d, maps k-forms to $(k+1)$-forms and it can be decomposed as $d = \partial + \overline{\partial}$, where

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \quad \text{and} \quad \overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M).$$

Further, from $d^2 = 0$ one gets

$$\partial^2 + \partial\overline{\partial} + \overline{\partial}\partial + \overline{\partial}^2 = 0,$$

where by dimensional reasons gives the following identities

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \partial\overline{\partial} + \overline{\partial}\partial = 0.$$

Proposition 1.1.3 (The local $\partial\overline{\partial}$-lemma ([55], Prop. 8.8, p. 68))**.** *Let $\omega \in \Omega^2(M,\mathbb{R})\cap\Omega^{1,1}(M)$ be a real 2-form of type $(1,1)$ on a complex manifold M. Then ω is closed if and only if every $x \in M$ has an open neighbourhood U such that the restriction of ω to U equals $\frac{i}{2\pi}\partial\overline{\partial}\Phi$ for some smooth real function Φ on U.*

Proof. One implication is clear:

$$d\left(\frac{i}{2\pi}\partial\overline{\partial}\right) = \frac{i}{2\pi}(\partial+\overline{\partial})\partial\overline{\partial} = \frac{i}{2\pi}(\partial^2\overline{\partial} - \partial\overline{\partial}^2) = 0.$$

The other implication is more delicate and needs the complex counterpart of the Poincaré Lemma (see [31, p. 25] for a proof). □

Example 1.1.4. The *complex Euclidean space* $\mathbb{C}^N$ of complex dimension N, whose atlas is given by the identity map, is a complex manifold. When $N=\infty$, $\mathbb{C}^\infty$ denotes the Hilbert space $\ell^2(\mathbb{C})$ consisting of sequences of complex numbers z_i such that $\sum_{j=1}^{+\infty}|z_j|^2 < +\infty$.

Example 1.1.5. The *complex projective space* $\mathbb{C}P^N$, of complex dimension N, is the set of complex lines in $\mathbb{C}^{N+1}$ or, equivalently,

$$\mathbb{C}P^N = \frac{\mathbb{C}^{N+1}\setminus\{0\}}{\mathbb{C}^*},$$

where $\mathbb{C}^*$ acts by multiplication on $\mathbb{C}^{N+1}$. The *standard open covering* of $\mathbb{C}P^N$ is given by the $N+1$ open subsets

$$U_i := \{[z_0:\ldots:z_N]\,|\,z_i\neq 0\} \subset \mathbb{C}P^N, \tag{1.1}$$

with the coordinates maps

$$\varphi_i([z_0:\ldots:z_N]) = \left(\frac{z_0}{z_i},\ldots,\frac{z_{i-1}}{z_i},\frac{z_{i+1}}{z_i},\ldots,\frac{z_N}{z_i}\right).$$

The infinite dimensional case $\mathbb{C}P^\infty$ is defined by $(\ell^2(\mathbb{C}))\setminus\{0\})/\mathbb{C}^*$, i.e. two sequences of complex numbers are identified iff they differs by multiplication by a non zero complex number.

1.1.1 Complex blow-up

In complex geometry the blowing-up operation consists of replacing a point in a given space with the space of all complex lines pointing out of that point. It is a local operation and in the case of the blow-up of $\mathbb{C}^n$, of finite dimension $n \geq 2$, at the origin it can be explicitly written down as follows:

$$\tilde{\mathbb{C}}^n := \{(z, [t]) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \,|\, t_\alpha z_\beta - t_\beta z_\alpha = 0,\ 1 \leq \alpha, \beta \leq n\}. \tag{1.2}$$

If $(t'_1, \ldots, t'_n) \in [t]$ then there exists $\lambda' \in \mathbb{C} \setminus \{0\}$ such that $t' = \lambda' t$, thus

$$t'_\alpha z_\beta - t'_\beta z_\alpha = \lambda'(t_\alpha z_\beta - t_\beta z_\alpha), \quad \forall \alpha, \beta = 1, \ldots, n.$$

So (1.2) is well defined.

Remark 1.1.6. Since $t_\alpha z_\beta = t_\beta z_\alpha$ is symmetric in α, β, Definition (1.2) is equivalent to

$$\tilde{\mathbb{C}}^n := \{(z, [t]) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \,:\, t_\alpha z_\beta - t_\beta z_\alpha = 0,\ 1 \leq \alpha < \beta \leq n\}.$$

Proposition 1.1.7. *$\tilde{\mathbb{C}}^n$ is a closed submanifold of $\mathbb{C}^n \times \mathbb{C}P^{n-1}$ of complex dimension n.*

Proof. Let us define the continuous map

$$\tilde{F} \circ F : \mathbb{C}^n \times \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^{\binom{n}{2}},\ (z, t) \mapsto (t_1 z_2 - t_2 z_1, \ldots, t_{n-1} z_n - t_n z_{n-1})$$

where

$$\tilde{F} : \mathbb{C}^n \times \mathbb{C}P^{n-1} \to \mathbb{C}^{\binom{n}{2}},\ (z, [t]) \mapsto (t_1 z_2 - t_2 z_1, \ldots, t_{n-1} z_n - t_n z_{n-1})$$

and

$$F : \mathbb{C}^n \times \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \times \mathbb{C}P^{n-1},\ (z, t) \mapsto (z, [t]).$$

Thus $\tilde{\mathbb{C}}^n = \tilde{F}^{-1}(0, \ldots, 0)$ and this prove that $\tilde{\mathbb{C}}^n$ is a closed subset of $\mathbb{C}^n \times \mathbb{C}P^{n-1}$.

A system of charts for $\tilde{\mathbb{C}}^n$ is given as follows: for $i = 1, \ldots, n$ we take

$$\tilde{U}_i = (\mathbb{C}^n \times U_i) \cap \tilde{\mathbb{C}}^n, \tag{1.3}$$

where U_i, for $i = 1, \ldots, n$, are open subsets of $\mathbb{C}P^{n-1}$ defined as in (1.1), with coordinate maps

$$\varphi_i : \tilde{U}_i \to \mathbb{C}^n,\ ((z_1, \ldots, z_n), [t_1 : \ldots : t_n]) \mapsto \left(\frac{t_1}{t_i}, \ldots, \frac{t_{i-1}}{t_i}, z_i, \frac{t_{i+1}}{t_i}, \ldots, \frac{t_n}{t_i}\right),$$

for $i = 1, \ldots, n$, having as inverses the parametrizations $\varphi_i^{-1} : \mathbb{C}^n \to \tilde{U}_i$ that map $(w_1, \ldots, w_n) \in \mathbb{C}^n$ to

$$((w_i w_1, \ldots, w_i w_{i-1}, w_i, w_i w_{i+1}, \ldots, w_i w_n), [w_1 : \ldots : w_{i-1} : 1 : w_{i+1} : \ldots : w_n]) .$$

One finds $\varphi_i \circ \varphi_j^{-1}(w_1, \ldots, w_n)$ equals to

$$\left(\frac{w_1}{w_i}, \frac{w_2}{w_i}, \ldots, \frac{w_{i-1}}{w_i}, w_j w_i, \frac{w_{i+1}}{w_i}, \ldots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_{j+1}}{w_i}, \ldots, \frac{w_n}{w_i}\right),$$

which is obviously holomorphic on its domain of definition. □

There are two projection maps

$$p_1 : \tilde{\mathbb{C}}^n \to \mathbb{C}^n,$$
$$p_2 : \tilde{\mathbb{C}}^n \to \mathbb{C}P^{n-1},$$

given by the restriction to $\tilde{\mathbb{C}}^n$ of the canonical projections of $\mathbb{C}^n \times \mathbb{C}P^{n-1}$. One can prove [54, p. 233] that p_2 induces on $\tilde{\mathbb{C}}^n$ the structure of complex line bundle, whose fibre over $[t] \in \mathbb{C}P^{n-1}$ is the corresponding complex line $\langle t \rangle = \{\lambda t \mid \lambda \in \mathbb{C}\}$ in $\mathbb{C}^n$. In other words, this is the tautological line bundle over $\mathbb{C}P^{n-1}$ (see Example 2.1.7). Observe that p_1 is bijective when restricted to $p_1^{-1}(\mathbb{C}^n \setminus \{0\})$, while

$$p_1^{-1}(0) = \{(0, [t]) \in \tilde{\mathbb{C}}^n\} \simeq \mathbb{C}P^{n-1}.$$

Thus we may think of $\tilde{\mathbb{C}}^n$ as obtained from $\mathbb{C}^n$ by replacing the origin 0 by the space of all lines in $\mathbb{C}^n$ through 0. The manifold $p_1^{-1}(0)$ is called the *exceptional divisor* (see Section 2.2 and Remark 2.2.5 for terminology details) of the blow-up, and we will denote it by H.

Proposition 1.1.8. *$\tilde{\mathbb{C}}^n \setminus H$ is biholomorphic to $\mathbb{C}^n \setminus \{0\}$.*

Proof. The restriction map

$$p_r := p_{1|\tilde{\mathbb{C}}^n \setminus H} : \tilde{\mathbb{C}}^n \setminus H \to \mathbb{C}^n \setminus \{0\}, \quad (z, [t]) \mapsto z$$

is a biholomorphism having as inverse

$$p_r^{-1} : \mathbb{C}^n \setminus \{0\} \to \tilde{\mathbb{C}}^n \setminus H, \quad z \mapsto (z, [z])\,.$$

□

Remark 1.1.9 (Connected sum). The connected sum of two oriented manifold M_1 and M_2 is constructed by removing two small discs $B_j \subset M_j$, $j = 1, 2$, and then identifying the boundaries via a smooth map $\phi : \partial B_1 \to \partial B_2$ which extends to an orientation-preserving diffeomorphism from a neighbourhood of ∂B_1 to a neighbourhood of ∂B_2. This extension must interchange the inner and outer boundaries of the annuli. It is possible to prove [54, p. 235] that the blow-up $\tilde{\mathbb{C}}^n$ is diffeomorphic as an oriented manifolds to the connected sum $\mathbb{C}^n \# \overline{\mathbb{C}P^n}$, where $\overline{\mathbb{C}P^n}$ denotes the manifold $\mathbb{C}P^n$ with the orientation opposite to the standard complex structure of $\mathbb{C}P^n$.

Proposition 1.1.10. *$\tilde{\mathbb{C}}^n$ is simply connected.*

Proof. First observe that if M_1 and M_2 are connected manifolds of the same real dimension $n \geq 3$, by the Seifert-van Kampen Theorem we known that $\pi_1(M_1 \# M_2) = \pi_1(M_1) * \pi_1(M_2)$. Now, since $\mathbb{C}^n \# \overline{\mathbb{C}P^n}$ is homeomorphic to $\mathbb{C}^n \# \mathbb{C}P^n$, we find

$$\pi_1(\tilde{\mathbb{C}}^n) = \pi_1(\mathbb{C}^n \# \overline{\mathbb{C}P^n}) = \pi_1(\mathbb{C}^n \# \mathbb{C}P^n) = \pi_1(\mathbb{C}^n) * \pi_1(\mathbb{C}P^n).$$

Thus the fundamental group of $\tilde{\mathbb{C}}^n$ is trivial i.e. $\tilde{\mathbb{C}}^n$ is simply connected. □

Proposition 1.1.11. *$\tilde{\mathbb{C}}^n$ is no contractible.*

Proof. The n-th singular cohomology group with real coefficients of $\tilde{\mathbb{C}}^n$ can be easily compute in the following way:

$$H^n_{dR}(\tilde{\mathbb{C}}^n) = H^n_{dR}(\mathbb{C}^n \# \mathbb{C}P^n) = H^n_{dR}(\mathbb{C}^n) \oplus H^n_{dR}(\mathbb{C}P^n) = 0 \oplus \mathbb{R} = \mathbb{R} \neq 0,$$

and the thesis follows. □

1.2 Kähler manifolds

Definition 1.2.1. A *Hermitian metric* on an almost complex manifold (M, J) is a Riemannian metric g such that $g(X, Y) = g(JX, JY)$, for all $X, Y \in TM$. The *fundamental* 2*-form* of a Hermitian metric is defined by $\omega(X, Y) := g(JX, Y)$. A *Hermitian manifold* is a couple (M, g) where M is an almost complex manifold and g a Hermitian metric on M.

Every almost complex manifold admits Hermitian metrics. Simply choose an arbitrary Riemannian metric $\tilde{g}$ and define $g(X, Y) := \tilde{g}(X, Y) + \tilde{g}(JX, JY)$. Let z_α be holomorphic coordinates on complex Hermitian manifold and denote by $g_{\alpha\bar{\beta}}$ the coefficients of the metric tensor in these local coordinates. In that case the fundamental form is given by

$$\omega = \frac{i}{2\pi} \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta.$$

Suppose that the fundamental form ω of a complex Hermitian manifold is closed. The local $\partial\overline{\partial}$-lemma yields the existence in some neighbourhood of each point of a real function Φ such that $\omega = \frac{i}{2\pi}\partial\overline{\partial}\Phi$, which in local coordinates reads

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 \Phi}{\partial z_\alpha \overline{\partial} \bar{z}_\beta}.$$

This particularly simple expression for the metric tensor in terms of one single real function deserves the following:

Definition 1.2.2. A Hermitian metric g on an almost complex manifold (M, J) is called a *Kähler metric* if J is a complex structure (i.e M is a complex manifold) and the fundamental form ω is closed (i.e. $d\omega = 0$). A local real function Φ satisfying $\omega = \frac{i}{2\pi}\partial\overline{\partial}\Phi$ is called a *local Kähler potential* of the metric g. The pair (M, ω) is called *Kähler manifold.*

A Kähler manifold (M, ω) can be seen as a symplectic manifold[1] with the additional requirements that M is a complex manifold and ω is positive. The

[1] A symplectic manifold is a smooth manifold M equipped with a closed non degenerate differential 2-form ω called the symplectic form.

latter means that the matrix $g_{\alpha\bar{\beta}}$ is positive definite.

The rest of this section is devoted to the construction of various examples of Kähler manifolds.

Example 1.2.3 (*The complex Euclidean space*). The complex space $\mathbb{C}^N$ of complex dimension $N \leq \infty$ endowed with the *Euclidean metric* g_0. With respect to the canonical holomorphic coordinates the Kähler form ω_0 is given by

$$\omega_0 = \frac{i}{2\pi} \sum_{\alpha=1}^{N} dz_\alpha \wedge d\bar{z}_\alpha = \frac{i}{2\pi} \partial\bar{\partial}|z|^2.$$

Thus

$$\Phi : \mathbb{C}^N \to \mathbb{R},\ z \mapsto |z|^2$$

is a global Kähler potential for the canonical Hermitian metric on $\mathbb{C}^N$.

Example 1.2.4 (*The complex projective space*). The complex projective space $\mathbb{C}P^N, N \leq \infty$ endowed with the *Fubini-Study metric* g_{FS}. In homogeneous coordinates $[t_0 : \ldots : t_N]$, the fundamental form ω_{FS} is given by

$$\omega_{FS} = \frac{i}{2\pi} \partial\bar{\partial} \log(|t_0|^2 + \cdots + |t_N|^2).$$

Thus

$$\Phi : U_0 \to \mathbb{R},\ z \mapsto \log\left(1 + \sum_{\alpha=1}^{N} |z_i|^2\right).$$

is a local Kähler potential for the Fubini-Study metric on $U_0 \subset \mathbb{C}P^N$ with respect to affine coordinates $z_i = t_i/t_0$ for every $i \neq 0$.

Example 1.2.5 (*The complex torus*). The quotient $\mathbb{C}^n/\mathbb{Z}^{2n}$, where $\mathbb{Z}^{2n} \subset \mathbb{R}^{2n} = \mathbb{C}^n$ is the natural inclusion, can be endowed with a complex structure and it is called *complex torus*. Since ω_0 (see Example 1.2.3) is invariant by translations it descends to a globally define Kähler form on $\mathbb{C}^n/\mathbb{Z}^{2n}$ which makes $(\mathbb{C}^n/\mathbb{Z}^{2n}, \omega_0)$ into a Kähler manifold.

Example 1.2.6 (*The complex hyperbolic space*). The complex hyperbolic space $\mathbb{C}H^N$, of complex dimension $N \leq \infty$, defined by

$$\mathbb{C}H^N = \left\{ (z_1, \ldots, z_N) \in \mathbb{C}^N \mid \sum_{\alpha=1}^{N} |z_j|^2 < 1 \right\}$$

endowed with the *hyperbolic metric* g_{hyp}. For this metric there exists a global Kähler potential given by

$$\Phi : \mathbb{C}H^N \to \mathbb{R},\, z \mapsto -\log\left(1 - \sum_{\alpha=1}^{N} |z_i|^2\right),$$

thus the Kähler form ω_{hyp} is given by

$$\omega_{hyp} = \frac{i}{2\pi}\partial\overline{\partial}\log\left(\frac{1}{1-|z|^2}\right).$$

1.3 Projectively induced Kähler metrics

In his seminal paper Eugenio Calabi [13] gives a complete answer to the problem of existence and uniqueness of a Kähler immersion of a Kähler manifold into finite or infinite dimensional complex space forms. A *complex space form* is a Kähler manifold of constant holomorphic sectional curvature. If the manifold is complete and simply connected, then up to holomorphic isometries the complex space forms are described by the following three models: *the complex Euclidean space* (Example 1.2.3), *the complex projective space* (Example 1.2.4) and *the complex hyperbolic space* (Example 1.2.6).

For the specific case of Kähler manifolds that admit a Kähler immersion into the complex projective space we give the following definition:

Definition 1.3.1. A Kähler metric g on a complex manifold M is said to be *projectively induced* if there exists a Kähler immersion of (M, g) into the complex projective space $(\mathbb{C}P^N, g_{FS}), N \leq +\infty$, endowed with the Fubini-Study metric g_{FS}.

Since we are interested in projectively induced metrics, we are going to summarize Calabi's results dealing with these particular Kähler immersions.
Let g be a projectively induced Kähler metric on a Kähler manifold (M, ω) and let $\omega_{|U} = \frac{i}{2\pi}\partial\overline{\partial}\Phi$. Then the Kähler potential Φ is real analytic and it extends to a sesquianalytic function $\tilde{\Phi}$ defined on a neighbourhood $W \subset U \times \overline{U}$ of the diagonal $M \times \overline{M}$, where $\overline{M}$ is the conjugate of M. Consider

the *Calabi's diastasis function* D_g, defined on W by

$$D_g(x,y) = \tilde{\Phi}(x,\bar{x}) + \tilde{\Phi}(y,\bar{y}) - \tilde{\Phi}(x,\bar{y}) - \tilde{\Phi}(y,\bar{x}).$$

By the $\partial\bar{\partial}$-lemma, a Kähler potential is defined up to an addition with the real part of a holomorphic function, therefore the diastasis is independent from the potential chosen.
It is easy to see that $D_g(x,y)$ is symmetric and once one of its two entries is fixed it is a Kähler potential for g.

Example 1.3.2. Consider the complex projective space $\mathbb{C}P^N$, $N \leq +\infty$, endowed with the Fubini-Study form ω_{FS}. The diastasis can be written in terms of the coordinates in $\mathbb{C}^{N+1}$ as

$$D_{FS}(\pi(z),\pi(w)) = \log \frac{||z||^2||w||^2}{|\langle z,w\rangle|^2},$$

where $\pi : \mathbb{C}^{N+1}\setminus\{0\} \to \mathbb{C}P^N$ is the canonical projection and $\langle\cdot,\cdot\rangle$ is the standard Hermitian metric on $\mathbb{C}^{N+1}$. One can prove (see [43, Ex. 2.1, p. 356] for more details) that $D_{FS} > 0$, where it is defined, and that $e^{-D_{FS}(\pi(z),\pi(w))}$ is globally defined and smooth on $\mathbb{C}P^N \times \mathbb{C}P^N$ and it equals 1 on the diagonal.

The diastasis function peculiarity is underlined by the following theorem.

Theorem 1.3.3 (Hereditary property ([13], Prop. 6, p. 4))**.** *If f is a local Kähler immersion of a real analytic manifold S into $\mathbb{C}P^N$, then the diastasis $D^S(p,z)$ of S around a point $p \in f^{-1}([1:0:\ldots:0])$ is defined on $S \setminus f^{-1}(\mathbb{C}P^N \setminus U_0)$, where U_0 is the affine chart $\{Z_0 \neq 0\}$, and it is equal to*

$$D^{\mathbb{C}\mathrm{P}^N}([1:0:\ldots:0],\cdot)\circ f = \log(1+|f(z)|^2).$$

The diastasis function plays a key role in the achievement of Calabi's results, as shown by the following.

Theorem 1.3.4 (Calabi's criterion ([13], Theor. 8, p. 18))**.** *There exists a local Kähler immersion around a point p of a real analytic Kähler manifold with diastasis $D(p,z)$ into $\mathbb{C}P^N$ if and only if the matrix of coefficients in the power expansion around p of*

$$e^{D(p,z)-1} = \sum_{i,j} b_{ij} z^{m_j} \bar{z}^{m_j}$$

is positive semidefinite of rank at most N (we also say that the metric is 1*-resolvable of rank N at p).*

Theorem 1.3.5 (Global character of projectively induced metrics ([13], Theor. 10, p. 19))**.** *In a connected Kähler manifolds endowed with a projectively induced metric, each point admits a neighbourhood where is defined a local Kähler immersion into $\mathbb{C}P^N$.*

Theorem 1.3.6 (Immersion's extension ([13], Theor. 11, p. 19))**.** *If a Kähler metric is defined on a simply connected manifold M then a local Kähler immersion $f : V \subset M \to \mathbb{C}P^N$ can be extended to a global one. This immersion is also injective if and only if $D(p, q) = 0$ only for $p = q$.*

1.4 Ricci-flat Kähler metrics

Let (M, g) be a Kähler manifold, of dimension $2n$, with Levi–Civita covariant derivate ∇. We denote by R the *Riemannian curvature tensor*, defined as

$$R(X, Y, Z, T) := g(R^{\nabla}(X, Y)Z, T)$$

for any tangent vector X, Y, Z and T, where R^{∇} is the curvature tensor of ∇. A Riemannian metric is called *flat* if its Riemannian curvature tensor vanishes. Using Frobenius's integrability theorem, one can show that this is equivalent to the fact that M is locally isometric to an Euclidean space.
The *Ricci tensor* of (M, g) defined by

$$\mathrm{Ric}(X, Y) := \mathrm{Tr}\{V \mapsto R^{\nabla}(V, X)Y, \},$$

can be expressed in local coordinates as

$$\mathrm{Ric}_{\alpha\bar{\beta}} = -\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta} \log\det(g_{\alpha\bar{\beta}}). \tag{1.4}$$

where $\det(g_{\alpha\bar{\beta}})$ denotes the determinant of the matrix $(g_{\alpha\bar{\beta}})$ expressing the metric g. The associated *Ricci form* is defined by

$$\mathrm{Ric}(g)(X, Y) := \mathrm{Ric}(JX, Y),$$

for any tangent vector X and Y, where J is the complex structure of M. The Ricci form is one of the most important objects on Kähler manifold. Among its properties we mention that the Ricci form is closed and in local coordinates it can be expressed as

$$\mathrm{Ric}(g) = -i\partial\overline{\partial}\log\det(g_{\alpha\bar{\beta}}). \tag{1.5}$$

A Kähler metric g is called *Kähler-Einstein* if the Ricci form of g is proportional to the Kähler form ω associated to g, that is, if there exists a real constant λ such that

$$\mathrm{Ric}(g) = \lambda\omega.$$

If $\lambda = 0$, the metric g is called *Ricci-flat*.
The *scalar curvature* of a Kähler manifold (M, g) is the trace of the Ricci tensor

$$\mathrm{scal}_g := \mathrm{Tr}(\mathrm{Ric}),$$

whose expression in local coordinates is given by

$$\mathrm{scal}_g = -\sum_{\alpha,\bar{\beta}=1}^{n} g^{\alpha\bar{\beta}}\mathrm{Ric}_{\alpha\bar{\beta}}, \tag{1.6}$$

where $(g^{\alpha\bar{\beta}})$ denotes the inverse matrix of $(g_{\alpha\bar{\beta}})$. From now on we will write *cscK metric* as short form for constant scalar curvature Kähler metrics. Finally we set

$$|\mathrm{Ric}|^2 = \sum_{\alpha,\bar{\beta}=1}^{n} |\mathrm{Ric}_{\alpha\bar{\beta}}|^2. \tag{1.7}$$

Chapter 2

Geometric quantization of Kähler manifolds

The main theme of this chapter is the interplay between the *geometric quantization* of a Kähler manifold and the *realization* of a Kähler manifold as a Kähler submanifold of some complex projective space endowed with the Fubini-Study metric. Sections 2.1 and 2.2 provide an introduction to holomorphic Hermitian line bundles and to the interaction between divisors and line bundles, respectively. Lines bundles are a key ingredient in definition of a geometric quantization: we will give it in Section 2.3. Section 2.4 touches on the main definitions of this thesis: that of regular quantization of Kähler manifold and that of balanced metric. Section 2.6 provides the computation of the epsilon function for the complex Euclidean space. We will mainly refer to [31, 55] and [58] for the background material and the notations of this chapter.

2.1 Holomorphic Hermitian line bundles

Let M be a complex manifold and let $\pi : L \to M$ be a complex line bundle over M (i.e. each fibre $\pi^{-1}(x) = L_x$ is a 1-dimensional vector space over $\mathbb{C}$).

Definition 2.1.1. L is a *holomorphic line bundle* if it admits a trivialization with holomorphic transition functions.

More precisely, if $\pi : L \to M$ is a holomorphic line bundle, there exist an open covering $\mathcal{U}$ of M and for each $U \in \mathcal{U}$ a diffeomorphism

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U \times \mathbb{C},$$

such that the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\psi_{U_\alpha}} & U \times \mathbb{C} \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle pr_{U_\alpha}} & \\ U_\alpha & & \end{array}$$

and for every intersecting U_α and U_β one has

$$\psi_\alpha \circ \psi_\beta(x, v) = (x, g_{\alpha\beta}(x)v),$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$, $x \mapsto (\psi_\alpha \circ \psi_\beta^{-1})(x)$ are nonvanishing holomorphic functions, called *transition function*, satisfying

$$g_{\alpha\beta} \cdot g_{\beta\alpha} = 1 \text{ on } U_\alpha \cap U_\beta, \quad g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma. \qquad (2.1)$$

One can easily prove that a complex line bundle $\pi : L \to M$ is holomorphic if and only if there exists a complex structure on L as manifold such that the projection π is a holomorphic map.

Remark 2.1.2. Given a collection of functions $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ satisfying the identities (2.1), one can construct a holomorphic line bundle L with transition functions $\{g_{\alpha\beta}\}$. Collections $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ of transition functions define the same line bundle if and only if there exist functions $f_\alpha \in \mathcal{O}^*(U_\alpha)$ satisfying $g'_{\alpha\beta} = (f_\alpha / f_\beta) g_{\alpha\beta}$.

We can give to the set of holomorphic line bundles on M the structure of a group, multiplication being by tensor product and inverses by dual bundles. More precisely, if L is given by data $\{g_{\alpha\beta}\}$, L' by $\{g'_{\alpha\beta}\}$ we have

$$L \otimes L' \sim \{g_{\alpha\beta} g'_{\alpha\beta}\}, \quad L^* \sim \{g_{\alpha\beta}^{-1}\}.$$

This group is called the *Picard group* of M and denoted by $\mathrm{Pic}(M)$.

We denote by $\Gamma(L)$ the set of smooth global sections of a holomorphic line bundle over M:

$$\Gamma(L) = \{s : M \to L \,|\, \pi \circ s = id_M,\ s \text{ is smooth}\}.$$

Definition 2.1.3. A *holomorphic structure* $\overline{\partial}$ on a complex line bundle is an operator

$$\overline{\partial} : \Gamma(L) \to \Gamma(\Omega^{0,1} \otimes L)$$

satisfying the Leibniz rule and such that $\overline{\partial}^2 = 0$.

This operator maps a smooth section $s \in \Gamma(L)$ to the smooth section $\overline{\partial} s$ of the bundle $\Omega^{0,1} \otimes L \to M$.

Theorem 2.1.4. *A complex line bundle $L \to M$ is holomorphic if and only if it has a holomorphic structure $\overline{\partial}$.*

See [55, Section 9.2, p. 72] for a proof of the above theorem and for more details.

Definition 2.1.5. A smooth global section $s \in \Gamma(L)$ is said to be *holomorphic* if

$$\overline{\partial} s = 0.$$

The space of global holomorphic sections on a holomorphic line bundle L is denoted by $H^0(L)$.

Example 2.1.6 (*Trivial line bundle*). It is defined as the complex line bundle $\pi : M \times \mathbb{C} \to M$, $(z,t) \mapsto z$ whose fibre L_z over some point $z \in M$ is the one dimensional complex vector space $\pi^{-1}(z) = \{z\} \times \mathbb{C} \simeq \mathbb{C}$.

Example 2.1.7 (*Tautological line bundle*). On the complex projective space there is some distinguished holomorphic line bundle called the *tautological line bundle* $\mathcal{O}(1)$. It is defined as the complex line bundle $\pi : \mathcal{O}(1) \to \mathbb{C}P^n$ whose fibre over some point $[z] \in \mathbb{C}P^n$ is the complex line $\langle z \rangle$ in $\mathbb{C}^{n+1}$. We consider the canonical holomorphic charts (U_i, φ_i) on $\mathbb{C}P^n$ and the local trivializations $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$ of $\mathcal{O}(1)$ defined by $\psi_i([z], w) = ([z], w_i)$. It is easy to compute the transition functions:

$$\psi_i \circ \psi_j^{-1}([z], \lambda) = ([z], g_{ij}([z])\lambda), \text{ where } g_{ij}([z]) = \frac{z_i}{z_j}.$$

This shows that the tautological bundle of $\mathbb{C}P^n$ is holomorphic. The dual of the tautological line bundle of $\mathbb{C}P^n$, denoted by $\mathcal{O}(-1)$, is called the *hyperplane line bundle of* $\mathbb{C}P^n$. Note that the fibre $\mathcal{O}(-1) \to \mathbb{C}P^n$ over some point $[z] \in \mathbb{C}P^n$ is the set of $\mathbb{C}$-linear maps on the line that determines $[z]$ in $\mathbb{C}P^n$.

Let $\pi : L \to M$ be a line bundle over a smooth manifold M (i.e. each fibre is a 1-dimensional vector space over $\mathbb{R}$).

Definition 2.1.8. A *connection* on L is a $\mathbb{C}$-linear differential operator

$$\nabla : \Gamma(L) \to \Omega(L)$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s,$$

for every complex valued function f on M and for every $s \in \Gamma(L)$, where $\Omega(L)$ denotes the space of L-valued 1-forms (i.e. smooth global sections of $T^*_{\mathbb{C}}M \otimes L$)[1].

The *curvature* $\mathrm{curv}(L, \nabla)$ of the connection ∇ is the closed complex 2-form on M satisfying

$$\mathrm{curv}(L, \nabla)(X, Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s, \tag{2.2}$$

for every $X, Y \in \Gamma(T_{\mathbb{C}}M)$ and for every $s \in \Gamma(L)$, where $\nabla_X s := (\nabla s)(X)$. Let L^+ be the complement in L of the zero section. More explicitly, if $\sigma : U \to L^+$ is a trivialising section over some open set $U \subset M$, we define the *local connection form* $\beta \in \Omega^1(U)$ by

$$\nabla \sigma = \beta \otimes \sigma. \tag{2.3}$$

It follows, by (2.2), that on U we have

$$\mathrm{curv}(L, \nabla) = d\beta. \tag{2.4}$$

[1] We denote by $T^*_{\mathbb{C}}M = \Omega^{1,0}(M) + \Omega^{0,1}(M)$ the dual of the complexification of the tangent bundle $T_{\mathbb{C}}M = T^{1,0}M + T^{0,1}M$.

The de Rham cohomology class $[\text{curv}(L, \nabla)] \in H^2(M, \mathbb{C})$ of the closed 2-form $\text{curv}(L, \nabla)$ does not depend on ∇ (see [55, Lemma 16.2, p. 114]). Furthermore, one can show [55, Theor. 16.3, p. 115] that the real cohomology class

$$c_1(\nabla) := \left[\frac{i}{2\pi}\text{curv}(L, \nabla)\right] \tag{2.5}$$

is equal to the image of $c_1(L)$ in $H^2(M, \mathbb{R})$, where $c_1(L)$ is the *first Chern class of* L. The comprehensive theory of Chern classes can be found in [37, Ch. 12].

Definition 2.1.9. Two holomorphic line bundles $\pi_i : L_i \to M$, over M are said to be *isomorphic* if there exists a holomorphic map $\psi : L_1 \to L_2$ such that $\pi_2 \circ \psi = \pi_1$, which is linear on the fibres.

Proposition 2.1.10 ([31])**.** *Two holomorphic line bundles* L_1 *and* L_2 *over a simply connected complex manifold with the same first Chern class, i.e.* $c_1(L_1) = c_1(L_2)$*, are isomorphic.*

Let $L \to M$ be a complex line bundle over a complex manifold M.

Definition 2.1.11. A *Hermitian structure* $\mathfrak{h}$ on L is a smooth field of Hermitian products on the fibres of L, that is, for every $x \in M$, there exists a map $h : L_x \times L_x \to \mathbb{C}$ which satisfies

- $h(u, v)$ is $\mathbb{C}$-linear in u for every $v \in L_x$.
- $h(u, v) = \overline{h(v, u)}$ for every $u, v \in L_x$ ($\mathbb{C}$-anti-linearity in the second variable).
- $h(u, u) > 0$ for every $u \neq 0$ (non-degenerate).
- $h(u, v)$ is a smooth function on M for every smooth sections u and v of L.

A complex line bundle endowed with a Hermitian structure is called *Hermitian line bundle*. A holomorphic line bundle endowed with a Hermitian structures is called *holomorphic Hermitian line bundle*.

Using a partition of the unity it is possible to prove that every complex line bundle admits a Hermitian structure (see [55, p. 78]).
Given a Hermitian structure $\mathfrak{h}$ on L, for every pair $s, t \in \Gamma(L)$, we will often write $h(s,t)(x)$ to mean $h(s(x), t(x))$.
Let $(L_1, \mathfrak{h}_1)$ and $(L_2, \mathfrak{h}_2)$ be two Hermitian line bundles over M. One can define a Hermitian structure $\mathfrak{h}_1 \otimes \mathfrak{h}_2$ on the complex line bundle $L_1 \otimes L_2$ by

$$(h_1 \otimes h_2)((s_1 \otimes t_1), (s_2 \otimes t_2))(x) := h_1(s_1, s_2)(x) h_2(t_1, t_2)(x), \tag{2.6}$$

for any $s_1, s_2 \in \Gamma(L_1), t_1, t_2 \in \Gamma(L_2)$ and for all $x \in M$.
Next consider Hermitian line bundles with connection ∇. We say that ∇ is an $\mathfrak{h}$-*connection* (or compatible with $\mathfrak{h}$) if

$$Xh(s,t)(x) = h(\nabla_X s, t)(x) + h(s, \nabla_X t)(x)$$

for every $s, t \in \Gamma(L), x \in M$ and for every vector field X on M.

Proposition 2.1.12 ([41], Prop. 1.2.3, p. 18)**.** *The curvature of a* $\mathfrak{h}$-*connection is a purely imaginary closed* 2*-form.*

Proof. Let σ be a trivialising section over an open set $U \subset M$ and let $\beta \in \omega^1(U)$ be the local connection form given by (2.3). If ∇ is a $\mathfrak{h}$-connection, then

$$\begin{aligned}(d\log(h(\sigma,\sigma)(x)))\,(X) &= \frac{Xh(\sigma,\sigma)(x)}{h(\sigma,\sigma)(x)} = \frac{h(\nabla_X\sigma,\sigma)(x) + h(\sigma,\nabla_X\sigma)(x)}{h(\sigma,\sigma)(x)} \\ &= \frac{h(\beta(X)\sigma,\sigma)(x) + h(\sigma,\beta(X)\sigma)(x)}{h(\sigma,\sigma)(x)} \\ &= (\beta(X) + \overline{\beta(X)})(x),\end{aligned} \tag{2.7}$$

for every vector field X on M. The above formula has to be true for every section σ, so if we suppose that $h(\sigma,\sigma)(x) = 1$, we get that $\beta + \overline{\beta} = 0$. Hence, by (2.4),

$$\mathrm{curv}(L,\nabla) + \overline{\mathrm{curv}(L,\nabla)} = d\beta + d\overline{\beta} = d(\beta + \overline{\beta}) = 0.$$

□

The decomposition of 1-forms into type $(1,0)$ and $(0,1)$ induces a decomposition $\nabla = \nabla^{1,0} + \nabla^{0,1}$, where the operators $\nabla^{1,0} : \Gamma(L) \to \Gamma(\Omega^{1,0}(M) \otimes L)$ and $\nabla^{0,1} : \Gamma(L) \to \Gamma(\Omega^{0,1}(M) \otimes L)$ satisfy the Leibniz rule.

Definition 2.1.13. A connection ∇ in a Hermitian line bundle with holomorphic structure $\overline{\partial}$ is said to be *holomorphic* if $\nabla^{0,1} = \overline{\partial}$.

Theorem 2.1.14 ([55], Theor. 10.3, p. 79)**.** *For every Hermitian structure* $\mathfrak{h}$ *in a Hermitian line bundle there exists a unique holomorphic* $\mathfrak{h}$*-connection* ∇ *called the* Chern connection*.*

Proposition 2.1.15. *The curvature of the Chern connection equals*

$$-\partial\overline{\partial}\log(h(\sigma(x), \sigma(x))),$$

where $\sigma : U \to L^+$ *is a trivializing holomorphic section.*

Proof. Let β be the local connection form on U. Since σ and ∇ are holomorphic, we have

$$\nabla_X\sigma = \nabla^{1,0}_X\sigma + \nabla^{0,1}_X\sigma = 0,$$

for every vector field $X \in \Gamma(T^{0,1}M)$. Thus $\nabla_X\sigma = \beta(X)\sigma = 0$ and β is a form of type $(1,0)$ on U. It follows by (2.7) that

$$\beta = \partial\log(h(\sigma(x), \sigma(x))),$$

and by (2.4) that

$$\mathrm{curv}(L, \nabla) = d\beta = (\partial + \overline{\partial})\partial\log(h(\sigma(x), \sigma(x))) = -\partial\overline{\partial}\log(h(\sigma(x), \sigma(x))). \tag{2.8}$$

Suppose that $\tau : V \to L^+$ is a trivialising holomorphic section on V, such that $U \cap V \neq \emptyset$. Then one has $\tau = f\sigma$ for a suitable holomorphic function $f : U \cap V \to \mathbb{C}$, thus

$$\partial\overline{\partial}\log(h(\tau(x), \tau(x))) = \partial\overline{\partial}\log(|f(x)|^2 h(\sigma(x), \sigma(x))) = \partial\overline{\partial}\log(h(\sigma(x), \sigma(x))),$$

showing that (2.8) does not depend on the chosen trivializing holomorphic section. □

2.2 Divisors and line bundles

Closely related to holomorphic line bundles is the concept of a divisor on a complex manifold. We report here the main material needed in the sequel, one can see [32, 35, 66] for a more detailed discussion of divisors and subvarieties.

Let M be a complex manifold of dimension n.

Definition 2.2.1. An *analytic subvariety* of M is a closed subset $H \subset M$ such that for any point $p \in M$ there exists an open neighbourhood $p \in U \subset M$ such that $U \cap H$ is the zero set of finitely many holomorphic functions $f_1, \ldots, f_k \in \mathcal{O}(U)$.

An analytic subvariety H is *irreducible* if it can not be written as the union of two proper analytic subvarieties.

Definition 2.2.2. Let H be a compact analytic subvariety of a complex manifold $\tilde{M}$. If there exists a complex manifold M and a holomorphic mapping $p_r : \tilde{M} \to M$ such that $p_r(H) = \{m_0\}$, with $m_0 \in M$, and $p_r : \tilde{M} \setminus H \to M \setminus \{m_0\}$ is a biholomorphism mapping, we say that H is an *exceptional* subvariety of $\tilde{M}$.

Definition 2.2.3. An *analytic hypersurface* of M is an analytic subvariety $H \subset M$ of codimension 1, i.e for any point $p \in H \subset M$, H is given in a neighborhood of p as the zero set of a single non-trivial holomorphic function f. f is called a *local defining function* for H near p, ad is unique up to multiplication by a function not vanishing at p.

Definition 2.2.4. A *divisor* D on M is a formal linear combination

$$D := \sum a_i H_i$$

with $H_i \subset M$ irreducible hypersurfaces and $a_i \in \mathbb{Z}$. A divisor D is called *effective* if $a_i \geq 0$ for all i.

In the last definition we want to assume that the sum is locally finite, i.e. for any $p \in M$ there exists an open neighbourhood U such that there exist only finitely many coefficients $a_i \neq 0$ with $H_i \cap U \neq \emptyset$. If M is compact, this reduces to finite sums.

Remark 2.2.5. Every hypersurface defines a divisor $\sum H_i$, where H_i are the irreducible component of H. An irreducible hypersurfaces H clearly induces the divisor $D = H$. In the construction of the blow-up $\tilde{\mathbb{C}}^n$ of $\mathbb{C}^n$ at the origin the manifold $p^{-1}(0) \simeq \mathbb{C}P^{n-1}$ is a compact irreducible analytic hypersurface of $\tilde{\mathbb{C}}^n$ namely it is an exceptional divisor.

Definition 2.2.6. Let $H \subset M$ be an irreducible analytic hypersurface with local defining function f around some $p \in H$. For every holomorphic function g around p, the *order* $\mathrm{ord}_{H,p}(g)$ of g along H at p is defined to be the largest positive integer a such that

$$g = f^a \cdot h$$

where h is holomorphic around p $(h(p) \neq 0)$.

One can see that $\mathrm{ord}_{H,p}(g)$ is independent of p (see, e.g. [31, Ch. 1, p. 130]), thus we can define the *order* $\mathrm{ord}_H(g)$ of g along H to be the order of g with respect to H at any point $p \in H$.
Divisors can also be described in sheaf-theoretic terms as follows: consider the exact sequence of multiplicative sheaves

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0$$

where $\mathcal{O}^*$ is the subsheaf of nonzero holomorphic functions on M and $\mathcal{M}^*$ is the sheaf of non-trivial meromorphic function on M. Then a *divisor* D on M is a global section of the quotient sheaf $\mathcal{M}^*/\mathcal{O}^*$. A global section on $\mathcal{M}^*/\mathcal{O}^*$ is given by an open covering $\{U_\alpha\}$ of M and meromorphic functions (sections of $\mathcal{M}^*$) f_α defined in U_α such that

$$\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta),$$

moreover

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma.$$

The line bundle given by the transition function $g_{\alpha\beta} = f_\alpha/f_\beta$ (cf. Remark 2.1.2) is called the *associated line bundle of* D, and denoted by $[D]$. Thus a divisor gives rise to an equivalence class of line bundles represented by the cocycle $\{g_{\alpha\beta}\}$ and two different divisors give the same class if they differ multiplicatively by a global meromorphic function (see [31, Ch. 1, p. 131-132] or [66, Ch. 3, p. 107] for more details).

2.3 Geometric quantization of Kähler manifolds

Definition 2.3.1. A *geometric quantization* of a Kähler manifold (M, ω) is a triple $(L, \nabla, \mathfrak{h})$ such that

$$\operatorname{curv}(L, \nabla) = -2\pi i\omega \tag{2.9}$$

where L is a holomorphic Hermitian line bundle over M and $\mathfrak{h}$ is a Hermitian structure such that ∇ is the Chern connection. The line bundle L is called *quantum line bundle* of (M, ω). A Kähler manifold (M, ω) is said to be *quantizable* if it admits a geometric quantization.

The above definition makes sense by Proposition 2.1.12. By Theorem 2.1.14 the space of $\mathfrak{h}$-connections and the space of holomorphic connections on a holomorphic line bundle L intersect in one point; this means that to describe a geometric quantization of a Kähler manifold (M, ω) it is enough to specify the holomorphic Hermitian line bundle $(L, \mathfrak{h})$ over M satisfying (2.9). We will use the notation $\operatorname{curv}(L, \mathfrak{h})$ to denote the curvature of the Chern connection associated to the holomorphic Hermitian line bundle $(L, \mathfrak{h})$. Finally, Proposition 2.1.15 gives a way to check if a holomorphic Hermitian line bundle $(L, \mathfrak{h})$ over M defines a geometric quantization of (M, ω): simply by choosing a trivialising holomorphic section $\sigma : U \to L^+$ and verifying if

$$\omega = -\frac{i}{2\pi}\partial\bar{\partial}\log(h(\sigma(x), \sigma(x))).$$

Not all Kähler manifolds admit a geometric quantization. In terms of cohomology classes a necessary and sufficient condition is expressed by the following theorem (see also [65]).

Theorem 2.3.2 (The integrality condition ([38], Prop. 2.1.1 , p. 133))**.** *A Kähler manifold (M, ω) admits a geometric quantization $(L, \mathfrak{h})$ if and only if $c_1(L) = [\omega]$ (i.e. if and only if ω is* integral*).*

Example 2.3.3 (*The flat space*). Let $\mathbb{C}^N$ be the complex Euclidean space endowed with the Kähler form $\omega_0 = \frac{i}{2\pi}\partial\overline{\partial}|z|^2$. Consider the trivial bundle $L = \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ (see Example 2.1.6) and for each $z \in \mathbb{C}^N$ define the map

$$h : L_z \times L_z \to \mathbb{C},\ ((z, t_1), (z, t_2)) \mapsto e^{-|z|^2} t_1 \overline{t_2}.$$

The above map induces a Hermitian structure $\mathfrak{h}$ on L that defines a geometric quantization of $(\mathbb{C}^N, \omega_0)$. Indeed, if $\sigma(z) = (z, f(z))$ is a global holomorphic section of L, where $f : \mathbb{C}^N \to \mathbb{C}$ is a holomorphic function, then by Proposition 2.1.15, one obtains

$$\mathrm{curv}(L, \mathfrak{h}) = -\partial\overline{\partial}\log(h(\sigma(z), \sigma(z))) = -\partial\overline{\partial}\log(e^{-|z|^2}|f(z)|^2) = -2\pi i\omega_0.$$

Example 2.3.4 (*The projective space*). Let $\mathbb{C}P^N$ be the complex projective space endowed with the Fubini-Study form $\omega_{FS} = \frac{i}{2\pi}\partial\overline{\partial}\log(|Z_0|^2 + \cdots + |Z_N|^2)$, where $[Z_0 : \ldots : Z_N]$ are homogeneous coordinates. One can show that ω_{FS} is an integral Kähler form (see [31]) and so, from Theorem 2.3.2 there exists a Hermitian line bundle $(L, \mathfrak{h})$ such that $\mathrm{curv}(L, \mathfrak{h}) = -2\pi i\omega_{FS}$. This line bundle is the hyperplane line bundle $\mathcal{O}(1)$ (see Example 2.1.7). The space $H^0(\mathcal{O}(1))$ of global holomorphic sections on $\mathcal{O}(1)$ can be identified with the space of homogeneous polynomials of degree one (i.e. linear forms) in $N + 1$ variables [31, p. 164–167].

Example 2.3.5 (*The hyperbolic space*). Let $\mathbb{C}H^N$ be the complex hyperbolic space endowed with the hyperbolic form $\omega_{hyp} = \frac{i}{2\pi}\partial\overline{\partial}\log(1 - |z|^2)^{-1}$. Consider the trivial bundle $L = \mathbb{C}H^N \times \mathbb{C} \to \mathbb{C}H^N$ and for each $z \in \mathbb{C}H^N$ define the map

$$h : L_z \times L_z \to \mathbb{C},\ ((z, t_1), (z, t_2)) \mapsto (1 - |z|^2) t_1 \overline{t_2}.$$

The above map induces a Hermitian structure $\mathfrak{h}$ on L that defines a geometric quantization of $(\mathbb{C}H^N, \omega_{hyp})$. Indeed, if $\sigma(z) = (z, f(z))$ is a global holomorphic section of L, where $f : \mathbb{C}H^N \to \mathbb{C}$ is a holomorphic function, then by Proposition 2.1.15, one obtains

$$\mathrm{curv}(L, \mathfrak{h}) = -\partial\overline{\partial}\log(h(\sigma(z), \sigma(z))) = -\partial\overline{\partial}\log((1-|z|^2)|f(z)|^2) = -2\pi i\omega_{hyp}.$$

Definition 2.3.6. Two holomorphic Hermitian line bundles $(L_1, \mathfrak{h}_1)$ and $(L_2, \mathfrak{h}_2)$ over the same Kähler manifold (M, ω) are said to be *equivalent* if there exists an isomorphism of holomorphic line bundle $\psi : L_1 \to L_2$ such that $\psi^* \mathfrak{h}_2 = \mathfrak{h}_1$. The equivalence class of $(L, \mathfrak{h})$ is denoted by $[(L, \mathfrak{h})]$.

Let $\mathcal{L}(M, \omega)$ be the set of all geometric quantizations of the Kähler manifold (M, ω). In view of Definition 2.3.6, one can define

$$\mathrm{curv}([(L, \mathfrak{h})]) := \mathrm{curv}(L, \mathfrak{h}),$$

and then the set $\mathcal{L}(M, \omega)$ can be partitioned in equivalence classes $[(L, \mathfrak{h})]$. When M is simply connected all geometric quantizations on (M, ω) are equivalent ([41, p. 25]), therefore $\mathcal{L}(M, \omega)$ consists of a single equivalence class.
Let $(L, \mathfrak{h}) \in [(L, \mathfrak{h})]$ be a geometric quantization of a Kähler manifold (M, ω) and let us fix the following notations:

- $\mathrm{Aut}(M) := \{f : M \to M \,|\, f \text{ is biholomorphic}\}$;
- $\mathrm{Isom}(M, \omega) := \{f : (M, \omega) \to (M, \omega) \,|\, f \in C^\infty(M),\ f^*\omega = \omega\}$;
- $\mathrm{Aut}(L, \mathfrak{h}) := \{\hat{f} : L \to L \,|\, \hat{f} \text{ is biholomorphic, } \mathbb{C}\text{-linear on fibres, } \hat{f}^* \mathfrak{h} = \mathfrak{h}\}$.

Definition 2.3.7. A *lifting* of a map $f \in \mathrm{Aut}(M) \cap \mathrm{Isom}(M, \omega)$ is a map $\hat{f} \in \mathrm{Aut}(L, \mathfrak{h})$ such that the following diagram is commutative

$$\begin{array}{ccc} (L, \mathfrak{h}) & \xrightarrow{\hat{f}} & (L, \mathfrak{h}) \\ \pi \downarrow & & \downarrow \pi \\ (M, \omega) & \xrightarrow{f} & (M, \omega) \end{array}$$

The group of all maps f which admit a lifting $\hat{f}$ is denoted by $D_{[(L,\mathfrak{h})]}(M)$.

Proposition 2.3.8 ([41], Prop. 1.5.1, p. 27)**.** *Let* $(L, \mathfrak{h}) \in [(L, \mathfrak{h})]$ *be a geometric quantization of a simply connected Kähler manifold* (M, ω)*. Then the group* $D_{[(L,\mathfrak{h})]}(M)$ *equals* $\mathrm{Aut}(M) \cap \mathrm{Isom}(M, \omega)$*.*

We conclude this Section by outlining some important facts which we need in the sequel.

Lemma 2.3.9. *Let M be a simply connected complex manifold and $g : M \to \mathbb{R}$ a smooth function satisfying $\partial\overline{\partial}g = 0$. Then there exists a holomorphic function $f : M \to \mathbb{C}$ such that $g = \Re(f)$, where $\Re(f)$ denotes the real part of f.*

Proof. Since $\partial\bar{\partial}g = d(\partial g) = 0$ and the manifold is simply connected, there exists a function $h : M \to \mathbb{C}$ such that $\partial g = dh$. Since ∂g is of type $(1, 0)$, this implies that $\bar{\partial}h = 0$, i.e. h is a holomorphic function on M. By the reality of g, follows $dg = \partial g + \bar{\partial}g = d(h + \bar{h})$. Thus, up to a constant, $g = h + \bar{h}$, i.e. $g = \Re(f)$, where $f := 2h$. □

Lemma 2.3.10. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. Suppose that there exists a rotation invariant*[2] *function $g : \mathbb{C}^n \to \mathbb{C}$ such that $\Re(f) = g$. Then f is constant.*

Proof. Since f is holomorphic it can be represented, at the origin of $\mathbb{C}^n$, by a convergent power series of the form

$$f(z) = f(z_1, \ldots, z_n) = \sum_{\alpha_1,\ldots,\alpha_n}^{\infty} a_{\alpha_1,\ldots,\alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$

with $a_{\alpha_1,\ldots,\alpha_n} \in \mathbb{C}$. Moreover, since g is a rotation invariant function it can be represented by the convergent series

$$g(z) = \sum_{\beta_1,\ldots,\beta_n}^{\infty} b_{\beta_1,\ldots,\beta_n} |z_1|^{2\beta_1} \cdots |z_n|^{2\beta_n},$$

with $b_{\beta_1,\ldots,\beta_n} \in \mathbb{C}$. Therefore we must have

$$\Re(f) = \frac{1}{2}\left(\sum_{\alpha_1,\ldots,\alpha_n}^{\infty} a_{\alpha_1,\ldots,\alpha_n} z_1^{\alpha_1} \cdots z_n^{\alpha_n} + \sum_{\alpha_1,\ldots,\alpha_n}^{\infty} \bar{a}_{\alpha_1,\ldots,\alpha_n} \bar{z}_1^{\alpha_1} \cdots \bar{z}_n^{\alpha_n}\right) =$$
$$= \sum_{\beta_1,\ldots,\beta_n}^{\infty} b_{\beta_1,\ldots,\beta_n} |z_1|^{2\beta_1} \cdots |z_n|^{2\beta_n},$$

[2] A function $f : \mathbb{C}^n \to \mathbb{C}$ is said to be *rotation invariant* (resp. *radial*) in $(z_1, \ldots, z_n)$ if it only depends on $|z_1|^2, \ldots, |z_n|^2$ (resp $|z_1|^2 + \cdots + |z_n|^2$).

and this happens if and only if $\Re(f) = \frac{1}{2}(a_{0,\dots,0} + \bar{a}_{0,\dots,0}) = b_{0,\dots,0}$ and $a_{\alpha_1,\dots,\alpha_n} = 0$ for all $(\alpha_1, \dots, \alpha_n) \neq (0, \dots, 0)$, i.e f is constant. □

Definition 2.3.11. A Kähler manifold is said to be *homogeneous* if the group $\mathrm{Aut}(M) \cap \mathrm{Isom}(M, \omega)$ acts transitively on M.

Lemma 2.3.12. *Let M be a homogeneous Kähler manifold and $g : M \to \mathbb{R}$ a real valued function on M invariant under the group $Aut(M) \cap Isom(M, \omega)$. Then g is constant.*

Proof. g invariant under the group $\mathrm{Aut}(M) \cap \mathrm{Isom}(M, \omega)$ means that

$$g(f(x)) = g(x) \tag{2.10}$$

for all $f \in \mathrm{Aut}(M) \cap \mathrm{Isom}(M, \omega)$ and $x \in M$. Since M is homogeneous, for any $x, y \in M$ there exists a map $f \in \mathrm{Aut}(M) \cap \mathrm{Isom}(M, \omega)$ such that $f(x) = y$. Therefore $g(x) = g(y)$ for any $x, y \in M$, so g is constant. □

The following lemma shows how to extend L^2-*bounded holomorphic functions* to holomorphic sections of holomorphic line bundles.

Lemma 2.3.13 (Extension section lemma ([45], Lemma 4.1, p. 44))**.** *Let (M, ω) be a Kähler manifold of complex dimension n. Assume that there exists an analytic subvariety $H \subset M$ such that the restriction of L to $M \setminus H$ is the trivial holomorphic line bundle and let $\sigma : M \setminus H \to L$ be a trivializing holomorphic section. Let f be a holomorphic function on $M \setminus H$ such that*

$$\int_{M \setminus H} |f(x)|^2 h(\sigma(x), \sigma(x)) \frac{\omega^n}{n!} < \infty.$$

Then f extends to a (unique) global holomorphic section, namely there exists $s \in H^0(L)$ such that $s(x) = f(x)\sigma(x)$ for all $x \in M \setminus H$.

2.4 Regular quantization of Kähler manifolds

2.4.1 The epsilon function

Let $(L, \mathfrak{h})$ be a geometric quantization of a Kähler manifold (M, ω). Consider the space

$$\mathcal{H}_{\mathfrak{h}} = \left\{ s \in H^0(L) \mid \int_M h(s(x), s(x)) \frac{\omega^n(x)}{n!} < \infty \right\},$$

that is the space of global holomorphic sections of L which are bounded with respect to the inner product

$$\langle s, t \rangle_{\mathfrak{h}} := \int_M h(s(x), t(x)) \frac{\omega^n(x)}{n!} \tag{2.11}$$

for $s, t \in H^0(L)$. If M is compact $\mathcal{H}_{\mathfrak{h}} = H^0(L)$.
One can show that $\mathcal{H}_{\mathfrak{h}}$ is a complex vector space such that every Cauchy sequence is convergent with respect the distance

$$\mathrm{dist}(s, t) = ||s - t||_{\mathfrak{h}} = \sqrt{\langle s - t, s - t \rangle_{\mathfrak{h}}}$$

for $s, t \in H^0(L)$, i.e. $(\mathcal{H}_{\mathfrak{h}}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$ is a complex Hilbert space. An introduction to the theory of Hilbert spaces can be found in [61]. We just recall here that a sequence (finite or infinite) $\{s_j\}_{j=0,\dots,N}$ ($\dim \mathcal{H}_{\mathfrak{h}} = N + 1 \leq \infty$) is an *orthonormal basis* for the Hilbert space $(\mathcal{H}_{\mathfrak{h}}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$ if

- it is *orthonormal*: $\langle s_j, s_k \rangle_{\mathfrak{h}} = 0$ whenever $j \neq k$ and $||s_j||_{\mathfrak{h}} = 1$ for all j;
- it is *total*: if $\langle s, s_k \rangle_{\mathfrak{h}} = 0$ for all k necessarily s is the trivial section.

It is possible to prove that $(\mathcal{H}_{\mathfrak{h}}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$ admits an orthonormal basis, i.e. $(\mathcal{H}_{\mathfrak{h}}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$ is a separable Hilbert space (see [12] for a proof).

Definition 2.4.1. Let $(L, \mathfrak{h})$ be a geometric quantization of a Kähler manifold (M, ω) and $\{s_j\}_{j=0,\dots,N}$ ($\dim \mathcal{H}_{\mathfrak{h}} = N+1 \leq \infty$) be an orthonormal basis for the Hilbert space $(\mathcal{H}_{\mathfrak{h}}, \langle \cdot, \cdot \rangle_{\mathfrak{h}})$. The *epsilon function* of the pair $(L, \mathfrak{h})$ is a smooth real valued function on M defined, for any $x \in M$, by

$$\epsilon_{(L,\mathfrak{h})}(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x)). \tag{2.12}$$

In the literature the epsilon function was first introduced under the name of η-function by Rawnsley in [58] later renamed as θ-*function* in [12]. It is not hard to see that in each coordinate neighborhood the series (2.12) is independent of the choice of orthonormal basis $\{s_j\}_{j=0,\dots,N}$ and that $\epsilon_{(L,\mathfrak{h})}$ does not depend on the chosen representative $(L,\mathfrak{h})$ in the class $[(L,\mathfrak{h})] \in \mathcal{L}(M,\omega)$. In the case of simply connected manifolds the epsilon function depends only on the Kähler form ω, since $\mathcal{L}(M,\omega)$ consists of a single equivalence class. In this case we will often write ϵ_ω (or ϵ_g) instead of $\epsilon_{(L,\mathfrak{h})}$.

Lemma 2.4.2 ([12])**.** *The epsilon function $\epsilon_{(L,\mathfrak{h})}$ is invariant under the group $Aut(M) \cap Isom(M,\omega)$, i.e. $F^*(\epsilon_{(L,\mathfrak{h})}) = \epsilon_{(L,\mathfrak{h})}$ for every $F \in Aut(M) \cap Isom(M,\omega)$.*

Proposition 2.4.3. *Let $(L,\mathfrak{h})$ be a geometric quantization of a simply connected homogeneous Kähler manifold (M,ω). Then the function $\epsilon_{(L,\mathfrak{h})}$ is constant.*

Proof. From Proposition 2.3.8 and Lemma 2.4.2, the function ϵ_ω is invariant under the group $\mathrm{Aut}(M) \cap \mathrm{Isom}(M,\omega)$ and since the manifold is homogeneous, from Lemma 2.3.12, it is forced to be constant. □

2.4.2 The coherent states map

Let $(L,\mathfrak{h})$ be a geometric quantization of a Kähler manifold (M,ω) and $\{s_j\}_{j=0,\dots,N}$ ($\dim \mathcal{H}_\mathfrak{h} = N+1 \leq \infty$) be an orthonormal basis for the Hilbert space $(\mathcal{H}_\mathfrak{h}, \langle\cdot,\cdot\rangle_\mathfrak{h})$ and let $\sigma : U \to L^+$ be a trivialising holomorphic section. Suppose that for all $x \in M$ there exists $s_{j_0} \in \{s_j\}_{j=0,\dots,N}$ such that $s_{j_0}(x) \neq 0$. Under these assumptions, one can define the holomorphic map

$$\varphi_\sigma : U \to \mathbb{C}^{N+1} \setminus \{0\},\ x \mapsto \left(\frac{s_0(x)}{\sigma(x)}, \dots, \frac{s_N(x)}{\sigma(x)}\right). \tag{2.13}$$

If $\tau : V \to L^+$ is another trivialising holomorphic section then there exists a non-vanishing holomorphic function f on $U \cap V$ such that $\sigma(x) = f(x)\tau(x)$. Therefore

$$\varphi_\tau(x) = f(x)\varphi_\sigma(x)$$

for all $x \in U \cap V$, and the map (2.13) induces a holomorphic map, called *the coherent states map*, on to the whole M

$$\varphi : M \to \mathbb{C}P^N,\ x \mapsto [s_0(x), \ldots, s_N(x)],$$

whose local expression in the open set U is given by (2.13).
The following theorem can be found, for example, in [12, 58]. Here we propose a different proof.

Theorem 2.4.4. *Let ω_{FS} be the Fubini-Study form on $\mathbb{C}P^N$, and let φ be the coherent states map associated to (M, ω). Then*

$$\varphi^*(\omega_{FS}) = \omega + \frac{i}{2\pi}\partial\overline{\partial}\log \epsilon_{(L,\mathfrak{h})}. \tag{2.14}$$

Proof. Consider the map

$$\varphi_{|U} : U \to U_0,\ x \mapsto \left[1, \frac{s_1(x)}{s_0(x)}, \ldots, \frac{s_N(x)}{s_0(x)}\right]$$

that is the local expression (2.13) of the coherent states map in a trivialising open set U that contains a point p such that, up to unitary transformation of $\mathbb{C}P^N$, $\varphi(p) = [1, 0, \ldots, 0] \in U_0 = \{[z_0 : \ldots : z_N] \mid z_0 \neq 0\} \subset \mathbb{C}P^N$. Then

$$\begin{aligned}
\varphi^*_{|U}(\omega_{FS})(x) = \varphi^*_{|U}&\left(\frac{i}{2\pi}\partial\overline{\partial}\log\left(1 + \sum_{k=1}^N |z_k|^2\right)\right)(x) = \\
&= \frac{i}{2\pi}\partial\overline{\partial}\log\left(1 + \sum_{k=1}^N \left|\frac{s_k(x)}{s_0(x)}\right|^2\right) = \\
&= \omega + \frac{i}{2\pi}\partial\overline{\partial}\log\left(1 + \sum_{k=1}^N \left|\frac{s_k(x)}{s_0(x)}\right|^2\right) - \omega,
\end{aligned}$$

where ω is restricted to U. Under the previous assumptions the local expression of ω in U is given by $\omega = -\frac{i}{2\pi}\partial\overline{\partial}\log h(\sigma(x), \sigma(x))$, then one find

$$\varphi^*_{|U}(\omega_{FS})(x) = \omega + \frac{i}{2\pi}\partial\overline{\partial}\log\left[h(\sigma(x), \sigma(x))\left(1 + \sum_{k=1}^N \left|\frac{s_k(x)}{s_0(x)}\right|^2\right)\right].$$

For all $j = 0, \dots, N$ there exist non-vanishing holomorphic functions g_j on U such that $s_j = g_j \sigma$, therefore

$$\begin{aligned}
\varphi^*_{|U}(\omega_{FS})(x) &= \omega + \frac{i}{2\pi}\partial\bar{\partial}\log\left[h(\sigma(x),\sigma(x))\left(1+\sum_{k=1}^{N}\left|\frac{g_k(x)}{g_0(x)}\right|^2\right)\right] = \\
&= \omega + \frac{i}{2\pi}\partial\bar{\partial}\log\left[h(\sigma(x),\sigma(x))\left(\frac{\sum_{k=1}^{N}|g_k(x)|^2}{|g_0(x)|^2}\right)\right] = \\
&= \omega + \frac{i}{2\pi}\partial\bar{\partial}\log\left[h(\sigma(x),\sigma(x))\left(\sum_{k=1}^{N}|g_k(x)|^2\right)\right] = \\
&= \omega + \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{k=1}^{N}|g_k(x)|^2 h(\sigma(x),\sigma(x))\right) = \\
&= \omega + \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{k=1}^{N}h(g_k(x)\sigma(x),g_k(x)\sigma(x))\right) = \\
&= \omega + \frac{i}{2\pi}\partial\bar{\partial}\log\epsilon_{(L,\mathfrak{h})}.
\end{aligned}$$

□

Corollary 2.4.5. *Let $(L, \mathfrak{h})$ be a geometric quantization of a Kähler manifold (M, ω). If $\epsilon_{(L,\mathfrak{h})}$ is a positive constant, then the coherent states map is a Kähler immersion.*

Proof. First of all, if $\epsilon_{(L,\mathfrak{h})}$ is a constant different from zero then for all $x \in M$ there exists $s_{j_0} \in \{s_j\}_{j=0,\dots,N}$ such that $s_{j_0}(x) \neq 0$ and consequently the coherent states map can be defined. By construction the coherent states map is holomorphic and, by Theorem 2.4.4, $\varphi^*(\omega_{FS}) = \omega$. Finally the map φ, that is isometric in each point of M, is an immersion. □

2.4.3 Balanced metrics

Let $(L, \mathfrak{h})$ be a geometric quantization of a Kähler manifold (M, ω) and g be the corresponding Kähler metric.

Definition 2.4.6. The metric g on M is called *balanced* if $\epsilon_{(L,\mathfrak{h})}$ is a positive constant.

The definition of balanced metric was originally given by Donaldson [20] in the case of compact quantizable Kähler manifold and generalized in [5] to the non compact case (see also [25, 30, 48]).
It immediately follows from Corollary 2.4.5 that a balanced metric is projectively induced via the coherent states map. Note that a projectively induced metric is not always balanced. We will give an explicit example of this fact in Chapter 4.

Question 1. *Given a Kähler manifold, what conditions have to exist, for having a balanced metric?*

The answer to the previous question is a really difficult matter. It is a still open problem finding a condition of existence of balanced metrics on a non compact manifold. In the compact case we have this fundamental result due to Donaldson [20].

Theorem 2.4.7. *Let $(L, \mathfrak{h})$ be a geometric quantization of a compact Kähler manifold (M, g), $g \in c_1(L)$, such that g is cscK. Assume that $\frac{Aut(M,L)}{\mathbb{C}^*}$ is discrete. Then, for all sufficiently large integers m, there exists a unique balanced metric g_m on M, with $g_m \in c_1(L^m)$, such that $\frac{g_m}{m}$ C^∞-converges to g.*

The quotient space $\frac{Aut(M,L)}{\mathbb{C}^*}$ denotes the biholomorphisms group of M which lift to holomorphic bundles maps $L \to L$ modulo the trivial automorphism group $\mathbb{C}^*$, and L^m denotes the m-th tensor power of L.
Note that the assumption on the automorphism group in the theorem cannot be dropped. Indeed, from the point of view of the existence of balanced metrics, a result of Della Vedova and Zuddas [17] shows that there exist a large class of Kähler manifolds (M, g), where g is a cscK metric but mg is not balanced for all sufficiently large integers m.
Regarding the uniqueness of balanced metrics we have this result due to Arezzo, Loi and Zuddas [6].

Proposition 2.4.8. *Let g and $\tilde{g}$ be two balanced metrics whose associated Kähler forms are cohomologous. Then g and $\tilde{g}$ are isometric, i.e. there exists $F \in Aut(M)$ such that $F^*\tilde{g} = g$.*

Consider now the Kähler form $m\omega$ on M, where m is a natural number, and the associated Kähler metric mg. If ω is an integral form then $m\omega$ is integral for any positive integer m. Therefore one can consider the quantum line bundle $(L^m, \mathfrak{h}_m)$ for $(M, m\omega)$, where L^m is the m-th tensor power of L and $\mathfrak{h}_m$ is the m-th tensor power of $\mathfrak{h}$ defined generalizing (2.6):

$$h_m(s_1 \otimes \cdots \otimes s_m, t_1 \otimes \cdots \otimes t_m)(x) := h(s_1, t_1)(x) \cdots h(s_m, t_m)(x),$$

for any $s_1 \otimes \cdots \otimes s_m, t_1 \otimes \cdots \otimes t_m \in \Gamma(L^m)$ and for all $x \in M$.
In this contest it is interesting to study the balanced condition for the metric mg when m changes, namely study the constancy of the epsilon function $\epsilon_{(L^m, \mathfrak{h}_m)}$ for every positive integer m. The fact that g is balanced does not imply that mg is.

Example 2.4.9. Let (Σ_g, g_{hyp}) be a compact Riemannian surface of genus $g \geq 2$ equipped with the hyperbolic metric. It is well-known that the associated Kähler form ω_{hyp} is integral and if $(L, \mathfrak{h})$ is a geometric quantization of (Σ_g, g_{hyp}) then $\frac{Aut(\Sigma_g, L)}{\mathbb{C}^*}$ is finite. Then, by Theorem 2.4.7, for all sufficiently large integers m, there exists a unique balanced metric g_m on Σ_g, with $g_m \in c_1(L^m)$, such that $\frac{g_m}{m}$ C^∞-converges to g_{hyp}. If by contradiction g_m is balanced for all naturals m then, by Theorem 2.4.11 below, scal_{g_m} should be constant. On the other hand, by Proposition 2.4.8, there exists $F \in \mathrm{Aut}(\Sigma_g)$ such that $F^*(g_1) = g_{hyp}$. But g_1 is balanced, and hence it is projectively induced, and (Σ_g, g_{hyp}) is not projectively induced [6].

Definition 2.4.10. A geometric quantization $(L, \mathfrak{h})$ of a Kähler manifold (M, g) is called a *regular quantization* if mg is balanced for any (sufficiently large) natural number m, i.e. if $\epsilon_{(L^m, \mathfrak{h}_m)}$ is a positive constant for any (sufficiently large) natural number m.

Many authors (see, e.g. [3] and [7] and references therein) have been trying to understand what kind of properties are enjoyed by those Kähler manifolds which admit a regular quantization. Here we recall two facts. The first can be found in [43].

Theorem 2.4.11. *A Kähler metric which admits a regular quantization is a cscK metric.*

The second one follows immediately from Proposition 2.4.3:

Corollary 2.4.12. *A geometric quantization of a homogeneous and simply connected Kähler manifold is regular.*

Remark 2.4.13. Not all homogeneous manifolds admit a regular quantization. An example is given by taking the complex torus (Example 1.2.5) with the flat form ω_0. One can prove that this manifold is homogeneous and that admits a geometric quantization $(L, \mathfrak{h})$ (see [38, 42]). On the other hand a theorem in [63] asserts that this manifold can not be projectively induced and so the quantization $(L, \mathfrak{h})$ can not be regular.

Therefore, the following question naturally arises:

Question 2. *Is it true that a complete Kähler manifold (M, ω) which admits a regular quantization is necessarily homogeneous (and simply-connected)?*

Remark 2.4.14. The assumption of completeness is necessary otherwise one can construct regular quantizations on non-homogeneous Kähler manifolds obtained by deleting a measure zero set from a homogeneous Kähler manifold (see [45] for more details). The simply connected request is in brackets since one can prove that every homogeneous and projectively induced Kähler manifold is simply connected (see [18]).

In this thesis we give a negative answer to Question 2 in the non compact case by considering a complete cscK metric on the complex blow-up $\tilde{\mathbb{C}}^2$ (note that it is the first non-contractible example (cfr. Remark 3.2.8)). In the compact case this question is still open and of great interest also because the Kähler manifolds involved are projectively algebraic. We believe our results could be used to built regular quantizations of non-homogeneous compact Kähler manifolds.

2.5 TYCZ expansion

If in the above setting M is compact ($\dim M = n$), there exists a complete asymptotic expansion of the epsilon function introduced by D. Catlin [16] and S. Zelditch [68] independently:

$$\epsilon_{(L^m,\mathfrak{h}_m)}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}, \tag{2.15}$$

where $a_0(x) = 1$ and $a_j(x)$, $j = 1, \dots$ are smooth functions on M. This means that, for any non negative integers r, k the following estimate holds:

$$||\epsilon_{(L^m,\mathfrak{h}_m)}(x) - \sum_{j=0}^{k} a_j(x) m^{n-j}||_{C^r} \leq C_{k,r} m^{n-k-1},$$

where $C_{k,r}$ are constants depending on k, r and on the Kähler form ω, and $||\cdot||_{C^r}$ denotes the C^r norm. The expansion (2.15) is called *Tian-Yau–Catlin-Zelditch expansion* (TYCZ expansion in the sequel). Later on, Z. Lu [52], by means of Tian's peak section method, proved that each of the coefficients $a_j(x)$ is a polynomial of the curvature and its covariant derivatives at x of the metric g which can be found by finitely many algebraic operations. In particular, he computed the first three coefficients. The first two are given by:

$$\begin{cases} a_1(x) = \frac{1}{2}\mathrm{scal}_g \\ a_2(x) = \frac{1}{3}\Delta\mathrm{scal}_g + \frac{1}{24}(|R|^2 - 4|\mathrm{Ric}|^2 + 3\mathrm{scal}_g^2) \end{cases}, \tag{2.16}$$

where scal_g, R, Ric denote respectively the scalar curvature, the Riemannian curvature tensor and the Ricci tensor of (M, g) in local coordinates (see Section 1.4). When M is non compact, there is not a general theorem which assures the existence of an asymptotic expansion (2.15).

In [46] the authors address the problem of studying those Kähler manifolds whose TYCZ expansion is *finite*, namely the epsilon function is of the form:

$$\epsilon_{(L^m,\mathfrak{h}_m)}(x) = f_s(x)m^s + f_{s-1}(x)m^{s-1} + \dots + f_r(x)m^r, \quad f_j \in C^\infty(M),\ s, r \in \mathbb{Z}$$

showing the following:

Theorem 2.5.1 ([46], Theor. 1.1). *Let (M, g) be a Kähler manifold with integral Kähler form ω and of finite complex dimension n. Assume that the corresponding TYCZ expansion is finite. Then $\epsilon_{(L,\mathfrak{h})}(x)$ is forced to be a polynomial in m of degree n.*

2.6 The epsilon function for the complex Euclidean space

Let $\mathbb{C}^2$ be the complex Euclidean space endowed with the Kähler form $\omega_0 = \frac{i}{2\pi}\partial\bar{\partial}|z|^2$ and $L = \mathbb{C}^2 \times \mathbb{C}$ be the trivial bundle. The map

$$h_m : L_z^m \times L_z^m \to \mathbb{C}\,((z,t_1),(z,t_2) \mapsto e^{-m|z|^2} t_1\overline{t_2},$$

induces a Hermitian structures $\mathfrak{h}_m$ on L^m that defines a geometric quantization of $(\mathbb{C}^N, m\omega_0)$, where m is a positive natural number. In this case the space of global holomorphic sections in given by

$$H^0(L^m) = \{s(z) = (z, f(z))\,|\, f : \mathbb{C}^2 \to \mathbb{C} \text{ is holomorphic}\},$$

therefore the complex Hilbert space $\mathcal{H}_{\mathfrak{h}_m}$ is defined as

$$\mathcal{H}_{\mathfrak{h}_m} = \left\{ s \in H^0(L^m)\,|\, \langle s, s\rangle_{\mathfrak{h}_m} = \int_{\mathbb{C}^2} e^{-m|z|^2} |f(z)|^2 \frac{\omega_0^2}{2} < \infty \right\}.$$

Lemma 2.6.1. *The set*

$$s_J := \left((z_1,z_2), z_1^{j_1} z_2^{j_2}\right), \quad J := (j_1, j_2),\, j_1, j_2 \in \mathbb{N} \tag{2.17}$$

is an orthogonal sequence for the Hilbert space $(\mathcal{H}_{\mathfrak{h}_m}, \langle\cdot,\cdot\rangle_{\mathfrak{h}_m})$.

Proof. Since $\frac{\omega_0^2}{2!} = \det g_0 vol_{\mathbb{C}^2}$, one find

$$\langle s_J, s_K\rangle_{\mathfrak{h}_m} = \int_{\mathbb{C}^2} e^{-m|z|^2} z_1^{j_1} z_2^{j_2} \overline{z_1^{k_1} z_2^{k_2}} \left(\frac{i}{2\pi}\right)^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.$$

By passing to polar coordinates $z_1 = \rho_1 e^{i\vartheta_1}, z_2 = \rho_2 e^{i\vartheta_2}$ with $\rho_1, \rho_2 \in (0, +\infty), \vartheta_1, \vartheta_2 \in (0, 2\pi)$, one has

$$\langle s_J, s_K\rangle_{\mathfrak{h}_m} = \frac{1}{\pi^2} \int_0^{2\pi}\int_0^{2\pi}\int_0^{+\infty}\int_0^{+\infty} \zeta(\rho_1,\rho_2,\vartheta_1,\vartheta_2) d\rho_1 d\rho_2 d\vartheta_1 d\vartheta_2,$$

where

$$\zeta(\rho_1,\rho_2,\vartheta_1,\vartheta_2) = e^{-m(\rho_1^2+\rho_2^2)} \rho_1^{j_1+k_1+1} \rho_2^{j_2+k_2+1} e^{i(j_1-k_1)\vartheta_1} e^{i(j_2-k_2)\vartheta_2}.$$

So if $J \neq K$, at least one of the two integrals in the theta variable is zero. Indeed, if for example $j_1 \neq k_1$, one has

$$\int_0^{2\pi} e^{i(j_1-k_1)\vartheta_1} d\vartheta_1 = \frac{1}{i(j_1-k_1)} e^{i(j_1-k_1)}\Big]_0^{2\pi} = 0.$$

In the case $J = K$ we find

$$\langle s_J, s_J \rangle_{\mathfrak{h}_m} = 4 \int_0^{+\infty} \int_0^{+\infty} e^{-m(\rho_1^2+\rho_2^2)} \rho_1^{2j_1+1} \rho_2^{2j_2+1} d\rho_1 d\rho_2.$$

With the substitution $\rho_1 = r\cos\theta$, $\rho_2 = r\sin\theta$, $0 < r < +\infty$, $0 < \theta < \frac{\pi}{2}$ one finds a product of one variable integrals:

$$\langle s_J, s_J \rangle_{\mathfrak{h}_m} = 4 \int_0^{\frac{\pi}{2}} (\cos\vartheta)^{2j_1+1} (\sin\vartheta)^{2j_2+1} d\vartheta \int_0^{+\infty} e^{-mr^2} r^{2(j_1+j_2+1)+1} dr.$$

For the first integral [1, 6.1.1, p. 255] we find

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{2j_1+1} (\sin\theta)^{2j_2+1} d\theta = \frac{\Gamma(j_1+1)\Gamma(j_2+1)}{2\Gamma(j_1+j_2+2)} = \frac{j_1! j_2!}{2(j_1+j_2+1)!}.$$

For the second integral, by [1, 6.2.1, p. 258]

$$\int_0^{\infty} r^s e^{-mr^2}\, dr = \frac{\Gamma(\frac{s+1}{2})}{2m^{(\frac{s+1}{2})}},$$

we find

$$\int_0^{\infty} e^{-mr^2} r^{2(j_1+j_2+1)+1} dr = \frac{\Gamma(j_1+j_2+2)}{2m^{j_1+j_2+2}} = \frac{(j_1+j_2+1)!}{2m^{j_1+j_2+2}}.$$

Therefore

$$\langle s_J, s_J \rangle_{\mathfrak{h}_m} = 4\frac{j_1! j_2!}{2(j_1+j_2+1)!} \cdot \frac{(j_1+j_2+1)!}{2m^{j_1+j_2+2}} = \frac{j_1! j_2!}{m^{j_1+j_2+2}},$$

and this proves that (2.17) is an orthogonal sequence for $(\mathcal{H}_{\mathfrak{h}_m}, \langle\cdot,\cdot\rangle_{\mathfrak{h}_m})$. □

Lemma 2.6.2. *The set*

$$s_J := \left((z_1, z_2), \frac{z_1^{j_1} z_2^{j_2}}{\sqrt{\frac{j_1! j_2!}{m^{j_1+j_2+2}}}} \right), \quad J := (j_1, j_2),\ j_1, j_2 \in \mathbb{N} \tag{2.18}$$

is an orthonormal basis for the Hilbert space $(\mathcal{H}_{\mathfrak{h}_m}, \langle\cdot,\cdot\rangle_{\mathfrak{h}_m})$.

Proof. The sequence (2.18) is orthonormal by Lemma 2.6.1 and it is also total, indeed every global holomorphic section $s = (z, g(z))$, where $g : \mathbb{C}^2 \to \mathbb{C}$ is holomorphic, can be represented at the origin by a convergent power series of the form

$$g(z) = g(z_1, z_2) = \sum_{j_1, j_2 = 0}^{\infty} a_{j_1, j_2} z_1^{j_1} z_2^{j_2},$$

with complex coefficients a_{j_1, j_2}.
Then if $\langle s, s_K \rangle_{\mathfrak{h}_m} = 0$ for all $K = (k_1, k_2), k_1, k_2 \in \mathbb{N}$, so for each fixed natural pair (k_1, k_2), we must have

$$\begin{aligned} 0 &= \langle ((z_1, z_2), \sum_{j_1, j_2 = 0}^{\infty} a_{j_1, j_2} z_1^{j_1} z_2^{j_2}), ((z_1, z_2), \frac{z_1^{k_1} z_2^{k_2}}{||z_1^{k_1} z_2^{k_2}||^2_{\mathfrak{h}_m}}) \rangle_{\mathfrak{h}_m} = \\ &= \frac{1}{||z_1^{k_1} z_2^{k_2}||^2_{\mathfrak{h}_m}} \sum_{j_1, j_2 = 0}^{\infty} a_{j_1 j_2} \langle ((z_1, z_2), z_1^{j_1} z_2^{j_2}), ((z_1, z_2), z_1^{k_1} z_2^{k_2}) \rangle_{\mathfrak{h}_m} = \\ &= \frac{1}{||z_1^{k_1} z_2^{k_2}||^2_{\mathfrak{h}_m}} a_{k_1 k_2} ||z_1^{k_1} z_2^{k_2}||^2_{\mathfrak{h}_m} = a_{k_1 k_2}, \end{aligned}$$

therefore s is the trivial section and this proves the lemma. □

Theorem 2.6.3. *The metric mg_0 is balanced for any positive natural number m.*

Proof. By definition 2.4.1 and from (2.18), for the epsilon function one has:

$$\begin{aligned} \epsilon_{m\omega_0}(z) &= \sum_{j_1, j_2 = 0}^{\infty} e^{-m(|z_1|^2 + |z_2|^2)} \frac{|z_1|^{2j_1} |z_2|^{2j_2}}{||z_1^{j_1} z_2^{j_2}||^2_{\mathfrak{h}_m}} = \\ &= e^{-m(|z_1|^2 + |z_2|^2)} \sum_{j_1, j_2 = 0}^{\infty} \frac{|z_1|^{2j_1} |z_2|^{2j_2}}{j_1! j_2!} m^{j_1 + j_2 + 2} = \\ &= m^2 e^{-m(|z_1|^2 + |z_2|^2)} \sum_{j_1, j_2 = 0}^{\infty} \frac{|z_1|^{2j_1} |z_2|^{2j_2}}{j_1! j_2!} m^{j_1 + j_2} = \\ &= m^2 e^{-m(|z_1|^2 + |z_2|^2)} e^{m(|z_1|^2 + |z_2|^2)} = \\ &= m^2, \end{aligned}$$

and this proves the theorem. □

Corollary 2.6.4. *The quantization* $(L^m, \mathfrak{h}_m)$ *of* $(\mathbb{C}^2, \omega_0)$*is regular.*

Corollary 2.6.5. *All the coefficients* $a_j(x)$*, with* $j \geq 1$*, of the TYCZ expansion for the flat metric* g_0 *on* $\mathbb{C}^2$ *vanish.*

The above results can be easily generalized to $(\mathbb{C}^N, \omega_0)$.

Chapter 3

The Burns–Simanca metric

3.1 Preliminaries

Let $\tilde{\mathbb{C}}^2$ be the blow-up of $\mathbb{C}^2$ at the origin (cfr. 1.1.1) and and $p_r : \tilde{\mathbb{C}}^2 \setminus H \to \mathbb{C}^2 \setminus \{0\}$ the biholomorphic map defined as in the proof of the Proposition 1.1.8, where H is the exceptional divisor. Take on $\mathbb{C}^2 \setminus \{0\}$ the $(1,1)$-form given by

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}(|z|^2 + \log|z|^2). \tag{3.1}$$

We claim that the pull-back $p_r^*(\omega)$ of ω, a priori defined only on $\tilde{\mathbb{C}}^2 \setminus H$, extends in fact to all $\tilde{\mathbb{C}}^2$. The pull-back $p_r^*(\omega)$ is given in the coordinates (1.3) by

$$p_r^*(\omega) = \frac{i}{2\pi}\partial\bar{\partial}\left(|z_1|^2(1+|z_2|^2) + \log(1+|z_2|^2)\right),$$

on $\tilde{U}_1 \setminus H$, and

$$p_r^*(\omega) = \frac{i}{2\pi}\partial\bar{\partial}\left(|z_2|^2(1+|z_1|^2) + \log(1+|z_1|^2)\right),$$

on $\tilde{U}_2 \setminus H$. This shows that $p_r^*(\omega)$ extends to the whole $\tilde{\mathbb{C}}^2$, as claimed. Clearly on $\tilde{\mathbb{C}}^2 \setminus H$ this form is given in local coordinates by (3.1).
The metric associated to $p_r^*(\omega)$ has been discovered by Burns [10] when $n = 2$ (first described by Le Brun [39]) and by Simanca [60] when $n \geq 3$. It is known in literature as the *Burns–Simanca metric* (see [8] and [62]) and denoted here

by g_{BS}. The associated form is denoted here by ω_{BS}. One easily verifies that g_{BS} is a Kähler metric by checking that it is non degenerate and positive definite: we have

$$g_{BS} = \begin{pmatrix} 1 + \frac{|z_2|^2}{(|z_1|^2+|z_2|^2)^2} & -\frac{\bar{z}_1 z_2}{(|z_1|^2+|z_2|^2)^2} \\ -\frac{z_1\bar{z}_2}{(|z_1|^2+|z_2|^2)^2} & 1 + \frac{|z_1|^2}{(|z_1|^2+|z_2|^2)^2} \end{pmatrix} \tag{3.2}$$

and one concludes just by noticing that $(g_{BS})_{1\bar{1}} > 0$ and $\det(g_{BS}) = 1+\frac{1}{|z|^2} > 0$.

Proposition 3.1.1. *The followings properties hold:*

1. *g_{BS} is complete.*
2. *g_{BS} is zero scalar curvature but not Ricci-flat.*
3. *$(\tilde{\mathbb{C}}^2, g_{BS})$ is a non homogeneous manifold.*
4. *$(\tilde{\mathbb{C}}^2, g_{BS})$ is projectively induced.*

Proof. 1. It is sufficient to show that the length of divergent curves is infinite (see [19, Ex. 5, p. 153]). By definition, a divergent curve on $\tilde{\mathbb{C}}^2$ is a differentiable map $\alpha : [0, +\infty) \to \tilde{\mathbb{C}}^2$ such that for any compact set $K \subset \tilde{\mathbb{C}}^2$ there exists $t_0 \in [0, \infty)$ such that $\alpha(t) \notin K$ for all $t > t_0$. Since $H \simeq \mathbb{C}P^1$ is compact and $(\tilde{\mathbb{C}}^2 \setminus H, \omega_{BS})$ is isometric to $(\mathbb{C}^2 \setminus \{0\}, \omega)$ via the projection p_r, we are reduced to show that a divergent curve $\alpha : [0, +\infty) \to \mathbb{C}^2 \setminus \{0\}$ has infinite length with respect to (3.1). In order to show this, notice that

$$\omega = \frac{i}{2\pi}\partial\bar{\partial}(|z|^2 + \log|z|^2) = \frac{i}{2\pi}\partial\bar{\partial}(|z|^2) + \frac{i}{2\pi}\partial\bar{\partial}(\log|z|^2),$$

where the second addendum $\eta = \frac{i}{2\pi}\partial\bar{\partial}(\log|z|^2)$ is a positive-semidefinite form (one finds $\eta_{1\bar{1}} = \frac{|z_2|^2}{|z|^4} > 0$, $\eta_{2\bar{2}} = \frac{|z_1|^2}{|z|^4} > 0$ and $\det(\eta) = 0$) and the first addendum is the flat Euclidean form ω_0. Then, if $\|\cdot\|$ (resp. $\|\cdot\|_0$) denotes the norm with respect to ω (resp. with respect to ω_0), we clearly have

$$\|\alpha'(t)\| \geq \|\alpha'(t)\|_0.$$

It follows that

$$\int_0^\infty \|\alpha'(t)\|dt \geq \int_0^\infty \|\alpha'(t)\|_0 dt = +\infty$$

where the last equality follows from the fact that ω_0 is complete and then divergent curves on $\mathbb{C}^2$ have infinite length.
2. The inverse matrix of (3.2) is given by

$$g_{BS}^{-1} = \frac{|z|^2}{1+|z|^2} \begin{pmatrix} 1+\frac{|z_1|^2}{|z|^4} & \frac{\bar{z}_1 z_2}{|z|^4} \\ \frac{z_1 \bar{z}_2}{|z|^4} & 1+\frac{|z_2|^2}{|z|^4} \end{pmatrix}$$

and by recalling (1.4) one gets

$$\mathrm{Ric} = \begin{pmatrix} \frac{(1+|z|^2)|z|^2-|z_1|^2(1+2|z|^2)}{(1+|z|^2)^2|z|^4} & -\frac{z_1\bar{z}_2(1+2|z|^2)}{(1+|z|^2)^2|z|^4} \\ -\frac{\bar{z}_1 z_2(1+2|z|^2)}{(1+|z|^2)^2|z|^4} & \frac{(1+|z|^2)|z|^2-|z_2|^2(1+2|z|^2)}{(1+|z|^2)^2|z|^4} \end{pmatrix}$$

Finally, by (1.6), one finds

$$\mathrm{scal}_{g_{BS}} = 0.$$

For the second part, by (1.5) one finds

$$\mathrm{Ric}(g_{BS}) = -i\frac{|z|^2(2+|z|^2)}{(1+|z|^2)^2},$$

that is, clearly, different from 0.
3. Suppose $(\tilde{\mathbb{C}}^2, g_{BS})$ is homogeneous. The map $|\mathrm{Ric}|^2 : \tilde{\mathbb{C}}^2 \to \mathbb{R}$ is invariant under the group $\mathrm{Aut}(\tilde{\mathbb{C}}^2) \cap \mathrm{Isom}(\tilde{\mathbb{C}}^2, \omega_{BS})$ and, by recalling (1.7), a straightforward computation gives $|\mathrm{Ric}|^2 = \frac{2}{(1+|z|^2)^4}$, in contrast with Lemma 2.3.12.
4. A proof of this fact can be found in [47, Theor 1.3]. We will prove it for any $n \geq 2$ in Theorem 4.2.1. □

So the Burns–Simanca metric is a well-known and important example, both from mathematical and physical point of view, of non homogeneous complete, zero constant scalar curvature Kähler metric.
Here $(\tilde{\mathbb{C}}^2, g_{BS})$ plays a key role in trying to answer to Question 2. Indeed Proposition 3.1.1 makes us think, or at least makes us suspect, that the answer to Question 2 may be negative in the non compact case. Indeed, we will show in Theorem 3.2.1 that the Burns–Simanca metric g_{BS} on $\tilde{\mathbb{C}}^2$ admits a regular quantization.

3.2 The Burns–Simanca metric admits a regular quantization

Theorem 3.2.1. *Let $\tilde{\mathbb{C}}^2$ be the blow-up of $\mathbb{C}^2$ at the origin endowed with the Burns–Simanca metric g_{BS}. Then $(\tilde{\mathbb{C}}^2, g_{BS})$ admits a regular quantization such that $\epsilon_{mg_{BS}} = m^2$.*

In order to prove Theorem 3.2.1 consider the holomorphic line bundle $L \to \tilde{\mathbb{C}}^2$ such that $c_1(L) = [\omega_{BS}]$, where $c_1(L)$ is the first Chern class of L. Such line bundle exists since ω_{BS} is integral[1] and it is unique, up to isomorphisms of line bundle, since $\tilde{\mathbb{C}}^2$ is simply-connected (cf. Proposition 2.1.10).
The map

$$h_m(\sigma(x), \sigma(x)) = \frac{1}{|z|^{2m}} e^{-m|z|^2} |q|^2,$$

induces a Hermitian structures $\mathfrak{h}_m$ on L^m that defines a geometric quantization of $(\tilde{\mathbb{C}}^2, m\omega_{BS})$, where m is a positive natural number and

$$\sigma : U \subset \tilde{\mathbb{C}}^2 \setminus H \to L^m \setminus \{0\}, x \mapsto (z, q) \in U \times \mathbb{C}$$

is a trivialising holomorphic section.
Since $L^m_{|\tilde{\mathbb{C}}^2 \setminus H}$ is equivalent to the trivial bundle[2] $\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}$, one can find a natural bijection between the complex space $H^0(L^m)$ and the space of holomorphic functions on $\mathbb{C}^2$ vanishing at the origin with multiplicity greater or equal than m (see [31, Ch. 1, p. 136 et seq.] for more details). This bijection takes $s \in H^0(L^m)$ to the holomorphic function f_s on $\mathbb{C}^2$ obtained by restricting s to $\tilde{\mathbb{C}}^2 \setminus H \simeq \mathbb{C}^2 \setminus \{0\}$. Moreover H has zero measure in $\tilde{\mathbb{C}}^2$ by Sard's theorem [59, Theor 4.1, p. 885] (note that the inclusion $i : H \to \tilde{\mathbb{C}}^2$ is smooth and $\dim H < \dim \tilde{\mathbb{C}}^2$), then one gets

$$\begin{aligned} \langle s, s \rangle_{\mathfrak{h}_m} &= \int_{\tilde{\mathbb{C}}^2} h_m(s(x), s(x)) \frac{\omega_{BS}^2}{2!} = \\ &= \int_{\mathbb{C}^2 \setminus \{0\}} \frac{e^{-m|z|^2}}{|z|^{2m}} |f_s(z)|^2 \left(1 + \frac{1}{|z|^2}\right) d\mu(z) < \infty, \end{aligned} \tag{3.3}$$

[1] By Proposition 3.1.1 ω_{BS} is the pull-back of the Fubini-Study form that is integral.

[2] Every vector bundle over a contractible base space is trivial [53, Cor 15.22, p. 156] and in this case, $\tilde{\mathbb{C}}^2 \setminus H$ is contractible since it is biholomorphic to $\mathbb{C}^2 \setminus \{0\}$.

where $d\mu(z) = \left(\frac{i}{2\pi}\right)^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$. Therefore, in this case, the inclusion $\mathcal{H}_m \subseteq H^0(L^m)$ is indeed an equality, namely $\mathcal{H}_m = H^0(L^m)$. Recall that $\mathcal{H}_m$ denotes the space of global holomorphic sections s of L^m, which are bounded with respect to

$$\langle s, s \rangle_{\mathfrak{h}_m} = ||s||^2_{\mathfrak{h}_m} = \int_{\tilde{\mathbb{C}}^2} h_m(s(x), s(x)) \frac{\omega^2_{BS}}{2!}.$$

Lemma 3.2.2. *The set*

$$s_J := \Big((z_1, z_2), z_1^{j_1} z_2^{j_2}\Big), \quad J := \{(j_1, j_2), \mid j_1, j_2 \in \mathbb{N},\, j_1 + j_2 \geq m\} \tag{3.4}$$

is an orthogonal sequence for the Hilbert space $(\mathcal{H}_{\mathfrak{h}_m}, \langle \cdot, \cdot \rangle_{\mathfrak{h}_m})$.

Proof. By passing to polar coordinates $z_1 = \rho_1 e^{i\vartheta_1}$, $z_2 = \rho_2 e^{i\vartheta_2}$ with $\rho_1, \rho_2 \in (0, +\infty)$, $\vartheta_1, \vartheta_2 \in (0, 2\pi)$, one has

$$\langle s_J, s_K \rangle_{\mathfrak{h}_m} = \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{+\infty} \int_0^{+\infty} \zeta(\rho_1, \rho_2, \vartheta_1, \vartheta_2) d\rho_1 d\rho_2 d\vartheta_1 d\vartheta_2,$$

where

$$\zeta(\rho_1, \rho_2, \vartheta_1, \vartheta_2) = \frac{e^{-m(\rho_1^2+\rho_2^2)}}{(\rho_1^2+\rho_2^2)^{m+1}} (1+\rho_1^2+\rho_2^2) \rho_1^{j_1+k_1+1} \rho_2^{j_2+k_2+1} e^{i(j_1-k_1)\vartheta_1} e^{i(j_2-k_2)\vartheta_2}.$$

So if $J \neq K$, at least one of the two integrals in the theta variable is zero. Indeed, if for example $j_1 \neq k_1$, one has

$$\int_0^{2\pi} e^{i(j_1-k_1)\vartheta_1} d\vartheta_1 = \frac{1}{i(j_1-k_1)} e^{i(j_1-k_1)} \Big]_0^{2\pi} = 0.$$

In the case $J = K$ we find

$$\langle s_J, s_J \rangle_{\mathfrak{h}_m} = 4 \int_0^{+\infty} \int_0^{+\infty} \frac{e^{-m(\rho_1^2+\rho_2^2)}}{(\rho_1^2+\rho_2^2)^{m+1}} (1+\rho_1^2+\rho_2^2) \rho_1^{2j_1} \rho_2^{2j_2} \rho_1 \rho_2 d\rho_1 d\rho_2.$$

With the substitution $\rho_1 = r\cos\theta$, $\rho_2 = r\sin\theta$, $0 < r < +\infty$, $0 < \theta < \frac{\pi}{2}$ one finds a product of one variable integrals:

$$\langle s_J, s_J \rangle_{\mathfrak{h}_m} = 4 \int_0^{\frac{\pi}{2}} (\cos\theta)^{2j_1+1} (\sin\theta)^{2j_2+1} d\theta \cdot \int_0^{+\infty} r^{2(j_1+j_2-m)+1} (1+r^2) e^{-mr^2}\, dr.$$

For the first integral [1, 6.1.1, p. 255] we find

$$\int_0^{\frac{\pi}{2}} (\cos\theta)^{2j_1+1}(\sin\theta)^{2j_2+1} d\theta = \frac{\Gamma(j_1+1)\Gamma(j_2+1)}{2\Gamma(j_1+j_2+2)} = \frac{j_1!j_2!}{2(j_1+j_2+1)!}.$$

For the second integral, by [1, 6.2.1, p. 258]

$$\int_0^{\infty} r^s e^{-mr^2}\, dr = \frac{\Gamma(\frac{s+1}{2})}{2m^{(\frac{s+1}{2})}},$$

we find that

$$\int_0^{+\infty} r^{2(j_1+j_2-m)+1}(1+r^2)e^{-mr^2}\, dr \tag{3.5}$$

equals

$$\frac{\Gamma(j_1+j_2-m+1)}{2m^{j_1+j_2-m+1}} + \frac{\Gamma(j_1+j_2-m+2)}{2m^{j_1+j_2-m+2}} = \frac{(j_1+j_2-m)!(j_1+j_2+1)}{2m^{j_1+j_2-m+2}}.$$

Therefore

$$\langle s_J, s_J\rangle_{\mathfrak{h}_m} = \frac{j_1!j_2!}{(j_1+j_2)!}\frac{(j_1+j_2-m)!}{m^{j_1+j_2-m+2}},$$

and this proves that (3.4) is an orthogonal sequence for $(\mathcal{H}_{\mathfrak{h}_m}, \langle\cdot,\cdot\rangle_{\mathfrak{h}_m})$. □

Remark 3.2.3. The integral (3.5) converges if and only if $j_1+j_2 \geq m$. Indeed, when $j_1+j_2 < m$, since $j_1, j_2 \in \mathbb{N}$ and $m \in \mathbb{N}\setminus\{0\}$, one has $2(j_1+j_2-m)+1 < 0$ and (3.5) equals

$$\int_0^{+\infty} f(r)\, dr = \int_0^1 f(r)\, dr + \int_1^{+\infty} f(r)\, dr, \tag{3.6}$$

where

$$f(r) = \frac{(1+r^2)}{e^{mr^2} r^{|2(j_1+j_2-m)+1|}}.$$

Since $f(r)$ is positive, the functions

$$F_1(x) = \int_x^1 f(r)\, dr, \quad F_2(x) = \int_1^x f(r)\, dr,$$

are monotone (decreasing and increasing, respectively) in the x variable. Then there exists, finite or infinite, the limits

$$\lim_{x\to 0^+} F_1(x) = \lim_{x\to 0^+}\int_x^1 f(r)\, dr, \quad \lim_{x\to+\infty} F_2(x) = \lim_{x\to+\infty}\int_1^x f(r)\, dr.$$

For all $r \in (0,1]$ one has

$$f(r) \geq \frac{(1+r^2)}{e^m r^{|2(j_1+j_2-m)+1|}} = \frac{1}{e^m}\left(\frac{1}{r^{|2(j_1+j_2-m)+1|}} + \frac{r^2}{r^{|2(j_1+j_2-m)+1|}}\right).$$

Under the assumptions, $|2(j_1+j_2-m)+1| \geq 1$ and then

$$\int_0^1 \frac{1}{r^{|2(j_1+j_2-m)+1|}}\, dr = +\infty.$$

Finally one gets

$$\lim_{x\to 0^+} F_1(x) \geq \frac{1}{e^m}\int_0^1 \left(\frac{1}{r^{|2(j_1+j_2-m)+1|}} + \frac{r^2}{r^{|2(j_1+j_2-m)+1|}}\right) dr = +\infty,$$

and one concludes that the integral (3.6) is divergent.

Lemma 3.2.4. *The set*

$$s_J := \left((z_1,z_2), \frac{z_1^{j_1} z_2^{j_2}}{\sqrt{\frac{j_1!j_2!}{(j_1+j_2)!}\frac{(j_1+j_2-m)!}{m^{j_1+j_2-m+2}}}}\right), \tag{3.7}$$

where $J := \{(j_1,j_2), \mid j_1, j_2 \in \mathbb{N},\ j_1+j_2 \geq m\}$, is an orthonormal basis for the Hilbert space $(\mathcal{H}_{\mathfrak{h}_m}, \langle\cdot,\cdot\rangle_{\mathfrak{h}_m})$.

Proof. The sequence (3.7) is orthonormal by Lemma 3.2.2 and it is also total, indeed every global holomorphic section $s = (z, f_s(z))$, where $f_s : \mathbb{C}^2 \to \mathbb{C}$ is a holomorphic function vanishing at the origin with order greater or equal than m, can be represented at the origin by a convergent power series of the form

$$f_s(z) = f_s(z_1,z_2) = \sum_{j_1,j_2=0}^{\infty} a_{j_1,j_2} z_1^{j_1} z_2^{j_2}$$

with complex coefficients a_{j_1,j_2} and with $j_1+j_2 \geq m$. If $\langle s, s_K\rangle_{\mathfrak{h}_m} = 0$ for all $K = (k_1,k_2), k_1,k_2 \in \mathbb{N}$, then for each fixed natural pair (k_1,k_2), we must have

$$0 = \langle((z_1,z_2), \sum_{j_1,j_2=0}^{\infty} a_{j_1,j_2} z_1^{j_1} z_2^{j_2}), ((z_1,z_2), \frac{z_1^{k_1} z_2^{k_2}}{||z_1^{k_1} z_2^{k_2}||^2_{\mathfrak{h}_m}})\rangle_{\mathfrak{h}_m} =$$

$$= \frac{1}{||z_1^{k_1} z_2^{k_2}||^2_{\mathfrak{h}_m}} \sum_{j_1,j_2=0}^{\infty} a_{j_1 j_2} \langle((z_1,z_2), z_1^{j_1} z_2^{j_2}), ((z_1,z_2), z_1^{k_1} z_2^{k_2})\rangle_{\mathfrak{h}_m} =$$

$$= \frac{1}{||z_1^{k_1} z_2^{k_2}||^2_{\mathfrak{h}_m}} a_{k_1 k_2} ||z_1^{k_1} z_2^{k_2}||^2_{\mathfrak{h}_m} = a_{k_1 k_2},$$

therefore s has the power series expansion identically equal to zero, that is s is the trivial section and this proves the lemma. □

Remark 3.2.5. By Lemma 2.3.13 each s_J extends to a unique global holomorphic sections of the line bundle $L^m \to \tilde{\mathbb{C}}^2$. This leads an alternative way to prove that $H^0(L^m)$ is in fact the space of holomorphic functions on $\mathbb{C}^2$ vanishing at the origin with multiplicity greater or equal than m.

We are now ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. By Definition 2.4.1 and by Lemma 3.2.4, for the epsilon function one has:

$$\begin{aligned}
\epsilon_{mg_{BS}}(z) &= \sum_{\substack{j_1,j_2\geq 0\\ j_1+j_2\geq m}} \frac{e^{-m(|z_1|^2+|z_2|^2)}}{(|z_1|^2+|z_2|^2)^m}\frac{|z_1|^{2j_1}|z_2|^{2j_2}}{||z_1^{j_1}z_2^{j_2}||^2_{h_m}} = \\
&= \frac{e^{-m(|z_1|^2+|z_2|^2)}}{(|z_1|^2+|z_2|^2)^m}\sum_{\substack{j_1,j_2\geq 0\\ j_1+j_2\geq m}} \frac{(j_1+j_2)!|z_1|^{2j_1}|z_2|^{2j_2}}{j_1!j_2!(j_1+j_2-m)!} m^{j_1+j_2-m+2} = \\
&= \frac{e^{-m(|z_1|^2+|z_2|^2)}}{(|z_1|^2+|z_2|^2)^m}\sum_{\beta=m}^{\infty}\left[\sum_{\substack{j_1,j_2\geq 0\\ j_1+j_2=\beta}} \frac{(j_1+j_2)!|z_1|^{2j_1}|z_2|^{2j_2}}{j_1!j_2!}\right]\frac{m^{\beta-m+2}}{(\beta-m)!} = \\
&= \frac{e^{-m(|z_1|^2+|z_2|^2)}}{(|z_1|^2+|z_2|^2)^m}\sum_{\beta=m}^{\infty}(|z_1|^2+|z_2|^2)^{\beta}\frac{m^{\beta-m+2}}{(\beta-m)!} = \\
&= m^2 e^{-m(|z_1|^2+|z_2|^2)}\sum_{\beta=m}^{\infty}(|z_1|^2+|z_2|^2)^{\beta-m}\frac{m^{\beta-m}}{(\beta-m)!} = \\
&= m^2 e^{-m(|z_1|^2+|z_2|^2)}\sum_{\alpha=0}^{\infty}(|z_1|^2+|z_2|^2)^{\alpha}\frac{m^{\alpha}}{\alpha!} = \\
&= m^2,
\end{aligned}$$

and this proves the theorem. □

Corollary 3.2.6. *All the coefficients $a_j(x)$, with $j \geq 1$, of the TYCZ expansion for the Burns–Simanca metric vanish.*

Remark 3.2.7. Notice that the line bundle L in the proof above, is not trivial and the Kähler form ω_{BS} does not admit a global Kähler potential. Indeed if ω_{BS} admits a global potential then, since $\tilde{\mathbb{C}}^2$ is simply connected, we must have $c_1(L) = [\omega_{BS}] = [0]$ and, by Proposition 2.1.10, L is isomorphic to the trivial bundle $\tilde{\mathbb{C}}^2 \times \mathbb{C}$. Hence there exists a global section $s : \tilde{\mathbb{C}}^2 \to L$ such that $s(x) \neq 0$ for all $x \in \tilde{\mathbb{C}}^2$, in contrast with $s(x) = 0$ for any global section and for all $x \in H$.

Remark 3.2.8. It is worth pointing out that recently Bi-Feng-Tu [27] have constructed examples of regular quantizations on Fock–Bargmann–Hartogs domains in the complex Euclidean space equipped with a negative constant scalar curvature Kähler metric. Thus, they provide a negative answer to Question 2 in the non compact case when the scalar curvature is negative. Another important difference between Bi-Feng-Tu example and the Burns–Simanca metric is that, in the first case, the quantization bundle is trivial (the manifold is contractible) and the Kähler metric has a global Kähler potential. Moreover, the Burns–Simanca metric has been a fundamental ingredient in the construction of cscK metrics on compact Kähler manifold via blow-up procedures (see [8]). Thus we believe our Theorem 3.2.1 could be used to built regular quantizations of non-homogeneous compact Kähler manifolds.

Remark 3.2.9. Corollary 2.6.5 and Theorem 3.2.1 show that $(\mathbb{C}^2, g_0)$ and $(\tilde{\mathbb{C}}^2, g_{BS})$ have the same epsilon functions both equal to m^2. It could be interesting to find other examples of Kähler manifolds sharing this property and, more generally, to analyse to what extent the TYCZ coefficients determine the underlying Kähler manifold (cf. [6] for this last issue).

3.3 Berezin quantization

In order to obtain an interesting corollary of Theorem 3.2.1 we need to briefly recall some important tools about Berezin quantization. We will mainly refer to [9, 26].

The modern theory of quantization was developed in the second half of the 20th century and the term *quantization*, from the outset, was used in two ways. The first meaning referred to the discretization of the set of values of some physical quantity. The second meaning referred to a construction for passing from a classical mechanics system - which, loosely speaking, is something that concerns macroscopic objects and that we are familiar with from everyday's life – to the "corresponding" quantum system – which pertains to microscopic objects where things are subject to more complicated rules – which had the classical system as its limit as $\hbar \to 0$, where $\hbar$ is Planck's constant. Letting $\hbar$ to zero means going from a system of units in which a quantized object is described, to a system more an more appropriate to a classical description.

It is well-known however, that not every quantum system has a meaningful classical counterpart and moreover, different quantum systems may reduce to the same classical theory. Over the time, it became apparent that such a concept is not totally appropriate, both mathematically and physically. From the point of view of physics, it is more appropriate to understand quantization just as a correspondence between classical and quantum systems; that is, there may be quantum systems which have no classical counterpart, as well as different quantum systems corresponding to the same classical system. From the mathematical point of view, one even encounters obstacles of a different kind — namely, various "no-go" theorems show that there can exist no mathematical recipe that would fulfill all the axioms required by the physical interpretation.

As a result, nowadays we face the existence of many different quantization theories, ranging from geometric quantization, deformation quantization and various related operator-theoretic quantizations to Feynman path integrals, asymptotic quantization, or stochastic quantization, to mention just a few. No one of the existing approaches solves the quantization problem completely; on the other hand, on the mathematics side all these have evolved into rich theories of their own right, and with results of great depth and beauty.

In this section we propose a more general definition for quantization in the Berezin's approach in order to prove the following corollary

Corollary 3.3.1. $(\mathbb{C}^2 \setminus \{0\}, \omega_{BS})$ *admits a Berezin quantization.*

Let (M, ω) be a symplectic manifold and let $\mathcal{A}(M)$ denote the algebra (with respect to the usual operations of linear combination and multiplication) of differentiable complex-valued function on M and define a Poisson bracket $\{\cdot,\cdot\}$ for elements of $\mathcal{A}(M)$ as

$$\{f, g\} = \sum_{j,k=1}^{n} g^{jk} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k}, \tag{3.8}$$

where $\{g^{jk}\}_{j,k=1}^{n}$ is the inverse matrix to $\{g_{jk}\}_{j,k=1}^{n}$. Here g_{jk} is the metric matrix associated to ω and n is the real dimension of the manifold. In terms of the components g_{jk} the condition $d\omega = 0$ has the form

$$\frac{\partial g_{st}}{\partial x_k} + \frac{\partial g_{ks}}{\partial x_t} + \frac{\partial g_{tk}}{\partial x_s} = 0,$$

thus, if the function $f_1, f_2, f_3 \in \mathcal{A}(M)$ are twice differentiable, then the Jacobi identity for the Poisson bracket holds:

$$\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0.$$

The pair (M, ω) together with the algebra $\mathcal{A}(M)$ supplied with the Poisson bracket (3.8) will be called a *classical mechanics.*

Definition 3.3.2. An associative algebra $\mathcal{A}$ with involution $\tilde{\sigma}$ is said to be a *general quantization* of the classical mechanics (M, ω) if it possesses the following properties.

1. There exists a family of associative algebras $\mathcal{A}_\hbar$ with involution σ such that:

   (a) the parameter $\hbar$ (which plays the role of the Planck constant) ranges over a set E of positive reals with limit point 0 (0 does not belong to E);

   (b) the algebra $\mathcal{A}$ is a subalgebra of the direct sum $\oplus_{\hbar \in E} \mathcal{A}_\hbar$. It is convenient to represent the element of $\mathcal{A}$ in the form of functions $f(\hbar) : M \to \mathcal{A}_\hbar$ for fixed $\hbar \in E$. Involution and multiplication in $\mathcal{A}$ are related to involution and multiplication in $\mathcal{A}_\hbar$, respectively, by

   $$(\tilde{\sigma}(f))(\hbar) = \sigma(f(\hbar)), \quad (f_1 \tilde{*} f_2)(\hbar) = f_1(\hbar) * f_2(\hbar),$$

where $\tilde{*}$ and $*$ denote multiplication and involution in $\mathcal{A}$ and $\mathcal{A}_\hbar$.

2. There exists a homomorphism $\varphi : \mathcal{A} \to \mathcal{A}(M)$ such that the following properties, called *the correspondence principle*, hold:

   (a) for any pair of points $x_1, x_2 \in M$ there exists $f \in \mathcal{A}$ such that
   $$\varphi(f)(x_1) \neq \varphi(f)(x_2);$$

   (b) for $f, g \in \mathcal{A}$
   $$\varphi(\hbar^{-1}(f * g - g * f)) = -i\{\varphi(f), \varphi(g)\}, \quad \varphi(\sigma(f)) = \overline{\varphi(f)},$$
   where the bar denotes complex conjugation.

We call *Berezin quantization* a general quantization which possesses the following additional properties:

3. the algebra $\mathcal{A}_\hbar$ consists of functions $f(x), x \in M$;

4. the algebra $\mathcal{A}$ consists of functions $f(\hbar, x) \in \mathcal{A}_\hbar$ for fixed $\hbar$;

5. the homomorphism $\varphi : \mathcal{A} \to \mathcal{A}(M)$ is given by the formula
$$\varphi(f) = \lim_{\hbar \to 0} f(\hbar).$$

Two examples-quantization on a cylinder and on a torus, which illustrate the definition, can be found in [9, Section 6, p. 1144].
The following theorem is a reformulation of Berezin quantization result for a complex Kähler domain (see [24] and [44]) in terms of balanced metrics and Calabi's diastasis function.

Theorem 3.3.3. *Let $\Omega \subset \mathbb{C}^n$ be a complex domain equipped with a real analytic Kähler form ω and corresponding Kähler metric g. Then, (Ω, ω) admits a Berezin quantization if the following two conditions are satisfied:*

1. *mg is balanced for all sufficiently large m;*

2. *the function $e^{-D_g(x,y)}$ is globally defined on $\Omega \times \Omega$, $e^{-D_g(x,y)} \leq 1$ and $e^{-D_g(x,y)} = 1$ if and only if $x = y$.*

We are now in the position to prove Corollary 3.3.1.

Proof of Corollary 3.3.1. We are going to show that Conditions 1 and 2 of Theorem 3.3.3 are fulfilled by $(\mathbb{C}^2 \setminus \{0\}, \omega_{BS})$. Condition 1 follows by Theorem 3.2.1. For Condition 2 consider the holomorphic map

$$\varphi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}P^\infty$$

given by

$$(z_1, z_2) \mapsto \left[z_1, z_2, \dots, \sqrt{\frac{j+k}{j!k!}} z_1^j z_2^k, \dots \right], \ j+k \neq 0.$$

In [47, Theor 1.3] the authors prove that φ is an injective Kähler immersion from $(\mathbb{C}^2 \setminus \{0\}, g_{BS})$ into $(\mathbb{C}P^\infty, g_{FS})$. By Example 1.3.2, Calabi's diastasis function $D_{g_{FS}}$ of $\mathbb{C}P^\infty$ is such that $e^{-D_{g_{FS}}}$ is globally defined on $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ and by Theorem 1.3.3 we get that, for all $x, y \in \mathbb{C}^2 \setminus \{0\}$,

$$e^{-D_{FS}(\varphi(x), \varphi(y))} = e^{-D_{g_{BS}}(x,y)}$$

is globally defined on $\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}^2 \setminus \{0\}$. Since, by Example 1.3.2, $e^{-D_{FS}(p,q)} \leq 1$ for all $p, q \in \mathbb{C}P^\infty$ it follows that $e^{-D_{g_{BS}}(x,y)} \leq 1$ for all $x, y \in \mathbb{C}^2 \setminus \{0\}$ and since φ is injective one gets that $e^{-D_{g_{BS}}(x,y)} = 1$ iff $x = y$. Hence, also Condition 2 is satisfied and this concludes the proof of the corollary. □

Chapter 4

Balanced metrics on the blow-up of $\mathbb{C}^n$ at the origin

4.1 The generalized Burns–Simanca metric

Let $\tilde{\mathbb{C}}^n$ be the blow-up of $\mathbb{C}^n$ at the origin, $p_r : \tilde{\mathbb{C}}^n \setminus H \to \mathbb{C}^n \setminus \{0\}$ be the biholomorphic map and H the exceptional divisor arising by the blow-up construction (as in Section 1.1.1). Take on $\mathbb{C}^n \setminus \{0\}$ the $(1,1)$-form given by

$$\omega_{BS(n)} = \frac{i}{2\pi}\partial\bar{\partial}(|z|^2 + \log|z|^2). \tag{4.1}$$

where $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$. Notice that when $n = 2$, $\omega_{S(2)}$ equals the form ω in (3.1). The pull-back $p_r^*(\omega_{BS(n)})$ is given in the coordinates (1.3) by

$$p_r^*(\omega_{BS(n)}) = \frac{i}{2\pi}\partial\bar{\partial}\left(|z_i|^2(1 + |z|^2 - |z_i|^2) + \log(1 + |z|^2 - |z_i|^2)\right),$$

on $\tilde{U}_i \setminus H$, for $i = 1, \ldots, n$. This shows that $p_r^*(\omega_{BS(n)})$ extends to the whole $\tilde{\mathbb{C}}^n$. On $\tilde{\mathbb{C}}^n \setminus H$ this form is given in local coordinates by (4.1). The metric associated to $p_r^*(\omega_{BS(n)})$ is denoted here by $g_{BS(n)}$ and we will call it *generalized Burns–Simanca metric*. Clearly, when $n = 2$ the metric $g_{S(2)}$ equals the Burns-Simanca metric. Since

$$(g_{BS(n)})_{i\bar{i}} = 1 + \frac{|z|^2 - |z_i|^2}{|z|^4}, \quad (g_{BS(n)})_{i\bar{j}} = -\frac{z_j\bar{z}_i}{|z|^4}, \quad i, j = 1, \ldots n,$$

we have

$$(g_{BS(n)})_{1\bar{1}} > 0 \quad \text{and} \quad \det(g_{BS(n)}) = \left(1 + \frac{1}{|z|^2}\right)^{n-1},$$

i.e. $g_{BS(n)}$ is a Kähler metric on $\tilde{\mathbb{C}}^n$. Notice that the generalized Burns–Simanca metric is complete but its scalar curvature is not constant, as expressed by the following proposition:

Proposition 4.1.1. *The followings properties hold:*

1. *$g_{BS(n)}$ is complete.*

2. *$scal_{g_{BS(n)}}$ is not constant for any $n \geq 3$.*

Proof. 1. It is sufficient to show that the length of divergent curves is infinite (see [19, Ex. 5, p. 153]). Since $H \simeq \mathbb{C}P^{n-1}$ is compact and $(\tilde{\mathbb{C}}^n \setminus H, \omega_{BS(n)})$ is isometric to $(\mathbb{C}^n \setminus \{0\}, \omega_{BS(n)})$ via the projection p_r, we are reduced to show that a divergent curve $\alpha : [0, +\infty) \to \mathbb{C}^n \setminus \{0\}$ has infinite length with respect to (4.1). In order to show this, notice that

$$\omega_{BS(n)} = \frac{i}{2\pi}\partial\bar{\partial}(|z|^2 + \log|z|^2) = \frac{i}{2\pi}\partial\bar{\partial}(|z|^2) + \frac{i}{2\pi}\partial\bar{\partial}(\log|z|^2),$$

where the second addendum $\eta = \frac{i}{2\pi}\partial\bar{\partial}(\log|z|^2)$ is a positive-semidefinite form (one finds $\eta_{i\bar{i}} = \frac{|z|^2 - |z_i|^2}{|z|^4} > 0$, for $i = 1, \dots, n$ and $\det(\eta) = 0$) and the first addendum is the flat Euclidean form ω_0. Then, if $\|\cdot\|$ (resp. $\|\cdot\|_0$) denotes the norm with respect to $\omega_{BS(n)}$ (resp. with respect to ω_0), we clearly have

$$\|\alpha'(t)\| \geq \|\alpha'(t)\|_0.$$

It follows that

$$\int_0^\infty \|\alpha'(t)\| dt \geq \int_0^\infty \|\alpha'(t)\|_0 dt = +\infty$$

where the last equality follows from the fact that ω_0 is complete and then divergent curves on $\mathbb{C}^n$ have infinite length.

2. For $(z_1, 0, 0)$ one finds

$$g_{BS(n)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1+\frac{1}{|z_1|^2} & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1+\frac{1}{|z_1|^2} \end{pmatrix}$$

and

$$g^{-1}_{S(3)} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{|z_1|^2}{1+|z_1|^2} & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{|z_1|^2}{1+|z_1|^2} \end{pmatrix}.$$

By recalling (1.4) one gets:

$$\mathrm{Ric} = \begin{pmatrix} -\frac{1}{(1+|z_1|^2)^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{|z_1|^2+|z_1|^4} & \vdots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{|z_1|^2+|z_1|^4} \end{pmatrix}.$$

Finally, by (1.6) one finds

$$\mathrm{scal}_{g_{BS(n)}} = \frac{2-n}{(1+|z_1|^2)^2}, \quad \text{for} \quad (z_1, 0, \ldots, 0)$$

so the scalar curvature is not constant for any $n \geq 3$. □

4.2 On the balanced condition for the generalized Burns–Simanca metric

The construction in the proof of Theorem 3.2.1 stops to work when $\mathbb{C}^2$ is replaced by $\mathbb{C}^n, n \geq 3$ and the Burns-Simanca metric is replaced by the generalized Burns–Simanca metric $g_{BS(n)}$ on $\tilde{\mathbb{C}}^n$.
This is expressed by the following theorem.

Theorem 4.2.1. *Let $\tilde{\mathbb{C}}^n$ be the blow-up of $\mathbb{C}^n$ at the origin endowed with the generalized Burns–Simanca metric $g_{BS(n)}$. For any integer $m \geq 1$ the following statements hold*

1. $(\tilde{\mathbb{C}}^n, mg_{BS(n)})$ *is projectively induced for any $n \geq 2$,*

2. $mg_{BS(n)}$ *is not balanced for all $n \geq 3$.*

The theorem gives an example of Kähler metric g on the blow up of $\mathbb{C}^n$ at the origin such that mg is projectively induced but it is not balanced for any positive integer m.

Proof. 1. The holomorphic map

$$\varphi : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}P^\infty$$

given by

$$(z_1, \dots, z_n) \mapsto \left[z_1, \dots, z_n, \dots, \sqrt{\frac{j_1 + \cdots + j_n}{j_1! \cdots j_n!}} z_1^{j_1} \cdots z_n^{j_n}, \dots \right],$$

for $j_1 + \cdots + j_n \neq 0$, is a Kähler immersion from $(\mathbb{C}^n \setminus \{0\}, g_{BS(n)})$ into $(\mathbb{C}P^\infty, g_{FS})$. In point of fact

$$\varphi^*(\omega_{FS}) = \frac{i}{2\pi} \partial\bar{\partial} \log \left(\sum_{\substack{j_1, j_2, \dots, j_n \geq 0 \\ j_1 + \cdots + j_n \geq m}} \left(\frac{j_1 + \cdots + j_n}{j_1! \cdots j_n!} |z_1|^{2j_1} \cdots |z_n|^{2j_n} \right) \right) =$$

$$= \frac{i}{2\pi} \partial\bar{\partial} \log(e^{|z|^2} |z|^2) = \omega_{BS(n)}.$$

Since $\tilde{\mathbb{C}}^n$ is simply-connected, from Theorem 1.3.6 follows that φ extends to a Kähler immersion from $(\tilde{\mathbb{C}}^n, g_{BS(n)})$ into $(\mathbb{C}P^\infty, g_{FS})$. Similarly, one can show that $mg_{BS(n)}$ is projectively induced for any positive integer m. Indeed, by [13, Theor. 13 (B), p. 21] if a Kähler manifold can be Kähler immersed into $\mathbb{C}P^\infty$ then the same is true for (M, mg).
2. For an integer $m > 0$, consider the geometric quantization given by the holomorphic line bundle $L^m \to (\tilde{\mathbb{C}}^n, \omega_{BS(n)})$ such that $c_1(L^m) = m[\omega_{BS(n)}]$, equipped with the hermitian structure

$$h_m(\sigma(x), \sigma(x)) = \frac{1}{|z|^{2m}} e^{-m|z|^2} |q|^2.$$

where $\sigma : U \subset \tilde{\mathbb{C}}^n \setminus H \to L^m \setminus \{0\}, x \mapsto (z, q) \in U \times \mathbb{C}$ is a trivialising holomorphic section. As for the Simanca metric in the proof of Theorem 3.2.1 there is a natural bijection between the complex space $H^0(L^m)$ of global holomorphic sections and the space of holomorphic functions on $\mathbb{C}^n$ vanishing at the origin with order greater or equal than m. This bijection takes $s \in H^0(L^m)$ to the holomorphic function f_s on $\mathbb{C}^n$ obtained by restricting s to $\tilde{\mathbb{C}}^n \setminus H \simeq \mathbb{C}^n \setminus \{0\}$. Moreover, since H has zero measure in $\tilde{\mathbb{C}}^n$, one gets

$$\begin{aligned} \langle s, s \rangle_{\mathfrak{h}_m} &= \int_{\tilde{\mathbb{C}}^n} h_m(s(x), s(x)) \frac{\omega^n_{BS(n)}}{n!} = \\ &= \int_{\mathbb{C}^n \setminus \{0\}} \frac{e^{-m|z|^2}}{|z|^{2m}} |f_s(z)|^2 \left(1 + \frac{1}{|z|^2}\right)^{n-1} d\mu(z) < \infty, \end{aligned} \tag{4.2}$$

where $d\mu(z) = \left(\frac{i}{2\pi}\right)^n dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n$. Therefore $\mathcal{H}_m = H^0(L^m)$.

From (4.2) by passing to polar coordinates $z_1 = \rho_1 e^{i\vartheta_1}, \ldots, z_n = \rho_n e^{i\vartheta_n}$ with $\rho_1, \ldots \rho_n \in (0, +\infty), \vartheta_1, \ldots, \vartheta_n \in (0, 2\pi)$ one easily sees, as in the proof of Lemma 3.2.2, that the set

$$s_J := \left((z_1, \ldots, z_n), z_1^{j_1} \cdots z_n^{j_n}\right),$$

where, $J := \{(j_1, \ldots, j_n), \mid j_1, \ldots, j_n \in \mathbb{N},\, j_1 + \cdots + j_n \geq m\}$, is an orthogonal system for the Hilbert space $(\mathcal{H}_m, \langle \cdot, \cdot \rangle_{h_m})$. Moreover, by (4.2),

$$\langle s_J, s_J \rangle_{\mathfrak{h}_m} = 2^n \int_\Omega \zeta(\rho_1, \ldots, \rho_n) d\rho_1 \cdots d\rho_n,$$

where

$$\zeta(\rho_1, \ldots, \rho_n) = \frac{e^{-m(\rho_1^2 + \cdots + \rho_n^2)}}{(\rho_1^2 + \cdots + \rho_n^2)^{m+n-1}} (1 + \rho_1^2 + \cdots + \rho_n^2)^{n-1} \rho_1^{2j_1+1} \cdots \rho_n^{2j_n+1},$$

and $\Omega = \{(0,+\infty)^n \subset \mathbb{R}^n\}$. With the substitution

$$\begin{aligned}
&\rho_1 = r\cos(\vartheta_1)\\
&\rho_2 = r\sin(\vartheta_1)\cos(\vartheta_2)\\
&\rho_3 = r\sin(\vartheta_1)\sin(\vartheta_2)\cos(\vartheta_3)\\
&\vdots\\
&\rho_{n-1} = r\sin(\vartheta_1)\cdots\sin(\vartheta_{n-2})\cos(\vartheta_{n-1})\\
&\rho_n = r\sin(\vartheta_1)\cdots\sin(\vartheta_{n-2})\sin(\vartheta_{n-1})
\end{aligned}$$

with $0 < r < +\infty$, $0 < \vartheta_i < \frac{\pi}{2}$ for $i = 1,\dots,n-1$, one finds a product of one variable integrals:

$$\begin{aligned}
\langle s_J, s_J\rangle_{\mathfrak{h}_m} =& 2^n \int_0^{\frac{\pi}{2}} (\cos\theta_1)^{2j_1+1}(\sin\theta_1)^{2(j_2+\cdots+j_n+(n-1)-1)+1}d\theta_1\cdot\\
&\cdot\int_0^{\frac{\pi}{2}} (\cos\theta_2)^{2j_2+1}(\sin\theta_2)^{2(j_3+\cdots+j_n+(n-1)-2)+1}d\theta_2\cdot\\
&\cdot\int_0^{\frac{\pi}{2}} (\cos\theta_3)^{2j_3+1}(\sin\theta_3)^{2(j_4+\cdots+j_n+(n-1)-3)+1}d\theta_3\cdot\\
&\vdots\\
&\cdot\int_0^{\frac{\pi}{2}} (\cos\theta_{n-2})^{2j_{n-2}+1}(\sin\theta_{n-2})^{2(j_{n-1}+j_n+(n-1)-(n-2))+1}d\theta_{n-2}\cdot\\
&\cdot\int_0^{\frac{\pi}{2}} (\cos\theta_{n-1})^{2j_{n-1}+1}(\sin\theta_{n-1})^{2(j_n+(n-1)-(n-1))+1}d\theta_{n-1}\cdot\\
&\cdot\int_0^{+\infty} r^{2(j_1+\cdots+j_n-m)+1}(1+r^2)^{n-1}e^{-mr^2}\,dr
\end{aligned}$$

For the first $n-1$ integrals, by [1, 6.1.1, p. 255] we find

$$\int_0^{\frac{\pi}{2}} (\cos\theta_1)^{2j_1+1}(\sin\theta_1)^{2(j_2+\cdots+j_n+(n-1)-1)+1}d\theta_1 = \frac{j_1!(j_2+\cdots+j_n+n-2)!}{2(j_1+j_2+\cdots+j_n+n-1)!},$$
$$\int_0^{\frac{\pi}{2}} (\cos\theta_2)^{2j_2+1}(\sin\theta_2)^{2(j_3+\cdots+j_n+(n-1)-2)+1}d\theta_2 = \frac{j_2!(j_3+\cdots+j_n+n-3)!}{2(j_2+j_3+\cdots+j_n+n-2)!},$$
$$\vdots$$
$$\int_0^{\frac{\pi}{2}} (\cos\theta_{n-2})^{2j_{n-2}+1}(\sin\theta_{n-2})^{2(j_{n-1}+j_n+(n-1)-(n-2))+1}d\theta_{n-2} = \frac{j_{n-2}!(j_{n-1}+j_n+1)!}{2(j_{n-2}+j_{n-1}+j_n+2)!},$$
$$\int_0^{\frac{\pi}{2}} (\cos\theta_{n-1})^{2j_{n-1}+1}(\sin\theta_{n-1})^{2(j_n+(n-1)-(n-1))+1}d\theta_{n-1} = \frac{j_{n-1}!j_n!}{2(j_{n-1}+j_n+1)!}.$$

For the last integral we find

$$\int_0^{+\infty} r^{2(j_1+\cdots+j_n-m)+1}(1+r^2)^{n-1}e^{-mr^2}\,dr = \frac{(\hat{J}-m)!}{2}U(\hat{J}-m+1, \hat{J}-m+n+1, m),$$

where $\hat{J} = j_1+\cdots+j_n$ and

$$U(a,b,z) = \frac{1}{\Gamma(a)}\int_0^{\infty} e^{-zt}t^{a-1}(1+t)^{b-a-1}\,dt$$

is the *Confluent Hypergeometric Function of the second kind* (see [1, 13.2.5, p. 505]). Since

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(a+1-b)}{}_1F_1(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}{}_1F_1(a+1-b,2-b,z),$$

where ${}_1F_1(a,b,z)$ is the *Confluent Hypergeometric Function*, one gets

$$\langle s_J, s_J\rangle_{\mathfrak{h}_m} = m^{m-n-\hat{J}}\frac{j_1!\cdots j_n!\Gamma(\hat{J}-m+n)}{\Gamma(\hat{J}+n)}{}_1F_1(1-n, 1+m-n-\hat{J}, m). \tag{4.3}$$

Therefore the set

$$\left((z_1,\ldots,z_n), \frac{z_1^{j_1}\cdots z_n^{j_n}}{\sqrt{\langle s_J, s_J\rangle_{\mathfrak{h}_m}}}\right),$$

where $J := \{(j_1,\ldots,j_n), \mid j_1,\ldots,j_n \in \mathbb{N},\ j_1+\cdots+j_n \geq m\}$, is an orthonormal sequence for the Hilbert Space $(\mathcal{H}_m, \langle\cdot,\cdot\rangle_{\mathfrak{h}_m})$. A similar arguments as in the proof of Lemma 3.2.4 shows that this sequence is also total and so it is an orthonormal basis for the Hilbert Space $(\mathcal{H}_m, \langle\cdot,\cdot\rangle_{\mathfrak{h}_m})$. For the epsilon function (setting $\hat{J} - m = k$, $k \in \mathbb{N}$) follows that

$$\epsilon_{mg_{BS(n)}}(z) = \frac{e^{-mt}}{t^m} \sum_{k=0}^{\infty} \frac{m^{k+n} t^{k+m} (k+m+n-1)!}{(k+m)!(k+n-1)!\,{}_1F_1(1-n, 1-k-n, m)}, \tag{4.4}$$

where $t := |z|^2$. For $n = 2$, ${}_1F_1(-1, -1-k, m) = \frac{k+m+1}{k+1}$ and (4.4) simplifies to m^2, in agreement with Theorem 3.2.1. In general, the right-hand side is, in terms of the variable $x := mt$, equal to

$$e^{-x} m^n \sum_{k=0}^{\infty} \frac{x^k (k+m+n-1)!}{(k+m)!(k+n-1)!\,{}_1F_1(1-n, 1-k-n, m)},$$

which is independent of x only if

$$f(k, m.n) := \frac{k!(k+m+n-1)!}{(k+m)!(k+n-1)!\,{}_1F_1(1-n, 1-k-n, m)} \tag{4.5}$$

is independent of k. Indeed if

$$e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} f(k, m, n) = g(m, n),$$

where $g(m, n)$ is a positive function that depends only on m and n, then one has

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} (f(k, m, n) - g(m, n) = 0. \tag{4.6}$$

Since

$$\lim_{k\to\infty} {}_1F_1(1-n, 1-k-n, m) = 1,$$

for fixed values of m and n, one has $f(k, m, n) \geq 1$ and $g(m, n) \leq 1$. Then (4.6) implies $f(k, m, n) - g(m, n) = 0$, namely $f(k, m, n) = g(m, n)$ is independent of k.

Looking at the asymptotic as $k \to +\infty$, the expression (4.5) behaves as

$$C(k) = 1 + \frac{m(n-1)(n-2)}{2k^2} + O\left(\frac{1}{k^3}\right),$$

so it can be independent of k only for $n \in \{1, 2\}$. Indeed, setting

$$A(k) = \frac{k!(k+m+n-1)!}{(k+m)!(k+n-1)} = 1 + \frac{m(n-1)}{k} + \frac{(m(n-2)-n)(n-1)m}{2k^2} + \cdots$$

and

$$B(k) = \frac{1}{{}_1F_1(1-n, 1-k-n, m)} = 1 - \frac{m(n-1)}{k} + \frac{(n-1)m(nm+2(n-1))}{2k^2} + \cdots$$

one easily gets

$$C(k) = A(k)B(k) = 1 + \frac{m(n-1)(n-2)}{2k^2} + O\left(\frac{1}{k^3}\right).$$

□

Chapter 5

The Eguchi–Hanson metric

5.1 Preliminaries

We will mainly refer to [21, 23] for the essential concepts appearing in this section. First we fix some useful notation: let $(x_1, x_2, y_1, y_2) \in \mathbb{R}^4$ be the four Euclidean coordinates and

$$\begin{aligned}
x_1 &= r\cos\left(\frac{\vartheta}{2}\right)\cos\left(\frac{\psi+\phi}{2}\right),\\
x_2 &= r\sin\left(\frac{\vartheta}{2}\right)\cos\left(\frac{\psi-\phi}{2}\right),\\
y_1 &= r\cos\left(\frac{\vartheta}{2}\right)\sin\left(\frac{\psi+\phi}{2}\right),\\
y_1 &= r\sin\left(\frac{\vartheta}{2}\right)\sin\left(\frac{\psi-\phi}{2}\right),
\end{aligned}$$

with $r^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2$, be the four-dimensional polar coordinates, where the variables ϑ, ϕ, ψ are Euler angles on the three-sphere $\mathbb{S}^3$ with ranges

$$0 \le \vartheta \le \pi, \quad 0 \le \phi \le 2\pi, \quad 0 \le \psi \le 4\pi.$$

These coordinates are related to the complex coordinates $(z_1, z_2) \in \mathbb{C}^2$ by

$$z_1 = x_1 + iy_1 = r\cos\left(\frac{\vartheta}{2}\right)\exp\left(\frac{i}{2}(\psi+\phi)\right) \tag{5.1}$$

$$z_2 = x_2 + iy_2 = r\sin\left(\frac{\vartheta}{2}\right)\exp\left(\frac{i}{2}(\psi-\phi)\right). \tag{5.2}$$

These coordinates cover $\mathbb{R}^4$ (i.e. $\mathbb{C}^2$) except for the trivial coordinate singularity at $r = 0$. For r = constant different from zero, the surfaces are homeomorphic to $\mathbb{S}^3$ and the curves r, ϑ, ψ = constant correspond to the Hopf fibration of $\mathbb{S}^3$ [28]. Let σ_1, σ_2 and σ_3 be the Cartan–Maurer forms for $\mathrm{SU}(2) \approx \mathbb{S}^3$, which are defined by

$$\begin{aligned}
\sigma_1 &= \frac{1}{r^2}(x_1dy_2 - y_2dx_1 + y_1dx_2 - x_2dy_1),\\
\sigma_2 &= \frac{1}{r^2}(y_1dy_2 - y_2dy_1 + x_2dx_1 - x_1dx_2),\\
\sigma_3 &= \frac{1}{r^2}(x_2dy_2 - y_2dx_2 + x_1dy_1 - y_1dx_1).
\end{aligned}$$

Since

$$\begin{aligned}
dx_1 &= -\frac{r}{2}\left[\sin\left(\frac{\vartheta}{2}\right)\cos\left(\frac{\psi+\phi}{2}\right)d\vartheta + \cos\left(\frac{\vartheta}{2}\right)\sin\left(\frac{\psi+\phi}{2}\right)(d\psi+d\phi)\right],\\
dx_2 &= \frac{r}{2}\left[\cos\left(\frac{\vartheta}{2}\right)\cos\left(\frac{\psi-\phi}{2}\right)d\vartheta + \sin\left(\frac{\vartheta}{2}\right)\sin\left(\frac{\psi-\phi}{2}\right)(d\phi-d\psi)\right],\\
dy_1 &= \frac{r}{2}\left[-\sin\left(\frac{\vartheta}{2}\right)\sin\left(\frac{\psi+\phi}{2}\right)d\vartheta + \cos\left(\frac{\vartheta}{2}\right)\cos\left(\frac{\psi+\phi}{2}\right)(d\psi+d\phi)\right],\\
dy_2 &= \frac{r}{2}\left[\cos\left(\frac{\vartheta}{2}\right)\sin\left(\frac{\psi-\phi}{2}\right)d\vartheta + \sin\left(\frac{\vartheta}{2}\right)\cos\left(\frac{\psi-\phi}{2}\right)(d\psi-d\phi)\right],
\end{aligned}$$

by using Simpson's formulas, one finds

$$\sigma_1 = \frac{1}{2}(\sin\psi d\vartheta - \sin\vartheta\cos\psi d\phi), \tag{5.3}$$

$$\sigma_2 = \frac{1}{2}(-\cos\psi d\vartheta - \sin\vartheta\sin\psi d\phi), \tag{5.4}$$

$$\sigma_3 = \frac{1}{2}(d\psi + \cos\vartheta d\phi). \tag{5.5}$$

Now consider the metric

$$ds^2 = (1 - 1/r^4)^{-1}dr^2 + r^2(\sigma_1^2 + \sigma_2^2) + r^2(1 - 1/r^4)\sigma_3^2. \tag{5.6}$$

The metric (5.6) was introduced and studied for the first time by Eguchi and Hanson in 1978 in the contest of finding self-dual solutions to the Euclidean Einstein equations [21, 23]. It is known in literature as the *Eguchi–Hanson metric*. The authors in [23] compute the curvature components of this metric finding

$$
\begin{aligned}
R_0^1 = R_3^2 &= -\frac{2}{r^6}(e^1 \wedge e^0 + e^2 \wedge e^3),\\
R_0^2 = R_1^3 &= -\frac{2}{r^6}(e^2 \wedge e^0 + e^3 \wedge e^1),\\
R_0^3 = R_2^1 &= \frac{4}{r^6}(e^3 \wedge e^0 + e^1 \wedge e^2),
\end{aligned}
$$

where

$$
e^0 = f^{-1/2}dr, \quad e^1 = r\sigma_1, \quad e^2 = r\sigma_2, \quad e^3 = rf^{1/2}\sigma_3,
$$

and

$$
f = \sqrt{1 - \frac{1}{r^4}}.
$$

showing that metric is Ricci-flat.

In the following an alternative way of obtaining the Eguchi–Hanson metric into complex Kähler form on $\mathbb{C}^2 \setminus \{0\}$ (see [11, p. 17] and [29, p. 278]). We begin by considering a Kähler potential which satisfy the following Ricci-flatness condition:

$$
\det\left(\frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j}\right) = 1. \tag{5.7}
$$

If one assumes that Φ is a function of $x := |z_1|^2 + |z_2|^2$, (5.7) may integrate. In imposing this condition the metric then looks as

$$
g = \begin{pmatrix} \Phi' + |z_1|^2\Phi'' & \bar{z}_1 z_2 \Phi'' \\ z_1 \bar{z}_2 \Phi'' & \Phi' + |z_2|^2\Phi'', \end{pmatrix}
$$

and the condition (5.7) gives

$$
\Phi'(x\Phi'' + \Phi') = 1, \tag{5.8}
$$

that is a second-order nonlinear ordinary differential equation. Let $\Phi'(x) = v(x)$, which gives $\Phi''(x) = v'(x)$. So the equation (5.8) equals

$$
v'(x) = \frac{1 - v(x)^2}{xv(x)}.
$$

After dividing both sides by $(1-v(x)^2)/v(x)$ and by integrating both sides with respect to x, one finds

$$-\frac{1}{2}\log(1-v(x)^2) = \log(x) + c_1,$$

where c_1 is an arbitrary constant. Solving for $v(x)$:

$$v(x) = \frac{\sqrt{x^2 - e^{-2c_1}}}{x}, \quad \text{or} \quad v(x) = -\frac{\sqrt{x^2 - e^{-2c_1}}}{x}.$$

By simplifying the arbitrary constants and choosing the first positive solution, one has

$$v(x) = \frac{\sqrt{x^2 + c_1}}{x}.$$

By substituting back for $\Phi'(x) = v(x)$ and by integrating both sides with respect to x, finally one finds

$$\Phi(x) = \sqrt{x^2 + c_1} + \sqrt{c_1}\log(x) - \sqrt{c_1}\log(\sqrt{c_1}\sqrt{x^2 + c_1} + c_1) + c_2,$$

where c_2 is is an arbitrary constant. Letting $c_1 = c_2 = 1$ one has

$$\Phi(z) = \sqrt{|z|^4 + 1} + \log|z|^2 - \log(1 + \sqrt{|z|^4 + 1}). \tag{5.9}$$

The potential (5.9) leads directly to the Eguchi–Hanson metric (5.6) (see [33] for details). We will denote by

$$\omega_{EH} = \frac{i}{2\pi}\partial\overline{\partial}\Phi \tag{5.10}$$

the Eguchi–Hanson metric in the complex form and by g_{EH} the corresponding Ricci-flat Kähler metric.
The $\log|z|^2$ term in this form causes problems at $(z_1, z_2) = (0, 0)$ (i.e at $r = 1$). However, this apparent singularity can be removed by identifying

$$(z_1, z_2) \sim (-z_1, -z_2) \quad \text{or} \quad (x_1, x_2, y_1, y_2) \sim (-x_1, -x_2, -y_1, -y_2). \tag{5.11}$$

We next give an explanation of this fact. First, by defining $u^2 = r^2(1-1/r^4)$, one find

$$dr^2 = \frac{r^4(r^4-1)}{(r^4+1)^2}dr^2,$$

therefore the metric can be rewritten as

$$ds^2 = \frac{du^2}{(1+1/r^4)^2} + u^2\sigma_3^2 + r^2(\sigma_1^2 + \sigma_2^2).$$

Very near to the apparent singularity at $r = 1$ (i.e. $u = 0$), by recalling (5.3)–(5.4)–(5.5), one finds

$$ds^2 \sim \frac{1}{4}du^2 + \frac{1}{4}u^2(d\psi + \cos\vartheta d\phi)^2 + \frac{1}{4}(d\vartheta^2 + \sin^2\vartheta d\phi^2). \tag{5.12}$$

For fixed ϑ and ϕ, we obtain

$$ds^2 \sim \frac{1}{4}(du^2 + u^2 d\psi^2). \tag{5.13}$$

We therefore conclude that if the range $0 \leq \psi \leq 4\pi$ is changed to $0 \leq \psi \leq 2\pi$

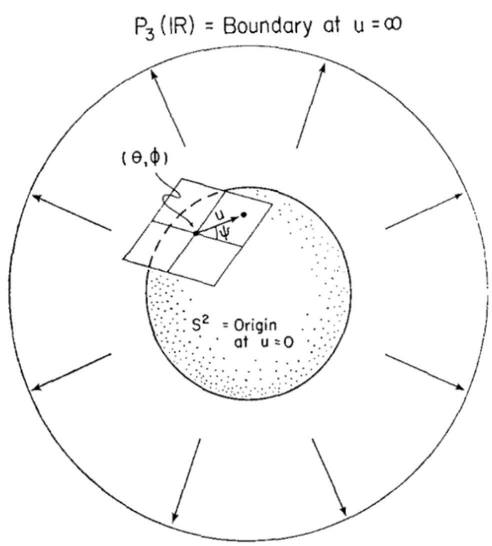

Figure 5.1: The manifold $M = T^*(\mathbb{C}P^1)$. Credits: [23]

(i.e. by the identification (5.11)), we can remove the apparent singularity at $r = 1$ and obtain a geodesically complete manifold (see [23, p. 92] for a full explanation of this fact). The global topology of the manifold described by the Eguchi–Hanson metric is now the following: for fixed $\mathbb{S}^2$ coordinates (ϑ, ϕ) the manifold has local topology $\mathbb{R}^2 \times \mathbb{S}^2$ indicated by (5.13). For r fixed (different from 1) the hypersurfaces are no longer topologically 3-spheres but 3-spheres with antipodal points identified, i.e. $\mathbb{R}P^3$. The boundary as $r \to \infty$ is thus the familiar group manifolds of $\mathbb{R}P^3$. As $u \to 0$ (i.e. $(z_1, z_2) \to (0,0)$) the manifold shrinks to $\mathbb{S}^2 \simeq \mathbb{C}P^1$. It can be shown [33] that the entire manifold M we have just described is in fact the cotangent bundle of the complex plane $\mathbb{C}P^1$ (Figure 5.1), and so we have

$$M = T^*(\mathbb{C}P^1), \quad \partial M = \mathbb{R}P^3.$$

Furthermore the Eguchi–Hanson metric can be made complete by considering the blow-up $\tilde{\mathbb{C}}^2$ of $\mathbb{C}^2$ at the origin and modding out by $\mathbb{Z}_2$ (see [39, p. 594] and references herein).

5.2 On the balanced condition for the Eguchi–Hanson metric

In [47] A. Loi, F. Salis and F. Zuddas study projectively induced Ricci-flat metrics and they prove that the Eguchi-Hanson metric g_{EH} is not projectively induced. Moreover, in the same paper they conjecture that mg_{EH} is not projectively induced for any positive integer m, and they give evidence of this fact for small values of the integer m.
The aim of this section is to provide the validity of the conjecture by restricting it to an interesting class of projectively induced metrics, namely the balanced metrics in the sense of Donaldson. Our main result is then the following:

Theorem 5.2.1. *The restriction of the metric mg_{EH} on $\mathbb{C}^2 \setminus \{0\}$ is not balanced for any positive integer m.*

Remark 5.2.2. A. Loi, M. Zedda and F. Zuddas, after my result contained in [14], showed that mg_{EH} is not projectively induced for any positive integer m (see [51, Cor. 1]. It immediately follows from Corollary 2.4.5 that a balanced metric is projectively induced via the coherent states map. This implies that Theorem 5.2.1 can be extended to the whole manifold $\tilde{\mathbb{C}}^2/\mathbb{Z}_2$. However I want to propose the proof with the explicit computation in the simplest case of the restriction of the metric mg_{EH} on $\mathbb{C}^2 \setminus \{0\}$. It is still an open question to directly prove that mg_{EH} on $\tilde{\mathbb{C}}^2/\mathbb{Z}_2$ is not balanced for any positive integer m.

In order to prove Theorem 5.2.1 consider the holomorphic line bundle $L \to (\mathbb{C}^2 \setminus \{0\}, \omega_{EH})$ such that $c_1(L) = [\omega_{EH}]_{dR}$. Such a line bundle exists since ω_{EH} is integral (see [51]). Moreover, L is unique, up to isomorphisms of line bundle, since $\mathbb{C}^2 \setminus \{0\}$ is simply-connected. Let m be a positive natural number and L^m be the m-th tensor power of L. This line bundle is isomorphic to the trivial bundle $\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}$. The map

$$h_m(\sigma(x), \sigma(x)) = e^{-m\sqrt{|z|^4+1}} \left(\frac{1 + \sqrt{|z|^4 + 1}}{|z|^2} \right)^m |q|^2,$$

induces a Hermitian structures $\mathfrak{h}_m$ on L^m that defines a geometric quantization of $(\mathbb{C}^2 \setminus \{0\}, m\omega_{EH})$, where

$$\sigma : U \subset \mathbb{C}^2 \setminus \{0\} \to L^m,\ x \mapsto (z, q) \in \mathbb{C}^2 \setminus \{0\} \times \mathbb{C}$$

is a trivialising holomorphic section. Moreover, one gets

$$\begin{aligned} \langle s, s \rangle_{\mathfrak{h}_m} &= \int_{\mathbb{C}^2 \setminus \{0\}} h_m(s(x), s(x)) \frac{\omega_{EH}^2}{2!} = \\ &= \int_{\mathbb{C}^2 \setminus \{0\}} e^{-m\sqrt{|z|^4+1}} \left(\frac{1 + \sqrt{|z|^4 + 1}}{|z|^2} \right)^m |f_s(z)|^2 d\mu(z) < \infty, \end{aligned} \tag{5.14}$$

where $d\mu(z) = \left(\frac{i}{2\pi}\right)^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$. We are now ready to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. Let $\mathcal{H}_m$ be the space of global holomorphic sections s of L^m which are bounded with respect to (5.14). By passing to polar coordinates $z_1 = \rho_1 e^{i\vartheta_1}, z_2 = \rho_2 e^{i\vartheta_2}$ with $\rho_1, \rho_2 \in (0, +\infty), \vartheta_1, \vartheta_2 \in (0, 2\pi)$ one easily sees, as in the proof of Lemma 3.2.2, that the set

$$s_J := \left((z_1, z_2), z_1^{j_1} z_2^{j_2} \right), \quad J := \{(j_1, j_2), \mid j_1, j_2 \in \mathbb{N},\ j_1 + j_2 \geq m\} \tag{5.15}$$

is an orthogonal sequence for the Hilbert space $(\mathcal{H}_{\mathfrak{h}_m}, \langle\cdot,\cdot\rangle_{\mathfrak{h}_m})$. More precisely, by (5.14) one finds

$$\langle s_J, s_J\rangle_{\mathfrak{h}_m} = 4\int_0^{+\infty}\int_0^{+\infty}\zeta(\rho_1,\rho_2)\,d\rho_1 d\rho_2,$$

where

$$\zeta(\rho_1,\rho_2) = e^{-m\sqrt{(\rho_1^2+\rho_2^2)^2+1}}\left(\frac{1+\sqrt{(\rho_1^2+\rho_2^2)^2+1}}{\rho_1^2+\rho_2^2}\right)^m \rho_1^{2j_1+1}\rho_2^{2j_2+1}.$$

With the substitution $\rho_1 = r\cos\theta$, $\rho_2 = r\sin\theta$, $0 < r < +\infty$, $0<\theta<\frac{\pi}{2}$ one finds a product of one variable integrals

$$\langle s_J, s_J\rangle_{\mathfrak{h}_m} = 4\int_0^{\frac{\pi}{2}}(\cos\theta)^{2j_1+1}(\sin\theta)^{2j_2+1}d\theta\cdot\int_0^{+\infty}f(r)\,dr,$$

where

$$f(r) = e^{-m\sqrt{r^4+1}}(1+\sqrt{r^4+1})^m r^{2(j_1+j_2-m+1)+1}.$$

For the first integral [1, 6.1.1, p. 255] we find

$$\int_0^{\frac{\pi}{2}}(\cos\theta)^{2j_1+1}(\sin\theta)^{2j_2+1}d\theta = \frac{\Gamma(j_1+1)\Gamma(j_2+1)}{2\Gamma(j_1+j_2+2)} = \frac{j_1!j_2!}{2(j_1+j_2+1)!}$$

Hence one gets

$$\langle s_J, s_J\rangle_{\mathfrak{h}_m} = \frac{2j_1!j_2!}{(j_1+j_2+1)!}\int_0^{+\infty}f(r)\,dr. \tag{5.16}$$

The integral in the equation (5.16) converges if and only if $j_1+j_2\geq m$. The real function $f(r)$ is continuous, positive, $f(0)=0$, increasing on $[0,r_M]$, decreasing on $[r_M,+\infty)$, r_M is the point where the function reaches its maximum, and the series

$$\sum_{n=[r_m]}^{+\infty} e^{-m\sqrt{n^4+1}}(1+\sqrt{n^4+1})^m n^{2(j_1+j_2-m+1)+1}$$

converges by the ratio test ($[r]$ denotes the floor function). In point of fact

$$\lim_{n\to+\infty}\frac{e^{-m\sqrt{(n+1)^4+1}}(1+\sqrt{(n+1)^4+1})^m(n+1)^{2(j_1+j_2-m+1)+1}}{e^{-m\sqrt{n^4+1}}(1+\sqrt{n^4+1})^m n^{2(j_1+j_2-m+1)+1}} = 0.$$

Therefore the integral in (5.16) converges by the integral test in the case $j_1 + j_2 \geq m$. When $j_1 + j_2 < m$, since $j_1, j_2 \in \mathbb{N}$ and $m \in \mathbb{N} \setminus \{0\}$, one has $2(j_1 + j_2 - m) + 1 < 0$ and (5.16) equals

$$\int_0^{+\infty} f(r)\,dr = \int_0^1 f(r)\,dr + \int_1^{+\infty} f(r)\,dr, \tag{5.17}$$

where

$$f(r) = \frac{(1+\sqrt{r^4+1})^m}{e^{m\sqrt{r^4+1}} r^{|2(j_1+j_2-m)+1|}}, \quad r \in (0, +\infty).$$

Since $f(r)$ is positive, the functions

$$F_1(x) = \int_x^1 f(r)\,dr, \quad F_2(x) = \int_1^x f(r)\,dr,$$

are monotone (decreasing and increasing, respectively) in the x variable. Then there exists, finite or infinite, the limits

$$\lim_{x\to 0^+} F_1(x) = \lim_{x\to 0^+} \int_x^1 f(r)\,dr, \quad \lim_{x\to+\infty} F_2(x) = \lim_{x\to+\infty} \int_1^x f(r)\,dr.$$

For all $r \in (0, 1]$ one has

$$f(r) \geq \frac{2^m}{e^{\sqrt{2}m} r^{|2(j_1+j_2-m)+1|}}.$$

Under the assumptions, $|2(j_1 + j_2 - m) + 1| \geq 1$ and then

$$\int_0^1 \frac{1}{r^{|2(j_1+j_2-m)+1|}}\,dr = +\infty.$$

Finally one gets

$$\lim_{x\to 0^+} F_1(x) \geq \left(\frac{2}{e^{\sqrt{2}}}\right)^m \int_0^1 \frac{1}{r^{|2(j_1+j_2-m)+1|}}\,dr = +\infty,$$

and one concludes that the integral (5.17) is divergent in the case $j_1+j_2 < m$. This proves that the set

$$s_J := \left((z_1, z_2), \frac{z_1^{j_1} z_2^{j_2}}{\sqrt{\frac{2j!k!}{(j+k+1)!} \int_0^{+\infty} f(r)\,dr}} \right),$$

where $J := \{(j_1, j_2),\, |\, j_1, j_2 \in \mathbb{N},\, j_1 + j_2 \geq m\}$, forms an orthonormal basis for the Hilbert space $(\mathcal{H}_{\mathfrak{h}_m}, \langle \cdot, \cdot \rangle_{\mathfrak{h}_m})$.
Now suppose that there exists a positive integer m_0 such that $m_0 \omega_{EH}$ is balanced. Therefore by (2.14), we have

$$\frac{i}{2\pi} \partial\bar{\partial} \log \sum_{j_1+j_2 \geq m_0} \left| \frac{z_1^{j_1} z_2^{j_2}}{\sqrt{\langle s_J, s_J \rangle_{\mathfrak{h}_{m_0}}}} \right|^2 = m_0 \omega_{EH} = m_0 \frac{i}{2\pi} \partial\bar{\partial}\Phi,$$

where Φ is the Kähler potential (5.9). By Lemma 2.3.9 there exists a holomorphic function g on $\mathbb{C}^2$ such that

$$\log \left(\sum_{j_1+j_2 \geq m_0} \left| \frac{z_1^{j_1} z_2^{j_2}}{\sqrt{\langle s_J, s_J \rangle_{\mathfrak{h}_{m_0}}}} \right|^2 e^{-m_0 \Phi} \right) = \Re(g),$$

where $\Re(g)$ denotes the real part of g. By Lemma 2.3.10, g is forced to be a constant, and so

$$\sum_{j_1+j_2 \geq m_0} \left| \frac{z_1^{j_1} z_2^{j_2}}{\sqrt{\langle s_J, s_J \rangle_{\mathfrak{h}_{m_0}}}} \right|^2 = C e^{m_0 \Phi}, \tag{5.18}$$

where C is a real positive constant. A straightforward calculation shows that the series expansion of $e^{m_0\Phi}$ at $(z_1, z_2) = (0, 0)$ is given by

$$e^{m_0\Phi} \simeq \left(\frac{e}{2}\right)^{m_0} \sum_{s=0}^{m_0} \binom{m_0}{s} |z_1|^{2(m_0-s)} |z_2|^{2s} + \\ + \frac{m_0}{4} \left(\frac{e}{2}\right)^{m_0} \sum_{s=0}^{m_0+2} \binom{m_0+2}{s} |z_1|^{2(m_0+2-s)} |z_2|^{2s} + o(|z|^{2m_0+6}). \tag{5.19}$$

From (5.18) and (5.19), we find

$$\frac{|z_1|^{2m_0}}{\langle s_J, s_J \rangle_{\mathfrak{h}_{m_0}}} = C \left(\frac{e}{2}\right)^{m_0} |z_1|^{2m_0} \tag{5.20}$$

for $J = (j_1, j_2) = (m_0, 0)$, and

$$\frac{|z_1|^{2(m_0+2)}}{\langle s_{\hat{J}}, s_{\hat{J}} \rangle_{\mathfrak{h}_{m_0}}} = C \frac{m_0}{4} \left(\frac{e}{2}\right)^{m_0} |z_1|^{2(m_0+2)} \tag{5.21}$$

for $\hat{J} = (j_1, j_2) = (m_0 + 2, 0)$. By comparing (5.20)-(5.21), we must have

$$C = \left(\frac{2}{e}\right)^{m_0} \frac{1}{\langle s_J, s_J \rangle_{\mathfrak{h}_{m_0}}} = \left(\frac{2}{e}\right)^{m_0} \frac{4}{m_0 \left\langle s_{\hat{J}}, s_{\hat{J}} \right\rangle_{\mathfrak{h}_{m_0}}}. \tag{5.22}$$

From (5.16), by integrating, we find

$$\langle s_J, s_J \rangle_{\mathfrak{h}_{m_0}} = \frac{1}{m_0^2(m_0+1)} \left(\frac{e}{m_0}\right)^{m_0} \left(\Gamma(m_0+2, 2m_0) - m_0 \Gamma(m_0+1, 2m_0)\right), \tag{5.23}$$

and

$$\left\langle s_{\hat{J}}, s_{\hat{J}} \right\rangle_{\mathfrak{h}_{m_0}} = \frac{1}{m_0^4(m_0+3)} \left(\frac{e}{m_0}\right)^{m_0} \big(\Gamma(m_0+4, 2m_0) - 3m_0 \Gamma(m_0+3, 2m_0) + 2m_0^2 \Gamma(m_0+2, 2m_0) \big), \tag{5.24}$$

where $\Gamma(a, b) = \int_b^\infty t^{a-1} e^{-t} dt$ is the incomplete Gamma function. By substituting (5.23) in (5.22), we find

$$C = \left(\frac{2m_0}{e^2}\right)^{m_0} \frac{m_0^2(m_0+1)}{\Gamma(m_0+2, 2m_0) - m_0 \Gamma(m_0+1, 2m_0)} \tag{5.25}$$

and by substituting (5.24) in (5.22), one gets

$$C = \left(\frac{2m_0}{e^2}\right)^{m_0} \frac{4m_0^3(m_0+3)}{\Gamma(m_0+4, 2m_0) - 3m_0 \Gamma(m_0+3, 2m_0) + 2m_0^2 \Gamma(m_0+2, 2m_0)}. \tag{5.26}$$

Consider now the real function

$$\begin{aligned} f(x) = &\frac{x^2(x+1)}{\Gamma(x+2, 2x) - x\Gamma(x+1, 2x)} + \\ &- \frac{4x^3(x+3)}{\Gamma(x+4, 2x) - 3x\Gamma(x+3, 2x) + 2x^2 \Gamma(x+2, 2x)} \end{aligned} \tag{5.27}$$

for $x \in [0, +\infty)$ and its graph in Figure 5.2.

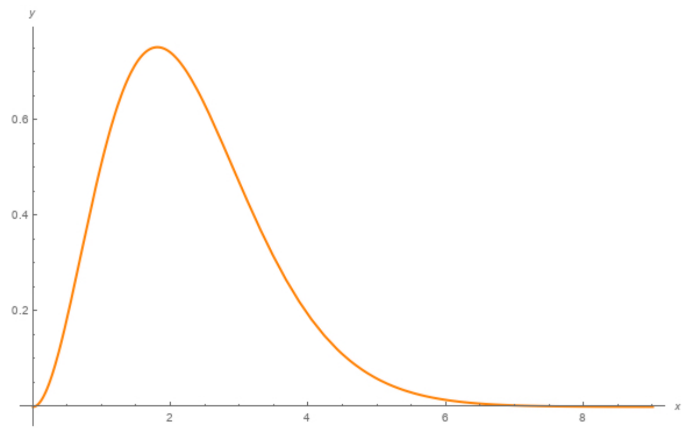

Figure 5.2: $y = f(x)$

The function $f(x)$ is monotone decreasing, after its maximum and tend to zero for $x \to +\infty$ (this facts follow by the asymptotic series representation for the incomplete Gamma function given in [2]). So, the value of the constant C in (5.25) is equal to the value in (5.26) if and only if $m_0 = 0$ in contrast with the positivity of m_0, yielding the desired contradiction. The proof of the theorem is complete.

Remark 5.2.3. In the version of the proof of Theorem 5.2.1 contains in [14] the function 5.27 includes errors in the calculation and consequently, its graph is not correct. These do not change the nature of the proof.

□

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