# Quantization of Kähler Manifolds

# and

# Holomorphic Immersions in Projective Spaces

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# Contents

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## Declaration

I declare that to the best of my knowledge the contents of this thesis are original and my work except where indicated otherwise.

## Summary

The main theme of this thesis is the interplay between the *geometric quantization* of a Kähler manifold $(M, \omega)$ and its realization as a *Kähler submanifold* of some complex projective space endowed with the Fubini-Study metric.

The thesis is divided into four chapters which are organized as follows.

In the first one, we shall recall the background about line bundles, connections, hermitian structures, curvature, prequantization and quantization of a Kähler manifold.

Chapter two provides many examples of geometric quantization of Kähler manifolds and these will form the basic material for the next chapters.

Chapter three deals with holomorphic isometric immersions of Kähler manifolds in complex space forms. We shall apply the results of Calabi to show that *the complex torus $V/\Lambda$ endowed with the flat form $\Omega_0^N$ and a Riemann surface $\Sigma_g$ endowed with the hyperbolic form $\omega_{hyp}$ cannot be Kähler submanifolds of any complex projective space* (see 3.5.6). The same result is proved (at least in the case where the target complex projective space is finite dimensional) for the regularized Kepler manifold (see 3.5.7).

In the second part of the chapter we shall study *Hartogs domains* $(D_F, \omega_F)$ in $\mathbb{C}^2$ (see 3.20). We shall give necessary and sufficient conditions for $(D_F, \omega_F)$ to admit a holomorphic and isometric immersion in a complex space form. Furthermore, we describe explicitly such an immersion in terms of the function $F$.

The last chapter is dedicated to the function *epsilon*, which plays a fundamental role in the theory of quantization of Kähler manifolds carried out in [5], [6], [7], [8] and [26]. Given a quantization of a Kähler manifold $(M, \omega)$, the function epsilon measures the obstruction for $\omega$ to be projectively induced via the coherent states map (see (4.2.2)). We give explicit formulae for the function epsilon in the case of the punctured plane $(\mathbb{C}^*, \omega^*)$ (see 4.5), the complex torus $(V/\Lambda, \Omega_0^N)$ (in terms of theta functions) and the Riemann surfaces $(\Sigma_g, \omega_{hyp})$ (see 4.6 and 4.7).

We then shall consider a geometric quantization $(L, h)$ for *generalized bounded domains* endowed with the Bergman metric. The main result is Theorem 4.8.9 which

gives necessary and sufficient conditions for the function epsilon to be constant.

At the end of the chapter we shall try to understand how the function epsilon varies with $\omega$ in a fixed cohomology class (see Section 4.9).

# Introduction

The main theme of this thesis is the interplay between the two following topics:

1) the *geometric quantization* of a Kähler manifold $(M, \omega)$;

2) the *realization* of a Kähler manifold $(M, \omega)$ as a *Kähler submanifold* of some complex projective space endowed with the Fubini-Study metric.

A geometric quantization of a Kähler manifold $(M, \omega)$ is a pair $(L, h)$, where $L$ is a holomorphic line bundle over $M$ and $h$ is a hermitian structure on $L$ such that its curvature satisfies $\mathrm{curv}(L, h) = -2\pi i \omega$. The curvature is calculated with respect to the *Chern connection*, i.e. the unique connection compatible with both the holomorphic structure and the hermitian structure $h$ (see Section 1.4).

Not all manifolds admit such a pair. In terms of cohomology classes, a Kähler manifold admits a geometric quantization if and only if the form $\omega$ is integral, i.e. its cohomology class $[\omega]_{dR}$, in the de Rham group, is in the image of the natural map $H^2(M, \mathbb{Z}) \hookrightarrow H^2(M, \mathbb{C})$. In particular, when $M$ is compact, the integrality of $\omega$ implies, by a well-known theorem of Kodaira, that $M$ is a projective algebraic manifold. This means that there exists a holomorphic embedding of $M$ into some complex projective space $\mathbb{P}^N(\mathbb{C})$, which is constructed by using global holomorphic sections of a suitable tensor power $L^k$ of $L$ (see Section 2.0.5).

In the framework of a geometric quantization of a Kähler manifold $(M, \omega)$, a natural map from $M$ into a complex projective space can be constructed as follows. Let $\mathcal{H}_h$ be the space of holomorphic sections of $L$ bounded with respect to the norm

$$\|s\|_h^2 =< s, s >_h= \int_M h(s(x), s(x)) \frac{\omega^n}{n!}(x),$$

where $n$ is the complex dimension of $M$. It turns out that $(\mathcal{H}_h, < \cdot, \cdot >_h)$ is a separable complex Hilbert space (see [5]). Under suitable conditions, one can define a holomorphic map, $\phi_{(L,h)} : M \to \mathbb{P}^N(\mathbb{C})$, the so called *coherent states map*, where $N+1$ ($N \leq \infty$) is the complex dimension of $\mathcal{H}_h$ (see Section 4.2). When $M$ is compact $\mathcal{H}_h$ coincides with the space of holomorphic sections of $L$ and $\phi_{(L,h)}$ is a holomorphic map from $M$ into a finite dimensional complex projective space.

Let $\Omega_{FS}^N$ be the Fubini-Study form in $\mathbb{P}^N(\mathbb{C})$. It is natural to consider the following:

**Problem 1** *Given a Kähler manifold* $(M, \omega)$ *(not necessarily compact), under which conditions does there exist a natural number* $N$ *(the case* $N = \infty$ *is not excluded) and a holomorphic immersion*

$$\phi : (M, \omega) \to (\mathbb{P}^N(\mathbb{C}), \Omega_{FS}^N)$$

*such that* $\phi^*\Omega_{FS}^N = \omega$*?*

The first systematic study of this Problem is due to Calabi ([9]) in his Ph.D. thesis. He studied the more general situation of holomorphic and isometric immersions of Kähler manifolds into finite or infinite dimensional *complex space forms*. His ingenious idea was to introduce, in the neighborhood $U$ of every point of $M$, a real analytic function $D_\omega : U \times U \to \mathbb{R}$ (see Section 3.1), which he christened *diastasis*, from the greek "distance", since in the case of the complex flat space it coincides with the square of the distance between two points. A deep analysis of the diastasis and its expansion in power series allowed Calabi to find necessary and sufficient conditions in order to attack Problem 1.

On the other hand, in the theory of geometric quantization, it is important to know when $\omega$ is projectively induced under the coherent states map. Working out the obstruction for $\phi_{(L,h)}$ to be a holomorphic isometric immersion one gets (see (4.2.2)):

$$\phi_{(L,h)}^*\Omega_{FS}^N = \omega + \frac{i}{2\pi}\partial\bar{\partial}\log \epsilon_{(L,h)}, \tag{1}$$

where $\epsilon_{(L,h)}$ is a smooth function on $M$.

This function is the central object in the study of the quantization of Kähler manifolds carried out in [5], [6], [7], [8] and [26]. When $\epsilon_{(L,h)}$ is constant, the quantization $(L, h)$ is said to be *regular*. One can calculate the function $\epsilon_{(L^k,h^k)}$ for every natural number $k$. Namely, one considers the Kähler form $k\omega$ on $M$ and $(L^k, h^k)$ the quantum line bundle for $(M, k\omega)$. If $(L^k, h^k)$ is regular for every $k$, then it is possible to apply a procedure called *quantization by deformation* (see [5], [6], [7] and [8]). From this, another Problem naturally arises:

**Problem 2** *Under which conditions does a Kähler manifold admit a regular quantization?*

Disregarding the applications to the theory of quantization, Problem 2 has its own intrinsic geometric interest. First of all, by formula (1), a solution to Problem 2 gives an answer to Problem 1. Secondly, we believe that the set of Kähler manifolds which admit a regular quantization, is a very special subset of the set of all projectively induced Kähler manifolds. For example, all the known cases are homogeneous and simply connected.

In fact, we conjecture that:

*a complete Kähler manifold which admits a regular geometric quantization has to be simply connected and homogeneous* (see Conjecture 1, Section 4.3).

One of the goals of this thesis is to study the geometric properties of the function epsilon and to give consistency to our conjecture.

The thesis is divided into four chapters which are organized as follows.

In the first one we shall recall the basic background of line bundles, connections, hermitian structures and curvature. We then describe the prequantization of a symplectic manifold, following Kostant's ideas [16].

We then pass to the quantization of a Kähler manifold $(M, \omega)$. In order to describe the set of all hermitian holomorphic line bundles $(L, h)$ with $\mathrm{curv}(L, h) = -2\pi i\omega$, one introduces the following notion: two geometric quantizations $(L_1, h_1)$ and $(L_2, h_2)$

of a Kähler manifold $(M, \omega)$ are said to be *equivalent* if there exists a holomorphic isomorphism $\psi$ between $L_1$ and $L_2$ such that $\psi^* h_2 = h_1$.

The set of all quantizations, denoted in this thesis by $\mathcal{L}_{hol}(M, \omega)$, can then be partitioned into equivalence classes.

The main result (see Theorem 1.4) is that the group of isomorphism classes of pairs $(L_0, h_0)$ with zero curvature, identifiable with $\mathrm{Hom}(\pi_1(M), S^1)$, acts simply transitively on $\mathcal{L}_{hol}(M, \omega)$.

Chapter two provides many examples of geometric quantization of Kähler manifolds and these will form the basic material for the next chapters. When $M$ is compact and not simply connected, a quantization of $M$ can be described in terms of the *factors of automorphy*. In particular, one can describe a quantization of: 1) the $N$ dimensional complex torus $V/\Lambda$ endowed with the flat form $\Omega_0^N$ (see 2.1.1) and 2) the Riemann surfaces $\Sigma_g$ endowed with the hyperbolic form $\omega_{hyp}$ (see 2.1.2). Since the manifolds under consideration are compact, we shall use the typical tools of algebraic geometry, for instance, the concept of theta functions.

Chapter three touches one of the main themes of this thesis: the holomorphic isometric immersions of Kähler manifolds in spaces of constant holomorphic sectional curvature (complex space forms). We will start by giving a detailed description of Calabi's work and we shall apply his results to show that the complex torus $(V/\Lambda, \Omega_0^N)$ and a Riemann surface $(\Sigma_g, \omega_{hyp})$ cannot be Kähler submanifolds of any complex projective space (see Proposition 3.5.6). The same result is proved (at least in the case when the target complex projective space is finite dimensional) for the regularized Kepler manifold (see Proposition 3.5.7).

In the second part of the chapter, we shall study *Hartogs domains* $D_F$ in $\mathbb{C}^2$ (see Section 3.6). Such domains are defined in terms of a function $F : [0, x_0) \to \mathbb{R}^+$ and they admit a natural Kähler form $\omega_F$ (see 3.21). In Theorems 3.6.5, 3.6.9 and 3.6.12 we shall give necessary and sufficient conditions for $(D_F, \omega_F)$ to admit a holomorphic and isometric immersion in a complex space form. Furthermore, we will describe explicitly such immersions in terms of the function $F$.

We conclude the chapter by proving that:

*if $F$ can be extended to an analytic function on $(-x_0, x_0)$ and $\omega_F$ is Kähler-Einstein then $(D_F, \omega_F)$ is holomorphically isometric (up to homotheties) to the hyperbolic two ball* (see Theorem 3.6.17).

The last chapter is the heart of this thesis. In fact all the tools and concepts developed in the previous chapters are used to study the function epsilon and to give consistency to conjecture 1.

We give explicit formulae for the function epsilon, in the case of the punctured plane $(\mathbb{C}^*, \omega^*)$ (see 4.5), the complex torus $(V/\Lambda, \Omega_0^N)$ (in terms of theta functions) and Riemann surfaces $(\Sigma_g, \omega_{hyp})$ (see 4.6 and 4.7).

We shall then consider a geometric quantization $(L, h)$ for *generalized bounded domains* endowed with the *Bergman metric*. The main result is Theorem 4.8.9 which compares the Hilbert space $(\mathcal{H}_h, \langle \cdot\,, \cdot \rangle_h)$ with the Hilbert space $(\mathcal{F}, (\cdot, \cdot))$ consisting of holomorphic $n$-forms $\alpha$ bounded with respect to $(\alpha, \alpha) = \int_M \alpha \wedge \bar{\alpha}$. We prove that *$\epsilon_{(L,h)}$ is a positive constant $\lambda$ if and only if the dimension of $\mathcal{H}_h$ equals the dimension of $\mathcal{F}$ and the $L^2$ scalar products are proportional, namely $\lambda \langle \cdot\,, \cdot \rangle_h = (\cdot, \cdot)$.* The main tool here is the *Calabi's rigidity Theorem* (see 3.2.4) which is used throughout this thesis. This theorem asserts that: *if a Kähler form $\omega$ on a complex manifold $M$ is projectively induced, then the dimension $N$ $(N \leq \infty)$ of the target complex projective space in which $(M, \omega)$ admits a full holomorphic isometric immersion depends only on $\omega$. Furthermore, if $\phi$ is a full holomorphic isometric immersion in the $N$-dimensional complex projective space, then all the other such maps are of the form $U \circ \phi$, for some projective unitary transformation $U$.*

In the last Section, we shall try to understand how the function epsilon varies with $\omega$ in a fixed cohomology class. More precisely, if $(M, \omega_0)$ is a quantizable, simply connected compact Kähler manifold, we will try to understand the relationship between $\epsilon_\omega$ and $\epsilon_{\omega_0}$ when $\omega$ varies in the set of all Kähler forms cohomologous to $\omega_0$.

Restricting ourself to the case when $(M, \omega_0)$ is a homogeneous simply connected compact Kähler manifold, we conjecture that: *if $\omega$ is cohomologous to $\omega_0$ and $\epsilon_\omega$ is constant, then $\omega = f^*\omega_0$ for some $f \in Aut(M)$* (see Conjecture 2 Section 4.9). The conjecture is true for the complex projective space endowed with the Fubini-Study form (see 4.9.6). However, for the case of the 1-dimensional complex projective space $\mathbb{P}^1(\mathbb{C})$, endowed with twice the Fubini-Study form $2\Omega^1_{FS}$, the situation is more complicated. In this case, we are able to show that Conjecture 2 partially holds (see Theorem 4.9.8). In fact we show that if $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ is a holomorphic map of the form

$$[(z_0, z_1)] \mapsto [(az_0^2, bz_0z_1, cz_1^2)], a, b, c \in \mathbb{C}^*$$

and $\omega = \phi^*\Omega^2_{FS}$ has constant epsilon, then $\phi$ is equivalent to the Veronese map $V_2$.

As a consequence of the previous theorem we will exhibit a family of projectively induced Kähler forms with a non-constant epsilon, which shows that Problem 1 is not equivalent to Problem 2, (see Remark 4.9.9).

# Chapter 1

# Geometric quantization of Kähler manifolds

We refer to [12] and [16] for the background material and the notations of this chapter.

## 1.1 Smooth and holomorphic line bundles

Let $M$ be a smooth manifold.

A $C^\infty$ line bundle $\pi : L \to M$ over $M$ is a complex vector bundle of rank 1, i.e. $\forall x \in M$ the fibre $L_x = \pi^{-1}(x)$ is a 1-dimensional vector space over $\mathbb{C}$.

Two line bundles $\pi_i : L_i \to M, i = 1, 2$, over $M$ are said to be $C^\infty$ *isomorphic* if there exists a smooth map $\psi : L_1 \to L_2$ such that $\pi_2 \circ \psi = \pi_1$, which is linear on the fibres. The isomorphism class of $L$ is denoted by $[L]$.

The set of isomorphism classes of line bundles over $M$ inherits a commutative group structure via the tensor product: the neutral element is given by the trivial bundle and the inverse of $L$ is given by the dual bundle $L^*$. This group is denoted, in analogy to the holomorphic case, by $\mathrm{Pic}^\infty(M)$. It has the following topological description.

Let $L^+$ be the complement in $L$ of the zero section. One can find an open covering $\{U_i\}_{i\in I}$ such that for every $i \in I$ there exists a trivialising section $s_i : U_i \to L^+$ and the intersection of any finite number of open sets of the covering is contractible. On the intersection $U_i \cap U_j$ of two open sets of the covering there exists a complex valued function $g_{ij}$ such that $s_i = g_{ij} s_j$. In the intersection $U_i \cap U_j \cap U_k$ of three open sets of the covering we have the cocycle relation

$$g_{ik} = g_{ij} g_{jk}. \tag{1.1}$$

This means that the $\{g_{ij}\}$ form a degree 1 Čech cocycle of the covering $\{U_i\}_{i \in I}$ with coefficients in the sheaf $\mathbb{C}_M^*$ of smooth non-vanishing complex valued functions on $M$. We refer to [16] for the proof of the following:

**Theorem 1.1.1** *The cohomology class of the Čech cocycle $\{g_{ij}\}$ in $H^1(M, \mathbb{C}_M^*)$ is independent of the open covering $\{U_i\}$ and of the choice of the trivialising sections $s_i$. Furthermore $Pic^\infty(M)$ and $H^1(M, \mathbb{C}_M^*)$ are isomorphic.*

Next consider the exponential exact sequence of sheaves (see [12, p. 37])

$$0 \to \mathbb{Z} \to \mathbb{C}_M \stackrel{e^{2\pi i}}{\to} \mathbb{C}_M^* \to 0$$

which gives the long exact sequence of cohomology groups

$$\cdots \to H^1(M, \mathbb{C}_M) \to H^1(M, \mathbb{C}_M^*) \stackrel{c_1}{\to} H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{C}_M) \to \cdots.$$

One can show that the cohomology groups $H^i(M, \mathbb{C}_M) = 0$ for $i > 0$ (see [12]) and so $H^1(M, \mathbb{C}_M^*)$ is isomorphic to $H^2(M, \mathbb{Z})$ via the map $c_1$, i.e. two $C^\infty$ line bundles $L_1$ and $L_2$ over $M$ are isomorphic if and only if $c_1(L_1) = c_1(L_2)$. If $L$ is a line bundle over $M$ then $c_1(L)$ is called its *first Chern class*.

Suppose now that $M$ is a *complex manifold*.

Two holomorphic line bundles $\pi_i : L_i \to M, i = 1, 2$, over $M$ are said to be *isomorphic* if there exists a *holomorphic* isomorphism of line bundles $\psi : L_1 \to L_2$. The isomorphism class of $L$ is denoted by $[L]_{hol}$. Take an open covering $\{U_i\}_{i \in I}$ such that the intersection of any finite number of open sets of the covering is contractible and for every open set $U_i$ of the covering there exists a trivialising holomorphic section $s_i : U_i \to L^+$. The functions $g_{ij}$ are in this case non-vanishing holomorphic functions on $U_i \cap U_j$ and the cocycle relation (1.1) means that $\{g_{ij}\}$ form a degree 1 Čech cocycle of the covering $\{U_i\}_{i\in I}$ with coefficients in the sheaf $\mathcal{O}^*_M$ of holomorphic non-vanishing functions on $M$. We have the analogous of 1.1.1, i.e. *the cohomology class of $\{g_{ij}\}$ in $H^1(M, \mathcal{O}^*{}_M)$ of the 1 Čech cocycle $\{g_{ij}\}$ is independent of the open covering $\{U_i\}$ and of the trivialising holomorphic sections $s_i$. Moreover, the Picard group Pic$(M)$ of isomorphism classes of holomorphic line bundles over $M$ is isomorphic to $H^1(M, \mathcal{O}^*{}_M)$.*

As in the $C^\infty$ case the tool to study the Picard group is the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_M \stackrel{e^{2\pi i}}{\to} \mathcal{O}^*_M \to 0$$

and the associated long exact sequence of cohomology groups

$$\cdots \to H^1(M, \mathcal{O}_M) \to H^1(M, \mathcal{O}^*_M) \stackrel{c_1}{\to} H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{O}_M) \to \cdots.$$

In the holomorphic case we cannot say anything about the map $c_1$ since $H^1(M, \mathcal{O}_M)$ and $H^2(M, \mathcal{O}_M)$ are, in general, different from zero. If we suppose that the manifold is simply connected then, from the Dolbeault Lemma (see [12]),

$$H^1(M, \mathcal{O}_M) = H^{0,1}_{\bar\partial}(M) = 0,$$

and so $H^1(M, \mathcal{O}^*_M)$ injects in $H^2(M, \mathbb{Z})$. In other words:

**Proposition 1.1.2** *Two holomorphic line bundles $L_1$ and $L_2$ over a simply connected complex manifold with $c_1(L_1) = c_1(L_2)$ are holomorphically equivalent, i.e. $[L_1]_{hol} = [L_2]_{hol}$.*

We conclude this Section by describing a natural exterior differential operator on the space $\Gamma(L)$ of the smooth sections of a holomorphic line bundle $L$ over $M$. The decomposition of the exterior differential $d = \partial + \bar\partial$ gives rise to a decomposition of the complexification of the tangent bundle $T_{\mathbb{C}}M = T^{1,0}M + T^{0,1}M$ and of its dual $T^*_{\mathbb{C}}M = \Omega^{1,0}(M) + \Omega^{0,1}(M)$ (see [12]). Let $\sigma : U \to L^+$ be a trivializing holomorphic section of $L$ over a open set $U$ and let $s$ be a smooth section of $L$. Then there exists a smooth complex valued function $f$ on $U$ such that $s = f\sigma$. Define

$$\bar\partial s := \bar\partial f \otimes \sigma. \qquad (1.2)$$

It is easy to see that definition (1.2) does not depend on the trivialising section $\sigma$ and so we can define a map

$$\bar{\partial} : \Gamma(L) \to \Gamma(\Omega^{0,1}(M) \otimes L) : s \mapsto \bar{\partial}s,$$

which maps a smooth section $s \in \Gamma(L)$ to the smooth section $\bar{\partial}s$ of the bundle $\Omega^{0,1}(M) \otimes L \to M$.

**Remark 1.1.3** *Notice that the space of global holomorphic sections of $L$, denoted by $H^0(L)$, is the subspace of all $s \in \Gamma(L)$ which satisfies*

$$\bar{\partial}s = 0.$$

## 1.2 Connection, curvature and hermitian structures

A *connection* $\nabla$ on a line bundle $L$ over a smooth manifold $M$ is a map

$$\nabla : \Gamma(L) \to \Gamma(T^*_{\mathbb{C}}M \otimes L)$$

satisfying:

$$\nabla(fs) = f\nabla s + df \otimes s,$$

for every $s \in \Gamma(L)$ and every complex valued function $f$ on $M$.

The *curvature* $\text{curv}(L, \nabla)$ of the connection $\nabla$ is the closed complex 2-form on $M$ satisfying:

$$\text{curv}(L, \nabla)(X, Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]}s, \tag{1.3}$$

$\forall X, Y \in \Gamma(T_{\mathbb{C}}M)$ and $\forall s \in \Gamma(L)$, where $\nabla_X s := (\nabla s)(X)$.

Let $\sigma : U \to L^+$ be a trivialising section over $U \subset M$. Consider the complex valued 1-form $\beta$ such that:

$$\nabla\sigma = \beta \otimes \sigma \tag{1.4}$$

It follows, by (1.3), that on $U$ we have:

$$\text{curv}(L, \nabla) = d\beta. \tag{1.5}$$

Suppose that $\widetilde{\nabla}$ is another connection on $L$. It is not difficult to see that $\mathrm{curv}(L, \widetilde{\nabla})$ is in the same de Rham cohomology class as $\mathrm{curv}(L, \nabla)$. Furthermore, one can show (see [16] and [12, p. 139]) that the de Rham cohomology class of $\frac{i}{2\pi}\,\mathrm{curv}(L, \nabla)$ is the first Chern class of $L$, i.e.

$$[\frac{i}{2\pi}\,\mathrm{curv}(L, \nabla)]_{dR} = c_1(L). \tag{1.6}$$

Let $L_1$ and $L_2$ be two $C^\infty$ line bundles over $M$ equipped with connections $\nabla^1$ and $\nabla^2$, respectively. One can define a connection on $L_1 \otimes L_2$, denoted by $\nabla^1 \otimes \nabla^2$, as follows: if $s$ and $t$ are smooth sections of $L_1$ and $L_2$, respectively, define

$$(\nabla^1 \otimes \nabla^2)(s \otimes t) := s \otimes \nabla^2 t + \nabla^1 s \otimes t.$$

It follows by (1.3) that

$$\mathrm{curv}(L_1 \otimes L_2, \nabla^1 \otimes \nabla^2) = \mathrm{curv}(L_1, \nabla^1) + \mathrm{curv}(L_2, \nabla^2).$$

Let us now introduce the concept of a hermitian structure on a line bundle $L$.

**Definition 1.2.1** *A hermitian structure on a $C^\infty$ line bundle $L$ over a smooth manifold $M$ is a smooth function $h : L \to \mathbb{R}^+ \cup \{0\}$ such that $h(\lambda q) = |\lambda|^2 h(q), \forall q \in L$, $\forall \lambda \in \mathbb{C}$ and $h(q) > 0$, $\forall q \in L^+$. A* hermitian *line bundle $(L, h)$ over $M$ is a $C^\infty$ line bundle equipped with a hermitian structure $h$.*

Given a hermitian structure $h$ on $L$ then, to every pair $s, t \in \Gamma(L)$, one can associate a smooth function on $M$ by the formula

$$h(s,t)(x) := \frac{s(x)}{\sigma(x)}\overline{\frac{t(x)}{\sigma(x)}}h(\sigma(x)), \tag{1.7}$$

where $\sigma$ is any trivialising section of $L$. It follows, by (1.7), that

$$h(\lambda s, \mu t)(x) = \lambda\bar{\mu} h(s(x), t(x)),\ \forall \lambda, \mu \in \mathbb{C}. \tag{1.8}$$

Conversely, given a map $h : \Gamma(L) \times \Gamma(L) \to \mathbb{C}$ which satisfies (1.8), one can define a hermitian structure on $L$ by

$$h(q) := h(s,s)(x),$$

where $s$ is any smooth section of $L$ with $s(x) = q$. In the sequel, we will write $h(s(x), s(x))$ to mean the function $h(s, s)(x)$.

Let $(L_1, h_1)$ and $(L_2, h_2)$ be two hermitian line bundles over $M$. One can define a hermitian structure $h_1 \otimes h_2$ on $L_1 \otimes L_2$ by:

$$(h_1 \otimes h_2)(s \otimes t, s \otimes t) := h_1(s, s)h_2(t, t), \ \forall s \in \Gamma(L_1), \forall t \in \Gamma(L_2).$$

Next consider *hermitian line bundles with connection.*

**Definition 1.2.2** *Let $(L, h)$ be a hermitian line bundle over a smooth manifold $M$. A connection $\nabla$ on $L$ is said to be* compatible *with $h$ or a $h$-*connection *if $\forall s, t \in \Gamma(L)$ and for every vector field $X$ on $M$*

$$Xh(s, t) = h(\nabla_X s, t) + h(s, \nabla_X t).$$

**Proposition 1.2.3** *The curvature of an $h$-connection $\nabla$ is a purely imaginary closed 2-form.*

**Proof:** Let $\sigma$ be a trivialising section on an open set $U$ of $M$ and let $\beta$ be the 1-form on $U$ given by (1.4). If $\nabla$ is a $h$-connection, then

$$d \log h(\sigma(x), \sigma(x)) = \frac{h(\nabla\sigma, \sigma)(x) + h(\sigma, \nabla\sigma)(x)}{h(\sigma(x), \sigma(x))} = (\beta + \bar{\beta})(x). \tag{1.9}$$

This formula has to be true for every section $\sigma$, so if we suppose that $h(\sigma(x), \sigma(x)) = 1$, we get that $\beta + \bar{\beta} = 0$. Hence, by (1.5),

$$\mathrm{curv}(L, \nabla) + \overline{\mathrm{curv}(L, \nabla)} = d\beta + d\bar{\beta} = d(\beta + \bar{\beta}) = 0.$$

□

Let now $M$ be a complex manifold and $L$ a holomorphic line bundle over $M$ endowed with a connection $\nabla$. The decomposition of 1-forms into type $(1, 0)$ and $(0, 1)$ induces a decomposition

$$\nabla = \nabla^{1,0} + \nabla^{0,1},$$

where $\nabla^{1,0} : \Gamma(L) \to \Gamma(\Omega^{1,0}(M) \otimes L)$ and $\nabla^{0,1} : \Gamma(L) \to \Gamma(\Omega^{0,1}(M) \otimes L)$.

**Definition 1.2.4** *A connection* $\nabla$ *on a holomorphic line bundle* $L$ *over a complex manifold* $M$ *is said to be* holomorphic *if* $\nabla^{0,1} = \bar{\partial}$ *(see (1.2)).*

**Remark 1.2.5** *Notice that, by Remark 1.1.3, if* $\nabla$ *is a holomorphic connection on a holomorphic line bundle* $L \to M$*, then the holomorphic sections* $s$ *of* $L$ *satisfy*

$$\nabla_X s = 0, \forall X \in \Gamma(T^{0,1}M).$$

## 1.3 Prequantization

Let $(M, \omega)$ be a symplectic manifold, i.e. $\omega$ is a non-degenerate real closed 2-form on $M$. From Proposition 1.2.3 the following definition makes sense:

**Definition 1.3.1** *A prequantization of a symplectic manifold* $(M, \omega)$ *is a triple* $(L, \nabla, h)$*, where* $L$ *is a* $C^\infty$ *line bundle over* $M$*, equipped with a hermitian structure* $h$ *and a* $h$*-connection* $\nabla$ *such that*

$$curv(L, \nabla) = -2\pi i\omega.$$

From now on, by a *triple* $(L, \nabla, h)$ on a smooth manifold $M$, we always mean a $C^\infty$ line bundle over $M$, endowed with a hermitian structure $h$ and a $h$-connection $\nabla$.

In order to study all the prequantizations of a symplectic manifold $(M, \omega)$ one gives the following:

**Definition 1.3.2** *Two triples* $(L_1, \nabla_1, h_1)$ *and* $(L_2, \nabla_2, h_2)$ *are said to be equivalent if there exists a* $C^\infty$ *isomorphism of line bundles* $\psi : L_1 \to L_2$ *such that* $\psi^*(\nabla_2) = \nabla_1$ *and* $\psi^*(h_2) = h_1$.

The equivalence class of $(L, \nabla, h)$ is denoted by $[(L, \nabla, h)]$. For a class $[(L, \nabla, h)]$ define

$$\mathrm{curv}([(L, \nabla, h)]) := \mathrm{curv}(L, \nabla).$$

It is not difficult to see that this is well defined, i.e. it does not depend on the representative in the equivalence class $[(L, \nabla, h)]$.

For any closed 2-form $\omega$ on $M$, let $\mathcal{L}(M,\omega)$ be the set of all triples $(L,\nabla,h)$ with $\mathrm{curv}(L,\nabla) = -2\pi i\omega$. The set $\mathcal{L}(M,\omega)$ can be partitioned into equivalence classes $[(L,\nabla,h)]$. We refer to [16] (cf. also Theorem 1.1.1 and (1.6)) for the proof of the following:

**Theorem 1.3.3** *$\mathcal{L}(M,\omega)$ is non-empty if and only if $\omega$ is integral, i.e. $[\omega]_{dR}$, the cohomology class of $\omega$ in the de Rham group, is in the image of the natural map $H^2(M,\mathbb{Z}) \hookrightarrow H^2(M,\mathbb{C})$.*

Let $\mathcal{L}(M,0)$ be the set of isomorphism classes of triples $(L_0,\nabla_0,h_0)$ with zero curvature. If $(L,\nabla,h)$ is in $\mathcal{L}(M,\omega)$ then $(L\otimes L_0, \nabla\otimes\nabla_0, h\otimes h_0)$ is in $\mathcal{L}(M,\omega)$. Therefore, one has defined an action of $\mathcal{L}(M,0)$ on $\mathcal{L}(M,\omega)$. The following theorem holds (see [16] for a proof):

**Theorem 1.3.4** *The set $\mathcal{L}(M,\omega)$ is a principal homogeneous space under the action of $\mathcal{L}(M,0)$, i.e. if $[(L_1,\nabla_1,h_1)]$ and $[(L_2,\nabla_2,h_2)]$ are two equivalence classes in $\mathcal{L}(M,\omega)$, then there exists a unique $[(L_0,\nabla_0,h_0)]$ in $\mathcal{L}(M,0)$ such that*

$$[(L_1\otimes L_0, \nabla_1\otimes\nabla_0, h_1\otimes h_0)] = [(L_2,\nabla_2,h_2)].$$

*Moreover $\mathcal{L}(M,0)$ can be identified with*

$$H^1(M,S^1) \cong \mathit{Hom}(\pi_1(M),S^1)$$

*the group of characters of the fundamental group of $M$.*

## 1.4 The Kähler case

Throughout this thesis we are interested in the quantization of Kähler manifolds. A Kähler manifold $(M,\omega)$ is a symplectic manifold with the additional requirements that $M$ is a complex manifold and $\omega$ is positive. The latter means that, if

$$\omega = \frac{i}{2\pi}\sum_{j,\bar{k}=1}^{n} g_{j\bar{k}}\, dz_j \wedge d\bar{z}_{\bar{k}} \tag{1.10}$$

is the expression of $\omega$ in local complex coordinates $z_j$, then the matrix $g_{j\bar{k}}$ is positive definite.

**Definition 1.4.1** *A geometric quantization, or simply, a quantization of a Kähler manifold* $(M, \omega)$ *is a prequantization* $(L, \nabla, h)$ *of* $(M, \omega)$*, with the additional conditions that* $L$ *is a* holomorphic *line bundle over* $M$ *and* $\nabla$ *is a* holomorphic connection *on* $L$*. The line bundle* $L$ *is called the quantum line bundle of* $(M, \omega)$*. A Kähler manifold* $(M, \omega)$ *is said to be quantizable if it admits a quantization.*

Notice that the holomorphic connection $\nabla$ of 1.4.1 is uniquely determined by the holomorphic line bundle $L$ and by the hermitian structure $h$. In fact the space of $h$-connections and the space of holomorphic connections on a holomorphic line bundle $L$ intersect in one point often called *the Chern connection* (see [12, p. 73]). This means that to describe a quantization of a Kähler manifold $(M, \omega)$ it is enough to specify a pair $(L, h)$, where $L$ is a holomorphic line bundle over $M$ and $h$ is a hermitian structure on $L$ satisfying

$$\mathrm{curv}(L, h) := \mathrm{curv}(L, \nabla) = -2\pi i\omega, \tag{1.11}$$

where $\nabla$ is the Chern connection.

Let $(L, h)$ be a quantization of a Kähler manifold $(M, \omega)$ and $\nabla$ the corresponding Chern connection. Let $\sigma : U \to L^+$ be a trivialising *holomorphic* section and let $\beta$ be the 1-form on $U$ given by (1.4).

For every vector field $X \in \Gamma(T^{0,1}M)$ we have (cf. Remark 1.2.5)

$$\nabla_X \sigma = \beta(X)\sigma = 0.$$

Thus $\beta$ is a form of type $(1, 0)$ on $U$. It follows by (1.9) that

$$\beta = \partial \log h(\sigma(x), \sigma(x))$$

and by (1.5) that

$$\mathrm{curv}(L, h) = d\beta = (\partial + \bar{\partial})\beta = -\partial\bar{\partial} \log h(\sigma(x), \sigma(x)). \tag{1.12}$$

In order to study all the quantizations of a Kähler manifold $(M, \omega)$ one gives the following:

**Definition 1.4.2** *Two holomorphic hermitian line bundles $(L_i, h_i) \stackrel{\pi_i}{\to} (M, \omega)$, $i = 1, 2$, over a Kähler manifold $(M, \omega)$ are called equivalent if there exists an isomorphism of holomorphic line bundles $\psi : L_1 \to L_2$, such that $\psi^* h_2 = h_1$. The equivalence class of $(L, h)$ is denoted by $[(L, h)]_{hol}$.*

Define

$$\mathrm{curv}([(L, h)]_{hol}) := \mathrm{curv}(L, h), \tag{1.13}$$

(see (1.11)). One can see that this is well defined, i.e. it does not depend on the representative of the equivalence class $[(L, h)]_{hol}$.

For every real closed form $\omega$ of type $(1, 1)$ on $M$, let $\mathcal{L}_{hol}(M, \omega)$ be the set of all pairs $(L, h)$ with $\mathrm{curv}(L, h) = -2\pi i\omega$. By (1.13), $\mathcal{L}_{hol}(M, \omega)$ can be partitioned in equivalence classes $[(L, h)]_{hol}$.

Now, we want to compare the set $\mathcal{L}(M, \omega)$ of all prequantizations of $(M, \omega)$, viewed as a symplectic manifold, to $\mathcal{L}_{hol}(M, \omega)$. We refer to ([17, p. 85]) for the proof of the following Lemma:

**Lemma 1.4.3** *Let $M$ be a complex manifold and $L$ a $C^\infty$ line bundle over $M$ equipped with a connection $\nabla$ such that $curv(L, \nabla)$ is purely of type $(1, 1)$. Then there exists a unique holomorphic structure on $L$ such that $\nabla$ is a holomorphic connection (see 1.2.4).*

Let $(L, \nabla, h)$ be a prequantization of a Kähler manifold $(M, \omega)$. Since $\omega$ is of type $(1, 1)$, from Lemma 1.4.3, one can endow $L$ with a unique holomorphic structure such that $\nabla$ is a holomorphic connection. Let denote by $L_{hol}$ the line bundle $L$ endowed with this holomorphic structure.

Therefore, one can define a map

$$\mathrm{hol} : \mathcal{L}(M, \omega) \to \mathcal{L}_{hol}(M, \omega) : (L, \nabla, h) \mapsto (L_{hol}, h).$$

**Proposition 1.4.4** *The map*

$$hol : \mathcal{L}(M,\omega) \to \mathcal{L}_{hol}(M,\omega)$$

*is a bijection between* $\mathcal{L}(M,\omega)$ *and* $\mathcal{L}_{hol}(M,\omega)$*. Moreover, the equivalence class of* $(L,\nabla,h)$ *in* $\mathcal{L}(M,\omega)$ *is mapped to the equivalence class of* $(L_{hol},h)$ *in* $\mathcal{L}_{hol}(M,\omega)$*.*

**Proof:** Let $(L,h)$ be in $\mathcal{L}_{hol}(M,\omega)$. If $\nabla$ is the Chern connection on $(L,h)$, then the triple $(L,\nabla,h)$ defines its inverse.

In order to prove the second part of the proposition, let $(L_1,\nabla^1,h_1)$ be equivalent to $(L_2,\nabla^2,h_2)$ in $\mathcal{L}(M,\omega)$. This means that there exists $\psi : L_1 \to L_2$ a $C^\infty$ isomorphism of line bundles such that $\psi^*(\nabla^2) = \nabla^1$ and $\psi^*(h_2) = h_1$. We claim that the map $\psi : L_{1hol} \to L_{2hol}$ is holomorphic. In fact, let $s_1$ be a holomorphic section of $L_{1hol}$, i.e. $\nabla^1_X s_1 = 0$, $\forall X \in \Gamma(T^{0,1}M)$ (cf. Remark 1.2.5). Then

$$\nabla^2_X(\psi s_1) = \psi(\nabla^1_X s_1) = 0,$$

i.e. $\psi s_1$ is a holomorphic section of $L_{2hol}$. This means that the map $\psi$ maps holomorphic sections of $L_{1hol}$ to holomorphic sections of $L_{2hol}$, and hence it is holomorphic. □

Applying 1.4.4 to the 0-form on $M$ one gets:

**Corollary 1.4.5** *Let* $\mathcal{L}_{hol}(M,0)$ *denote the group of equivalence classes of holomorphic hermitian line bundles* $(L_0,h_0)$ *with* $curv(L_0,h_0) = 0$*.*

*The map* $hol : \mathcal{L}(M,0) \to \mathcal{L}_{hol}(M,0)$ *is an isomorphism of groups mapping the equivalence class of* $(L_0,\nabla_0,h_0)$ *in* $\mathcal{L}(M,0)$ *to the equivalence class of* $(L_{0hol},h_0)$ *in* $\mathcal{L}_{hol}(M,0)$*.*

From 1.3.3, 1.3.4, 1.4.4 and 1.4.5 our results can be summarized as follows:

**Theorem 1.4.6** *A Kähler manifold* $(M,\omega)$ *admits a geometric quantization* $(L,h)$ *if and only if* $\omega$ *is integral. The set* $\mathcal{L}_{hol}(M,\omega)$ *of equivalence classes* $[(L,h)]_{hol}$ *is acted upon simply-transitively by* $\mathcal{L}_{hol}(M,0) \cong Hom(\pi_1(M),S^1)$*.*

We conclude this Section by describing a link between the geometric quantization of a Kähler manifold $(M, \omega)$ and the space of smooth functions $f$ on $M$ satisfying the functional equation $\partial\bar{\partial} f = 0$.

Let $\pi : L \to M$ be a holomorphic line bundle over a complex manifold $M$ and $\tilde{h}$ and $h$ two hermitian structures on $L$.

Let $\sigma : U \to L^+$ be a trivialising section of $L$. Define a smooth real valued function on $M$ by the formula

$$f(x) := \frac{\tilde{h}(\sigma(x), \sigma(x))}{h(\sigma(x), \sigma(x))} \tag{1.14}$$

We write $\tilde{h} = fh$ to mean that the function $f$ satisfies (1.14).

**Proposition 1.4.7** *Let $(L_i, h_i) \stackrel{\pi_i}{\to} (M, \omega)$ be two quantum line bundles over a Kähler manifold $(M, \omega)$. Suppose that there exists an isomorphism $\psi : L_1 \to L_2$ of holomorphic line bundles. Let $f$ be the smooth real valued function on $M$ such that $\psi^* h_2 = f h_1$. Then $\partial\bar{\partial} \log f = 0$.*

**Proof:** Since $(L_i, h_i)$ are two quantum line bundles over $(M, \omega)$, then $\mathrm{curv}(L_1, h_1) = \mathrm{curv}(L_2, h_2) = -2\pi i \omega$. On the other hand, since $\psi$ is holomorphic,

$$\mathrm{curv}(L_2, h_2) = \mathrm{curv}(L_1, \psi^* h_2) = \mathrm{curv}(L_1, f h_1).$$

Thus, if $\sigma$ is any trivialising holomorphic section of $L_1$, by (1.12) one obtains:

$$\begin{aligned} -\partial\bar{\partial} \log h_1(\sigma(x), \sigma(x)) &= \mathrm{curv}(L_1, h_1) = \mathrm{curv}(L_2, h_2) = -\partial\bar{\partial} \log f h_1(\sigma(x), \sigma(x)) \\ &= -\partial\bar{\partial} \log f - \partial\bar{\partial} \log h_1(\sigma(x), \sigma(x)) \end{aligned},$$

therefore $\partial\bar{\partial} \log f = 0$. □

The following, well-known, Lemmas will be of constant use throughout this thesis.

**Lemma 1.4.8** *Let $(M, \omega)$ be a compact Kähler manifold and $f$ a smooth function on $M$ satisfying $\partial\bar{\partial} f = 0$. Then $f$ is constant.*

**Proof:** In a Kähler manifold the equation $\partial\bar{\partial} f = 0$ is equivalent to the fact that $f$ is harmonic and so it is constant, since $M$ is compact. □

**Lemma 1.4.9** *Let $(M, \omega)$ be a simply connected complex manifold and $f$ a smooth real valued function on $M$ satisfying $\partial\bar{\partial} f = 0$. Then there exists a holomorphic function $k$ on $M$ such that $f = \Re(k)$, where $\Re(k)$ denotes the real part of $k$.*

**Proof:** Suppose

$$\partial\bar{\partial} f = d(\partial f) = 0.$$

Since the manifold is simply connected there exists a complex valued function $g$ on $M$ such that

$$\partial f = dg.$$

Since $\partial f$ is of type $(1,0)$, this implies that $\bar{\partial} g = 0$, i.e. $g$ is a holomorphic function on $M$. The reality of $f$ implies that $\bar{\partial} f = d\bar{g}$, i.e.

$$df = \partial f + \bar{\partial} f = d(g + \bar{g}).$$

Thus, up to a constant, $f = g + \bar{g}$, i.e. $f = \Re(k)$, where $k = 2g$. □

When $M$ is a simply connected Kähler manifold we know, from Theorem 1.4.6, that $\mathcal{L}_{hol}(M, \omega)$ consists of a single equivalence class. We can recover this result as follows. Let $(L_1, h_1)$ and $(L_2, h_2)$ be two quantizations of a simply connected Kähler manifold $(M, \omega)$. By (1.6), $[\omega]_{dR} = c_1(L_1) = c_1(L_2)$ and since the manifold is simply connected, it follows from 1.1.2 that there exists an isomorphism of holomorphic line bundles $\psi : L_1 \to L_2$. Let $f$ be the real valued function on $M$ such that $\psi^* h_2 = f h_1$. From 1.4.7 and 1.4.9 $\log f = \Re(k)$ for some holomorphic function $k$ on $M$. Then the map $\tilde{\psi} = e^{\frac{-k}{2}} \psi$ is a holomorphic isomorphism between $L_1$ and $L_2$ satisfying $\tilde{\psi}^* h_2 = h_1$. In fact,

$$\tilde{\psi}^*(h_2)(q) = h_2(e^{\frac{-k}{2}} \psi(q)) = e^{-\Re(k)} h_2(\psi(q)) = e^{-\Re(k)} f h_1(q) = h_1(q), \ \forall q \in L_1,$$

and so the claim.

## 1.5 The group $D_{[(L,h)]}(M)$

Let $(L,h) \in [(L,h)]_{hol} \in \mathcal{L}_{hol}(M,\omega)$ be a geometric quantization of a Kähler manifold $(M,\omega)$. Denote by $\mathrm{Aut}(L,h)$ the group of holomorphic diffeomorphisms $\tilde{F} : L \to L$, linear on the fibres, such that $\tilde{F}^*h = h$, by $\mathrm{Aut}(M)$ the group of holomorphic diffeomorphisms of $M$ and by $\mathrm{Isom}(M,\omega)$ the group of isometries of the Kähler manifold $(M,\omega)$, i.e. the group of smooth maps $F : M \to M$ such that $F^*\omega = \omega$.

For every $\tilde{F} \in \mathrm{Aut}(L,h)$ there exists a unique $F \in \mathrm{Aut}(M) \cap \mathrm{Isom}(M,\omega)$ which makes the following diagram commutative:

$$\begin{array}{ccc} (L,h) & \stackrel{\tilde{F}}{\to} & (L,h) \\ \pi\downarrow & & \pi\downarrow \\ (M,\omega) & \stackrel{F}{\to} & (M,\omega) \end{array}$$

The map $\tilde{F}$ is called a *lifting* of $F$. Denote by $D_{[(L,h)]}(M)$ the group of all maps $F : M \to M$ which admit a lifting $\tilde{F}$. This notation is justified from the fact that $D_{[(L,h)]}(M)$ really depends on the equivalence class $[(L,h)]_{hol}$.

Therefore, one can define a map

$$\mathrm{Aut}(L,h) \stackrel{P}{\to} D_{[(L,h)]}(M) : \tilde{F} \mapsto F.$$

It can be shown that the kernel of $P$ consists of constants of modulus one (see [26]). In other words, there exists an exact sequence of groups:

$$1 \to S^1 \to \mathrm{Aut}(L,h) \stackrel{P}{\to} D_{[(L,h)]}(M) \to 1.$$

For $\tilde{F} \in \mathrm{Aut}(L,h)$ and $s \in H^0(L)$ define

$$\tilde{F} \cdot s = \tilde{F} \circ s \circ F^{-1},$$

where $F = P(\tilde{F})$. It is not difficult to see that $\tilde{F} \cdot s$ is a holomorphic section of $L$ and this gives rise to a representation of $\mathrm{Aut}(L,h)$ on the space $H^0(L)$.

Notice that the group $D_{[(L,h)]}(M)$ does not act on $H^0(L)$ due to the ambiguity on the choice of $\tilde{F}$ with $P(\tilde{F}) = F$. On the other hand, this ambiguity is the multiplication by a constant, therefore it disappears when one considers projective representations.

We conclude that $D_{[(L,h)]}(M)$ *admits a projective representation in* $\mathbb{P}(H^0(L))$.

**Proposition 1.5.1** *Let* $(L,h)$ *be a geometric quantization of a simply connected Kähler manifold* $(M,\omega)$. *Then the group* $D_{[(L,h)]}(M)$ *is equal to* $Aut(M) \cap Isom(M,\omega)$.

**Proof:** By its very definition, $D_{[(L,h)]}(M)$ is a subgroup of $\mathrm{Aut}(M) \cap Isom(M,\omega)$. Let $F$ be in $\mathrm{Aut}(M) \cap \mathrm{Isom}(M,\omega)$. Take the pull-back $(F^*L, F^*h)$ and consider the natural diagram

$$\begin{array}{ccc} (F^*L, F^*h) & \xrightarrow{F^*} & (L,h) \\ \pi^* \downarrow & & \pi \downarrow \\ (M,\omega) & \xrightarrow{F} & (M,\omega) \end{array}$$

It follows that $\mathrm{curv}(F^*L, F^*h) = -2\pi i\omega$, i.e. $(F^*L, F^*h)$ is a quantum line bundle for $(M,\omega)$. Since $M$ is simply connected, by (1.6) and 1.1.2, there exists an isomorphism of holomorphic line bundles $\psi : L \to F^*L$ such that $\psi^*(F^*h) = h$. Then $\tilde{F} := F^* \circ \psi$ is the desired lifting of F. □

# Chapter 2

# Examples

### 2.0.1 The flat space $(\mathbb{C}^N, \Omega_0^N)$

Let $N$ be a natural number and $\Omega_0^N$ the Kähler form on $\mathbb{C}^N$ defined by:

$$\Omega_0^N = \frac{i}{2} \sum_{j=1}^{N} dz_j \wedge d\bar{z}_j,$$

where $(z_1, \ldots, z_N)$ are global coordinates on $\mathbb{C}^N$. The trivial bundle $L = \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ equipped with the hermitian structure:

$$h(z,t) = e^{-\pi \sum_{j=1}^{N} |z_j|^2} |t|^2, \ \forall z \in \mathbb{C}^N, \forall t \in \mathbb{C},$$

defines a geometric quantization of $(\mathbb{C}^N, \Omega_0^N)$. Indeed, if $\sigma(z) = (z, f(z))$ is a global holomorphic section of $L$, where $f(z)$ is a holomorphic function in $\mathbb{C}^N$ then, by (1.12), one obtains:

$$\mathrm{curv}(L,h) = -\partial\bar{\partial} \log h(\sigma(z), \sigma(z)) = -2\pi i \Omega_0^N.$$

Since $\mathbb{C}^N$ is simply connected, it follows, from 1.5.1, that

$$D_{[(L,h)]}(\mathbb{C}^N) = \mathrm{Aut}(\mathbb{C}^N) \cap \mathrm{Isom}(\mathbb{C}^N, \Omega_0^N).$$

Notice that the group $\mathrm{Aut}(\mathbb{C}^N) \cap \mathrm{Isom}(\mathbb{C}^N, \Omega_0^N)$ acts transitively on $(\mathbb{C}^N, \Omega_0^N)$, i.e. $(\mathbb{C}^N, \Omega_0^N)$ is a *homogeneous* Kähler manifold.

We are also interested in the case of the infinite dimensional space $\mathbb{C}^\infty$. This is the Hilbert space consisting of sequences of complex numbers $z_j$ satisfying

$$\sum_{j=1}^{+\infty} |z_j|^2 < \infty.$$

In analogy with the finite dimensional case, $\mathbb{C}^\infty$ can be endowed with the Kähler form

$$\Omega_0^\infty = \frac{i}{2}\sum_{j=1}^{+\infty} dz_j \wedge d\bar{z}_j.$$

### 2.0.2 The unit disk $(\mathbb{D}, \omega_{hyp})$

Let $\mathbb{D} = \{z \in \mathbb{C} |\ |z|^2 < 1\}$ be the unit disk endowed with the hyperbolic Kähler form

$$\omega_{hyp} = \frac{i}{\pi}\frac{dz \wedge d\bar{z}}{(1-|z|^2)^2}.$$

The trivial bundle $L := \mathbb{D} \times \mathbb{C} \to \mathbb{D}$, endowed with the hermitian structure $h$ given by

$$h(z,t) = (1-|z|^2)^2|t|^2,\ \forall z \in \mathbb{D}, \forall t \in \mathbb{C}, \tag{2.1}$$

is a quantization of $(\mathbb{D}, \omega_{hyp})$. Indeed, by (1.12),

$$\mathrm{curv}(L,h) = -2\partial\bar{\partial}\log(1-|z|^2) = \frac{2dz \wedge d\bar{z}}{(1-|z|^2)^2} = -2\pi i \omega_{hyp}. \tag{2.2}$$

Since $\mathbb{D}$ is simply connected, it follows from 1.5.1, that

$$D_{[(L,h)]}(\mathbb{D}) = \mathrm{Aut}(\mathbb{D}) \cap \mathrm{Isom}(\mathbb{D}, \omega_{hyp}).$$

This group is given by

$$SU(1,1) = \{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} |\ |a|^2 - |b|^2 = 1\},$$

(see [14]).

The example of the unit disk can be generalized by considering *the complex hyperbolic space*

$$\mathcal{D}_N = \{(z_1, \dots, z_N) \in \mathbb{C}^N |\ \|z\|^2 = \sum_{j=1}^{N} |z_j|^2 < 1\}$$

endowed with the hyperbolic form

$$\Omega^N_{hyp} = \frac{i}{2\pi}\partial\bar{\partial}\log\frac{1}{1-\|z\|^2}. \tag{2.3}$$

More generally, one can consider $\mathcal{D}_\infty$ the infinite dimensional complex hyperbolic space. This is defined as the set of sequences $z_j$ in $\mathbb{C}^\infty$ such that $\|z\|^2 = \sum_{j=1}^{+\infty}|z_j|^2 < 1$ and it can be endowed with the Kähler form

$$\Omega^\infty_{hyp} = \frac{i}{2\pi}\partial\bar{\partial}\log\frac{1}{1-\|z\|^2}. \tag{2.4}$$

### 2.0.3 The regularized Kepler manifold

Let $T_0^*S^n$ be the cotangent space to the $n$-dimensional sphere $S^n$ minus its zero section (see [25] and [26]). This can be realized as the space

$$X = \{(e,x) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid e \cdot e = 1, x \cdot e = 0, x \neq 0\},$$

where the " $\cdot$ " is the standard scalar product in $\mathbb{R}^{n+1}$. One can endow $X$ with the natural symplectic form

$$\Omega = d\theta,$$

where

$$\theta = \sum_{j=1}^{n+1} x_j de_j = x \cdot de$$

is called the *Liouville* form. The manifold $X$ can be further identified with the *isotropic cone*

$$C_n = \{z \in \mathbb{C}^{n+1} \mid z \cdot z = 0, z \neq 0\}$$

via the map

$$\psi : X \to C_n : (e,x) \mapsto |x|e + ix \in \mathbb{C}^{n+1}.$$

Therefore $X$ becomes a complex manifold via the pull-back of the complex structure of $\mathbb{C}^{n+1}$. One can show that the symplectic form $\Omega$ is, in fact, a Kähler form and it can be written as

$$\Omega = 2i\partial\bar{\partial}\sqrt{z \cdot \bar{z}} = 2i\partial\bar{\partial}|x|,$$

(see [26]). Since $\Omega$ is exact it is trivially integral and, from 1.4.6, there exists a quantization $(L, h)$ of $(X, \Omega)$. Restricting to the case $n \geq 3$, $X$ is simply connected and, since $[\Omega]_{dR} = c_1(L) = 0$ it follows from 1.1.2, that $L$ is holomorphically trivial over $X$. When $X$ is simply connected, it follows from Proposition 1.5.1 that

$$D_{[(L,h)]}(X) = \mathrm{Aut}(X) \cap \mathrm{Isom}(X, \omega).$$

The explicit description of this group can be found in [26].

### 2.0.4 The complex projective space

Given a natural number $N$, let $\mathbb{P}^N(\mathbb{C})$ be the complex projective space and $\pi : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N(\mathbb{C})$ the standard projection map. Let $U \subset \mathbb{P}^N(\mathbb{C})$ be an open set and $z : U \to \mathbb{C}^{N+1} \setminus \{0\}$ a *lifting* of $U$, i.e. a holomorphic map such that $\pi \circ z = id$. Define the differential form on $U$ by the formula

$$\Omega_{FS}^N = \frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2, \tag{2.5}$$

where $\|z\|^2 = \langle z, z \rangle$ is the standard form on $\mathbb{C}^{N+1}$. If $\tilde{z} : V \to \mathbb{C}^{N+1} \setminus \{0\}$ is another lifting then on $U \cap V$, there exists a non-zero holomorphic function $f$ such that $\tilde{z} = fz$. Therefore

$$\frac{i}{2\pi} \partial\bar{\partial} \log \|\tilde{z}\|^2 = \frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2 + \frac{i}{2\pi} \partial\bar{\partial} \log f + \frac{i}{2\pi} \partial\bar{\partial} \log \bar{f} = \Omega_{FS}^N,$$

and consequently $\Omega_{FS}^N$ defines a differential form on $\mathbb{P}^N(\mathbb{C})$, since it is independent of the lifting chosen.

Furthermore, one can show that $\Omega_{FS}^N$ is an integral Kähler form (see [12]) and so, from 1.4.6, there exists a hermitian line bundle $(L, h)$ such that $\mathrm{curv}(L, h) = -2\pi i \Omega_{FS}^N$. The line bundle $L$, known as *the hyperplane bundle*, is denoted by $\mathcal{O}_N(1)$.

The global holomorphic sections of $\mathcal{O}_N(1)$ can be identified with linear forms in the $N+1$ variables $(z_0, \ldots, z_N)$ (see [12, p. 164-167]). More generally, if $\mathcal{O}_N(k)$ denotes the $k$-th tensor power of $\mathcal{O}_N(1)$, then the space $H^0(\mathcal{O}_N(k))$ can be identified with the space of homogeneous polynomials of degree $k$ in $(z_0, \ldots, z_N)$. A combinatorial calculation shows that

$$\dim H^0(\mathcal{O}_N(k)) = \binom{N+k}{N}.$$

The group $\mathrm{Aut}(\mathbb{P}^N(\mathbb{C}))$ equals $\mathrm{PGL}(N+1, \mathbb{C})$, the projective linear group, and $\mathrm{Isom}(\mathbb{P}^N(\mathbb{C}), \Omega^N_{FS})$ equals $\mathrm{PU}(N+1)$ the group of projective unitary transformations (see [13] for a proof). Since $\mathbb{P}^N(\mathbb{C})$ is simply connected, it follows from 1.5.1, that

$$D_{[(\mathcal{O}_N(1),h)]}(\mathbb{P}^N(\mathbb{C})) = \mathrm{Aut}(\mathbb{P}^N(\mathbb{C})) \cap \mathrm{Isom}(\mathbb{P}^N(\mathbb{C}), \Omega^N_{FS}) = \mathrm{PU}(N+1).$$

Similar considerations can be done for the infinite dimensional complex projective space defined as follows (see [18, p. 280]). Two sequences $z_j$ and $w_j$ in $\mathbb{C}^\infty \setminus \{0\}$ are said to be equivalent if $z_j = \lambda w_j$ for some $\lambda \in \mathbb{C}^*$. The quotient space of $\mathbb{C}^\infty \setminus \{0\}$ by this equivalence relation is called the *infinite dimensional complex projective space* and it is denoted by $\mathbb{P}^\infty(\mathbb{C})$. Let $\pi : \mathbb{C}^\infty \setminus 0 \to \mathbb{P}^\infty(\mathbb{C})$ be the standard projection map. One can define a Kähler form on $\mathbb{P}^\infty(\mathbb{C})$, analogous to (2.5), by:

$$\Omega^\infty_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log\|z\|^2, \tag{2.6}$$

where $z : U \to \mathbb{C}^\infty \setminus 0$ is a lifting of an open set $U \subset \mathbb{P}^\infty(\mathbb{C})$.

One can also consider the projective space of a complex Hilbert space defined as follows. Let $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a separable complex Hilbert space. Define $\mathbb{P}(\mathcal{H})$ as the quotient $\mathcal{H} \setminus \{0\}/\!\sim$, where $s$ is equivalent to $t$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $s = \lambda t$. The space $\mathbb{P}(\mathcal{H})$ can be endowed with the Kähler form

$$\Omega_{\mathcal{H}} = \frac{i}{2\pi}\partial\bar{\partial}\log\|s\|^2,\ \ \|s\|^2 = \langle s, s\rangle \tag{2.7}$$

(see [26, p. 409]). A choice of a unitary basis $s_j$, $j = 0, 1, \ldots N$ $(N \leq \infty)$, of $(\mathcal{H}, \langle\,\cdot\,,\,\cdot\,\rangle)$ gives rise to a natural holomorphic diffeomorphism

$$b : \mathcal{H} \to \mathbb{C}^N : s \mapsto [(\ldots, \langle s, s_j\rangle, \ldots)]$$

which induces a map

$$b : \mathbb{P}(\mathcal{H}) \to \mathbb{P}^N(\mathbb{C}),$$

satisfying $b^*\Omega^N_{FS} = \Omega_{\mathcal{H}}$.

### 2.0.5 Projective algebraic manifolds

Let $M$ be a compact complex manifold. Suppose that there exists a natural number $N$ and a holomorphic embedding $\phi : M \to \mathbb{P}^N(\mathbb{C})$. In this case the manifold $M$ is called a projective algebraic manifold. It is not difficult to see that $\omega = \phi^*(\Omega^N_{FS})$ is a Kähler form and $L = \phi^*(\mathcal{O}_N(1))$ is a holomorphic line bundle over $M$. Furthermore, $L$ can be endowed with a hermitian structure $h$ such that $\mathrm{curv}(L, h) = -2\pi i\omega$ (see [12, pp. 148-149]) and so every projective algebraic manifold is quantizable.

If one starts from a compact quantizable Kähler manifold $(M, \omega)$, then $\omega$ is integral, and therefore, by a well-known theorem of Kodaira, $M$ can be embedded into some complex projective space using sections of a suitable tensor power $L^k$ of $L$ (see [12]). More precisely, let $N(k)+1$ be the complex dimension of $L^k$ and $(s_0, \dots, s_{N(k)})$ a basis for $H^0(L^k)$. Suppose that the so called *base point free condition* is satisfied, i.e. for every $x \in M$ there exists a holomorphic section $s$ of $H^0(L^k)$ such that $s(x) \neq 0$. For $\sigma : U \to L^+$ a trivialising holomorphic section of $L^k$ define

$$\phi_k : U \to \mathbb{C}^{N(k)+1} : x \mapsto (\frac{s_0(x)}{\sigma(x)}, \dots, \frac{s_{N(k)}(x)}{\sigma(x)}). \tag{2.8}$$

If $\tau : V \to L^+$ is another holomorphic trivialisation then, there exists a holomorphic function $f$ on $U \cap V$ such that $\sigma(x) = f(x)\tau(x)$. Therefore the map (2.8) can be extended to a map

$$\phi : M \to \mathbb{P}^{N(k)}(\mathbb{C}) : x \mapsto [(s_0(x), \dots, s_{N(k)}(x)]. \tag{2.9}$$

The above mentioned theorem of Kodaira says that *for k sufficiently large the base point free condition is satisfied and the map $\phi$ is an embedding.* Once that the map $\phi$ is given, one has $\phi^*(\mathcal{O}_N(k)) = L^k$, but in general $\phi^*(\Omega^N_{FS})$ is different from the Kähler form $k\omega$. This leads to the problem of holomorphic isometric immersions of Kähler manifolds in complex projective spaces, which will be the main theme of the next chapter.

## 2.1 Factors of automorphy and holomorphic line bundles

All the material contained in this Section is taken from Appendix A in [23].

Let $M$ be a complex manifold, and $p : \tilde{M} \to M$ its covering map. The topological space $\tilde{M}$ inherits the structure of a complex manifold as follows. If $(U_i, z_i)$ is a holomorphic coordinate covering of $M$ in which each set $U_i$ is simply connected then any connected component of $p^{-1}(U_i)$ is homeomorphic to $U_i$ under the projection $p$ and so $(p^{-1}(U_i), z_i \circ p)$ is a holomorphic coordinate covering of $\tilde{M}$.

The aim of this Section is to describe the holomorphic line bundles $L$ over $M$ in terms of the holomorphic line bundles $p^*(L)$ over $\tilde{M}$.

**Definition 2.1.1** *Let $\pi_1(M)$ be the fundamental group of $M$. A map $f : \pi_1(M) \times \tilde{M} \to \mathbb{C}^*$, holomorphic for any fixed $\lambda \in \pi_1(M)$, is called a* 1-cocycle *if it satisfies the cocycle relation*

$$f(\lambda\mu, \tilde{x}) = f(\lambda, \mu \cdot \tilde{x}) f(\mu, \tilde{x}), \ \forall \lambda, \mu \in \pi_1(M), \ \forall \tilde{x} \in \tilde{M},$$

*where $\mu \cdot \tilde{x}$ denotes the action of $\pi_1(M)$ on $\tilde{M}$.*

Under multiplication the 1-cocycles form an abelian group which is denoted by $Z^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}}))$, where $H^0(\mathcal{O}^*_{\tilde{M}})$ is the set of non-vanishing holomorphic functions on $\tilde{M}$. The elements of $Z^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}}))$ are called the *factors of automorphy.*

Denote by $B^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}}))$ the subgroup of $Z^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}}))$ consisting of $f$ such that $f(\lambda, \tilde{x}) = h(\lambda\tilde{x})h(\tilde{x})^{-1}$ for some $h \in H^0(\mathcal{O}^*_{\tilde{M}})$. Define the cohomology group as the quotient

$$H^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}})) = Z^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}}))/B^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}}))$$

**Theorem 2.1.2** *There is a canonical isomorphism*

$$\Phi : H^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}})) \to \ker\left(p^* : H^1(M, \mathcal{O}^*_M) \to H^1(\tilde{M}, \mathcal{O}^*_{\tilde{M}})\right)$$

**Proof:** Let $\{U_i\}_{i \in I}$ be an open covering of $M$ with the property that for every $i \in I$ there exists a connected set $W_i \subset p^{-1}(U_i)$ such that $p_i = p|_{W_i} : W_i \to U_i$ are biholomorphisms. For every $(i, j) \in I \times I$ and $x \in U_i \cap U_j$ there exists $\lambda_{ij} \in \pi_1(M)$ such that $p_j^{-1}(x) = \lambda_{ij} p_i^{-1}(x)$. Therefore $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$ for every $(i, j, k) \in I^3$. Let $f$ be an element in $Z^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}}))$. For $(i, j) \in I \times I$ and $x \in U_i \cap U_j$ define

$$g_{ij}(x) = f(\lambda_{ij}, p_i^{-1}(x)). \tag{2.10}$$

It follows immediately that $g_{ij}g_{jk} = g_{ik}$ for every $(i,j,k) \in I^3$, that is $g_{ij}$ defines a cocycle in $Z^1(M, \mathcal{O}^*_M)$. Hence we have constructed a homomorphism

$$\Phi : Z^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}})) \to Z^1(M, \mathcal{O}^*_M) : f \mapsto g_{ij}.$$

One can show that $\Phi$ is independent of the choices involved: the covering $U_i$ and the maps $p_i$. Furthermore, if $f$ belongs to $B^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}}))$, i.e. $f(\lambda\tilde{x}) = h(\lambda\tilde{x})h(\tilde{x})^{-1}$ for some $h \in H^0(\mathcal{O}^*_{\tilde{M}})$, then its image under $\Phi$ is given by:

$$g_{ij}(x) = h(\lambda_{ij}p_i^{-1}(x))h(p_i^{-1}(x))^{-1} = h(p_j^{-1}(x))h(p_i^{-1}(x))^{-1}.$$

This means that $\Phi$ induces a map

$$\tilde{\Phi} : H^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}})) \to H^1(M, \mathcal{O}^*_M)$$

between cohomology groups. Let $p^*(L) \in \ker\left(p^* : H^1(M, \mathcal{O}^*_M) \to H^1(\tilde{M}, \mathcal{O}^*_{\tilde{M}})\right)$ be a holomorphic trivial line bundle over $\tilde{M}$. The action $\tilde{x} \to \lambda \cdot \tilde{x}$ of $\pi_1(M)$ on $\tilde{M}$ induces a holomorphic automorphism $\rho_\lambda$ of $p^*(L)$ over this action

$$\begin{array}{ccc} p^*(L) & \xrightarrow{\rho_\lambda} & p^*(L) \\ \downarrow & & \downarrow \\ M & \longrightarrow & M \\ \lambda & \longmapsto & \lambda \cdot \tilde{x} \end{array}$$

Let $A : p^*(L) \to \tilde{M} \times \mathbb{C}$ be a trivialisation of $p^*(L)$ and $\Phi^A_\lambda = A \circ \rho_\lambda \circ A^{-1}$ the corresponding automorphism of the trivial bundle. The map $\Phi^A_\lambda(\tilde{x}, t)$ is necessarily of the form

$$(\lambda\tilde{x}, f^A(\lambda, \tilde{x})t), \tag{2.11}$$

for some $f^A : \pi_1(M) \times \tilde{M} \to \mathbb{C}^*$. Furthermore

$$\Phi^A_{\lambda\mu} = \Phi^A_\lambda \Phi^A_\mu,$$

i.e. $f^A(\lambda, \tilde{x}) \in Z^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}}))$. In other words, we have built a map, depending on the trivialisation $A$,

$$\Psi_A : \ker\left(p^* : H^1(M, \mathcal{O}^*_M) \to H^1(\tilde{M}, \mathcal{O}^*_{\tilde{M}})\right) \to Z^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}}))$$

which assigns $L$ to $f^A$. If $B : p^*(L) \to \tilde{M} \times \mathbb{C}$ is another trivialisation, then

$$\Phi_\lambda^B = B \circ \rho_\lambda \circ B^{-1} = B \circ A^{-1} \circ \Phi_\lambda^A \circ A \circ B^{-1}.$$

The map $B \circ A^{-1} : \tilde{M} \times \mathbb{C} \to \tilde{M} \times \mathbb{C}$ is necessarily of the form $B \circ A^{-1}(\tilde{x}, t) = (\tilde{x}, h(\tilde{x}t))$ with $h \in H^0(\mathcal{O}^*_{\tilde{M}})$. Hence

$$\Phi_\lambda^B(\tilde{x}, t) = (\lambda\tilde{x}, f^B(\lambda, \tilde{x})) = (\lambda\tilde{x}, h(\lambda\tilde{x}) f^A(\lambda, \tilde{x}) h(\tilde{x})^{-1} t),$$

which shows that $f^A(\lambda, \tilde{x})$ and $f^B(\lambda, \tilde{x})$ are cohomologous in $H^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}})$. Hence, the map $\Psi_A$ gives rise to a map

$$\Psi : \ker\left(p^* : H^1(M, \mathcal{O}^*_M) \to H^1(\tilde{M}, \mathcal{O}^*_{\tilde{M}})\right) \to H^1(\pi_1(M), H^0(\mathcal{O}^*_{\tilde{M}}),$$

which it can be seen to be the inverse of $\tilde{\Phi}$. □

By formula (2.11), one deduces that, if $L$ is a holomorphic line bundle over $M$ and $p^*L$ is holomorphically trivial over $\tilde{M}$, then the set of global holomorphic sections of $L$ can be identified with the set of holomorphic functions $s$ on $\tilde{M}$ satisfying the functional equation

$$s(\lambda\tilde{x}) = f^A(\lambda, \tilde{x}) s(x), \tag{2.12}$$

where $f^A(\lambda, \tilde{x})$ is the factor of automorphy corresponding to the trivialisation $A$.

We are interested in two cases which are treated in the next two subsections: the complex tori and the compact Riemann surfaces of genus $g$ greater or equal to two. In these cases the universal covering space is either $\mathbb{C}^n$ or the unit disk $\mathbb{D}$ and they have the important feature that any holomorphic line bundle over them is holomorphically trivial. In fact, let $\tilde{M}$ be either $\mathbb{C}^N$ or the unit disk $\mathbb{D}$. Consider the cohomology long exact sequence

$$\cdots \to H^1(\tilde{M}, \mathcal{O}_{\tilde{M}}) \to H^1(\tilde{M}, \mathcal{O}^*_{\tilde{M}}) \xrightarrow{c_1} H^2(\tilde{M}, \mathbb{Z}) \to \cdots.$$

Since $\tilde{M}$ is contractible, $H^2(\tilde{M}, \mathbb{Z}) = 0$ and $H^1(\tilde{M}, \mathcal{O}_{\tilde{M}}) = H^{0,1}_{\bar{\partial}}(\tilde{M}) = 0$. This implies that $H^1(\tilde{M}, \mathcal{O}^*_{\tilde{M}}) \cong \mathrm{Pic}(\tilde{M}) = 0$, and so the assertion.

### 2.1.1 Quantization of complex tori

We refer to [23, Chapter 1] for the material of this Section. Let $V$ be a complex vector space of complex dimension $N$ and $\Lambda$ a $2n$-lattice on $V$, i.e. a discrete subgroup of $V$ of rank $2n$. The quotient $V/\Lambda$ is a $N$-dimensional compact complex manifold called the *complex torus*. The canonical projection $p : V \to V/\Lambda$ is the universal covering map and the lattice $\Lambda$ can be identified with the fundamental group of $V/\Lambda$. Consider the set of maps $H : V \times V \to \mathbb{C}$ linear in the first factor and complex antilinear in the second factor, such that $ImH(\Lambda \times \Lambda) \subset \mathbb{Z}$. Take the group of semicharacters of $H$, i.e. the set of maps $\chi : \Lambda \to S^1$ such that

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)e^{\pi i ImH(\lambda,\mu)}, \ \forall \lambda, \mu \in \Lambda.$$

Define

$$A(\lambda, v) = \chi(\lambda)e^{\pi H(v,\lambda)+\frac{\pi}{2}H(\lambda,\lambda)}.$$

It follows immediately that $A(\lambda + \mu, v) = A(\lambda, v + \mu)A(\mu, v)$, thus $A(\lambda, v)$ is a factor of automorphy, i.e. $A(\lambda, v)$ belongs to $Z^1(\Lambda, H^0(\mathcal{O}_V^*))$. From Theorem 2.1.2 the class $[A(\lambda, v)] \in H^1(\Lambda, H^0(\mathcal{O}_V^*))$ defines a holomorphic line bundle $L$ on $V/\Lambda$ which is denoted by $L(H, \chi)$. Furthermore, it follows by (2.12) that the global holomorphic sections of $L(H, \chi)$ can be seen as holomorphic functions $\theta$ on $V$ satisfying

$$\theta(v + \lambda) = A(\lambda, v)\theta(v). \tag{2.13}$$

The functions $\theta$ satisfying (2.13) are called the *canonical theta functions*.

Let $H$ be a hermitian form on $V$, i.e. a map $H : V \times V \to \mathbb{C}$ complex linear in the first factor and complex antilinear in the second factor such that $H(v, v) \geq 0$ and $H(v, v) = 0$ if and only if $v = 0$. Define the differential form on $V$

$$\Omega_0^N := \frac{i}{2}\partial\bar{\partial}H.$$

It is easily seen that $\Omega_0^N$ is a Kähler form on $V$, invariant by translations. In particular, $\Omega_0^N$ is invariant by the action of the lattice $\Lambda$ and so it defines a Kähler form on $V/\Lambda$ , denoted by the same symbol $\Omega_0^N$.

**Proposition 2.1.3** *The Kähler manifold* $(V/\Lambda, \Omega_0^N)$ *admits a quantization if and only if*

$$ImH(\Lambda \times \Lambda) \subset \mathbb{Z}.$$

**Proof:** If $ImH(\Lambda\times\Lambda) \subset \mathbb{Z}$, then one can take the holomorphic line bundle $L = L(H, \chi)$ where $\chi$ is *any* semicharacter associated to $H$. In order to introduce a hermitian structure $h$ on $L$, one defines a function $F : V \to \mathbb{R}^+$, given by:

$$F(v) := e^{-\pi H(v,v)}.$$

A simple calculation shows that

$$F(v + \lambda) = e^{-2\pi\Re H(v,\lambda)-\pi H(\lambda,\lambda)}F(v),\ \forall v \in V, \forall\lambda \in \Lambda.$$

Let now $\theta$ be a holomorphic section of $L$. Define

$$h(\theta(v), \theta(v)) = F(v)|\theta(v)|^2.$$

It follows by (2.13) that the function $h$ is invariant under the action of the lattice, i.e.

$$h(\theta(v + \lambda), \theta(v + \lambda)) = h(\theta(v), \theta(v))\ \forall\lambda \in \Lambda.$$

Since $H$ is positive definite, $h$ defines a hermitian structure on $L$. Furthermore, by (1.12),

$$\mathrm{curv}(L(H, \chi), h) = -\partial\bar{\partial}\log h = -\partial\bar{\partial}\log F = \pi\partial\bar{\partial}H = -2\pi i\Omega_0^N.$$

This shows that $(L, h)$ is a quantization of $(V/\Lambda, \Omega_0^N)$. For the converse we refer to [23]. □

**Remark 2.1.4** *One can show that the condition* $ImH(\Lambda \times \Lambda) \subset \mathbb{Z}$ *is equivalent to the integrality of the Kähler form* $\Omega_0^N$*. Hence Proposition 2.1.3 could also be deduced from Theorem 1.3.3.*

*In the language of algebraic geometers, a complex torus with a Kähler form which admits a quantization is called an abelian variety. One can show that every* 1*-dimensional complex torus is an abelian variety. This is false even in dimension two (see [31, pp. 214-16]).*

The group of translations which is a subgroup of

$$\mathrm{Aut}(V/\Lambda) \cap \mathrm{Isom}(V/\Lambda, \Omega_0^N)$$

acts transitively on the torus and therefore, $(V/\Lambda, \Omega_0^N)$ is a homogeneous Kähler manifold.

The group $D_{[(L,h)]}(V/\Lambda)$ is finite. In fact, from Section 1.5, this group admits a projective representation in $\mathbb{P}(H^0(L))$, and it can be shown that the group of automorphism of an abelian variety which admits a projective representation is finite (see [12, p. 326] for a proof).

### 2.1.2 Quantization of Riemann surfaces

Let $\Sigma_g$ be a compact Riemann surface of genus $g \geq 2$. One can realize $\Sigma_g$ as the quotient $\mathbb{D}/\Gamma$ of the unit disk $\mathbb{D} \subset \mathbb{C}$ under the fractional linear transformations of a Fuchsian subgroup $\Gamma$ of

$$\mathrm{SU}(1,1) = \{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid \ |a|^2 - |b|^2 = 1 \}.$$

Here the action of $\gamma = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \Gamma$ is given by $z \mapsto \gamma(z) = \frac{az+b}{\bar{b}z+\bar{a}}$. It is immediate to check that the Kähler form

$$\omega_{hyp} = \frac{i}{\pi} \frac{dz \wedge d\bar{z}}{(1 - z\bar{z})^2}$$

is invariant under fractional linear transformations, so it defines a Kähler form on $\Sigma_g$, denoted by the same symbol $\omega_{hyp}$. Let $p : \mathbb{D} \to \mathbb{D}/\Gamma$ be the natural projection map and $s$ a holomorphic form of type $(1,0)$ on $\Sigma_g$. This means that if $(U_\alpha, z_\alpha)$ is a complex atlas for $\Sigma_g$ and $s(z) = s_\alpha dz_\alpha$ in $U_\alpha$, then $s_\alpha(z)$ are holomorphic functions on $U_\alpha$. For all $z \in U_\alpha \cap U_\beta$ one has $s_\alpha(z) dz_\alpha = s_\beta(z) dz_\beta$. Define $k_{\alpha\beta} := \frac{dz_\beta}{dz_\alpha}$; the chain rule for differentiation implies that $k_{\alpha\beta}(z) k_{\beta\gamma}(z) = k_{\alpha\gamma}(z)$ for every $z \in U_\alpha \cap U_\beta \cap U_\gamma$, so there exists a holomorphic line bundle K, called the *canonical bundle*, which has these functions as coordinates transformations. The line bundle $p^*(L)$ is holomorphically trivial and its global holomorphic sections are the form of type $(1,0)$ on $\mathbb{D}$. If $z$ denotes the global holomorphic coordinate on $\mathbb{D}$ then such a forms are given by $s(z)dz$ where

$s(z)$ is a holomorphic function on $\mathbb{D}$. Hence, the global holomorphic sections of $K$ can be seen as the 1-forms $s$ of type $(1,0)$ on $\mathbb{D}$ invariant by the action of $\Gamma$, i.e.

$$s(\gamma(z))d(\gamma(z)) = s(\gamma(z))\gamma'(z)dz = s(z)dz, \forall \gamma \in \Gamma, \tag{2.14}$$

where $\gamma'(z)$ denotes the derivative of $\gamma(z)$ with respect to $z$ (if $\gamma(z) = \frac{az+b}{bz+\bar{a}}$ then $\gamma'(z) = (\bar{b}z+\bar{a})^{-2}$). Thus, by (2.14), the factor of automorphy is given by $(\gamma(z)')^{-1}$. Define

$$h(s(z), s(z)) := (1-|z|^2)^2|s(z)|^2.$$

One can easily check that

$$(1-|\gamma(z)|^2) = |\gamma'(z)|(1-|z|^2), \tag{2.15}$$

and so, by (2.14),

$$h(s(\gamma(z)), s(\gamma(z))) = h(s(z), s(z)),\ \forall s \in H^0(K), \forall \gamma \in \Gamma.$$

Therefore $h$ defines a hermitian structure on $K$. Moreover,

$$\mathrm{curv}(K, h) = -2\pi i \omega_{hyp}$$

(see (2.2)), which shows that $(K, h)$ is a geometric quantization for the Riemann surface $(\Sigma_g, \omega_{hyp})$.

One can show that $\mathrm{Aut}(\Sigma_g)$ is a finite group (see [19, p. 88]) and so, a fortiori, the group

$$D_{[(L,h)]}(\Sigma_g) \subset \mathrm{Aut}(\Sigma_g) \cap \mathrm{Isom}(\Sigma_g, \omega_{hyp})$$

is finite.

# Chapter 3

# Holomorphic isometric immersions of Kähler manifolds in complex space forms

All the material in Sections 3.1, 3.2, 3.3 and 3.4 is taken from [9] to which we refer without further comments.

## 3.1 The Calabi's diastasis function

Let $(M, \omega)$ be a Kähler manifold. On a contractible open set $U \subset M$ the Kähler form $\omega$ can be written as $\omega = d\beta$ where $\beta$ is a real 1-form. One may write $\beta = \alpha + \overline{\alpha}$, for some form $\alpha$ of type $(1,0)$. Since $\omega$ is of type $(1,1)$,

$$\omega = d\beta = (\partial + \bar{\partial})(\alpha + \bar{\alpha}) = \partial\alpha + \bar{\partial}\alpha + \partial\bar{\alpha} + \bar{\partial}\bar{\alpha}$$

implies $\partial\alpha = 0$. Thus, from the Dolbeault Lemma, there exists a function $g$, defined on a possibly smaller open set $V \subset U$, such that

$$\omega = \bar{\partial}\partial g + \partial\bar{\partial}\bar{g} = \partial\bar{\partial}(\bar{g} - g) = \frac{i}{2\pi}\partial\bar{\partial}\Phi_\omega, \tag{3.1}$$

where $\Phi_\omega = -2\pi i(g - \bar{g})$ is a real valued function on $V$.

**Definition 3.1.1** *A function $\Phi_\omega$ satisfying (3.1) is called a* Kähler potential *of $\omega$.*

A Kähler potential $\Phi_\omega$ is not unique. If $\partial\bar{\partial} f = d(\partial f) = 0$ then locally there exists a holomorphic function $h$ such that $\partial f = dh$ (see Lemma 1.4.9). The reality of $f$ implies that $\bar{\partial} f = d\bar{h}$, and thus $df = d(h + \bar{h})$. Therefore $f$ has to be the real part of a holomorphic function. This means that *a Kähler potential is defined up to the sum with the real part of a holomorphic function.*

Throughout this chapter our interest is devoted to the study of holomorphic isometric immersions, denoted here by *h.i.i.*, of a Kähler manifold in *finite or infinite dimensional complex space forms*, i.e. spaces of constant holomorphic sectional curvature (see [20, p. 165]). There are three types of complex space forms : *flat, hyperbolic or elliptic* according as the holomorphic sectional curvature is *zero, negative, or positive.* Every $N$-dimensional complex space form (the case $N = \infty$ is not excluded) is, after multiplying by a suitable constant, locally holomorphically isometric to one of the following:

- the complex euclidean space $\mathbb{C}^N$ endowed with the Kähler form $\Omega_0^N$ of zero holomorphic sectional curvature (see 2.0.1);
- the unit ball $\mathcal{D}_N$ endowed with the Bergman form $\Omega_{hyp}^N$ of negative holomorphic sectional curvature (see 2.3 and 2.4);
- the $N$-dimensional complex projective space $\mathbb{P}^N(\mathbb{C})$ endowed with the Fubini-Study form $\Omega_{FS}^N$ (see 2.0.4) of positive holomorphic sectional curvature.

We refer to [9, Theorem 1 and 7] for the proof of the following:

**Theorem 3.1.2** *If a Kähler manifold $(M, \omega)$ can be holomorphically and isometrically immersed in a complex space form, then $\omega$ is real analytic.*

If $\omega$ is a real analytic Kähler form then, in a complex coordinate system $(z_1, \ldots, z_n)$ around a point $p_0$, a Kähler potential can be expanded in power series

$$\Phi_\omega(p) = \sum_{j,k=0}^{+\infty} \Phi_{jk} z(p)^{m_j} \overline{z(p)}^{m_k}.$$

Here we are using the following convention: we arrange every $n$-tuple of non-negative integers as the sequence $m_j = (m_{1,j}, m_{2,j}, \ldots, m_{n,j})_{j=0,1,\ldots}$ such that $m_0 = (0, \ldots, 0)$, $|m_j| \leq |m_{j+1}|$, with $|m_j| = \sum_{\alpha=1}^{n} m_{\alpha,j}$ and $z^{m_j} = \prod_{\alpha=1}^{n} (z_\alpha)^{m_{\alpha,j}}$. A Kähler potential can be complex analytically continued in an open neighborhood of the diagonal $W \subset V \times V$ as

$$\Phi_\omega(p, \bar{q}) = \sum_{j,k=0}^{+\infty} \Phi_{jk} z(p)^{m_j} \overline{z(q)}^{m_k}.$$

It is holomorphic in $p$ and antiholomorphic in $q$.

The *diastasis* function is defined by

$$D_\omega(p, q) = \Phi_\omega(p, \bar{p}) + \Phi_\omega(q, \bar{q}) - \Phi_\omega(p, \bar{q}) - \Phi_\omega(q, \bar{p}), \ \forall p, q \in W. \tag{3.2}$$

Since the Kähler potential is independent of the coordinate system chosen so is $D_\omega$. Furthermore, (3.2) shows that the ambiguity on the definition of a Kähler potential, defined up to the sum of the real part of a holomorphic function, drops out and so $D_\omega$ depends only on the Kähler form $\omega$.

The diastasis is real valued since $\overline{\Phi_\omega(p, \bar{q})} = \Phi_\omega(q, \bar{p})$, it is symmetric in $p$ and $q$ and $D(p, p) = 0$.

For $q \in M$, let $D_{\omega,q}$ be the function defined by

$$D_{\omega,q}(p) = D_\omega(p, q), \tag{3.3}$$

and let

$$\mathcal{M}_q^\omega = \{p \in M \mid D_{\omega,q} \text{ is defined}\} \tag{3.4}$$

its maximal domain of definition. The following proposition, for a proof of which we refer to [9], is the key to studying the h.i.i. of a Kähler manifold $(M, \omega)$ into a complex space form.

**Proposition 3.1.3** *Let $\phi : M \to N$ be a holomorphic immersion of a complex manifold $M$ into a complex manifold $N$. Suppose that $\omega_M$ and $\omega_N$ are analytic Kähler forms on $M$ and $N$, respectively. Then $\phi^*(\omega_N) = \omega_M$ if and only if $D_{\omega_M}(p, q) = D_{\omega_N}(\phi(p), \phi(q))$.*

## 3.2 H.i.i. in $(\mathbb{C}^N, \Omega_0^N)$

Throughout this Section, Sections 3.3 and 3.4, $N$ will be either a natural number or $\infty$. We start by analysing the diastasis of $\Omega_0^N$. From Example 2.0.1 a Kähler potential, globally defined in $\mathbb{C}^N$, is given by:

$$\Phi_{\Omega_0^N}(z, \bar{z}) = \pi \sum_{j=1}^{N} |z_j|^2 = \pi \|z\|^2.$$

Its complex analytic continuation in $\mathbb{C}^N \times \mathbb{C}^N$ is

$$\Phi_{\Omega_0^N}(z, \bar{w}) = \pi \sum_{j=1}^{N} z_j \bar{w}_j = \pi z \cdot \bar{w}.$$

Thus, by (3.2), the diastasis function can be calculated as

$$D_{\Omega_0^N}(z, w) = \pi \|z\|^2 + \pi \|w\|^2 - z \cdot \bar{w} - w \cdot \bar{z} = \pi \|z - w\|^2. \tag{3.5}$$

It is a globally defined real analytic function on $\mathbb{C}^N \times \mathbb{C}^N$, given by the square of the distance between the points $z$ and $w$ times $\pi$.

Suppose now that a Kähler manifold $(M, \omega)$ admits a h.i.i. in $(\mathbb{C}^N, \Omega_0^N)$. Then, from Theorem 3.1.2, $\omega$ is real analytic and so one can calculate its diastasis $D_\omega$. By (3.5) and 3.1.3 one can easily deduce:

**Corollary 3.2.1** *If a Kähler manifold $(M, \omega)$, admits a h.i.i. in $(\mathbb{C}^N, \Omega_0^N)$, then $D_\omega$ is a non-negative real analytic function on $M \times M$ and so $D_{\omega,q}$ is globally defined for all $q \in M$.*

We can carry out our analysis further by studying the power series expansion of the diastasis in a neighborhood of a point as follows. Let $p_0 \in M$ and suppose that there exists a neighborhood of it, say $U_{p_0}$, and a holomorphic map $\phi : U_{p_0} \to (\mathbb{C}^N, \Omega_0^N)$ which is an isometry with respect to the induced metric $\omega|_{U_{p_0}}$ on $U_{p_0}$, i.e. $\phi^*\Omega_0^N = \omega|_{U_{p_0}}$. Let $x = (x_1, \dots, x_n)$ be local coordinates around $p_0$ and let $(\phi_1, \dots, \phi_N)$ be the components of $\phi$. After a translation in $\mathbb{C}^N$ we can suppose, without lost of generality, that $p_0$ is mapped onto the origin of $\mathbb{C}^N$. It follows, from Proposition 3.1.3, and (3.5), that $D_\omega(p, p_0) = \sum_{\alpha=1}^{N} |\phi_\alpha(x(p))|^2$. The functions $\phi_\alpha(z(p))$ are holomorphic, so for $|x_\alpha| < \rho_\alpha, \alpha = 1, \ldots, N$, one can write $\phi_\alpha(x(p)) = \sum_{j=0}^{+\infty} \phi_j^\alpha x(p)^{m_j}$. Thus,

$$D_\omega(p, p_0) = \sum_{\alpha=1}^{N} \sum_{j,k=0}^{+\infty} \phi_j^\alpha \bar{\phi}_k^\alpha x(p)^{m_j} \overline{x(p)}^{m_k} = \sum_{j,k=0}^{+\infty} D_{jk} x(p)^{m_j} \overline{x(p)}^{m_k}, \tag{3.6}$$

where

$$D_{jk} := \sum_{\alpha=1}^{N} \phi_j^\alpha \bar{\phi}_k^\alpha \tag{3.7}$$

is a $\infty \times \infty$ matrix given by the product of the $\infty \times n$ matrix $\phi_j^\alpha$ and its transpose conjugate. Therefore it is semi-positive definite and its rank is at most $N$.

**Definition 3.2.2** *A real analytic Kähler form $\omega$ is said to be* resolvable *of rank $N$ at $p_0 \in M$ if the matrix $D_{jk}$ given by (3.6) is semi-positive definite and of rank $N$. If $N = \infty$ we say that $\omega$ is resolvable of infinite rank.*

A theorem of Calabi asserts that the resolvability condition is also sufficient for the existence of a h.i.i. of a neighborhood of $p_0$ into $(\mathbb{C}^N, \Omega_0^N)$. Furthermore, the concept of resolvability, even if it is defined locally, turns out to be a global concept. Hence, one can speak of a real analytic Kähler form being *resolvable* without specifying the point into consideration. The following theorem tells us that, in the simply connected case, a local h.i.i. can always be extended to a global one.

**Theorem 3.2.3** *A simply connected complex manifold $M$ endowed with a real analytic Kähler form $\omega$ admits a h.i.i. in $(\mathbb{C}^N, \Omega_0^N)$ if and only if $\omega$ is resolvable of rank at most $N$. The immersion is full if the rank is exactly $N$.*

The main ingredient needed to prove the previous theorem is given by the Local Rigidity Theorem which will be of constant use throughout this thesis:

**Theorem 3.2.4** *Let $U \subset M$ be a connected open set of $M$, $\phi : U \to \mathbb{C}^r$ and $\psi : U \to \mathbb{C}^s$ be two full holomorphic maps such that $\phi^*(\Omega_0^r) = \psi^*(\Omega_0^s)$ is a Kähler form on $U$. Then $r = s = N$ and there exists $T$ an isometry of $\mathbb{C}^N$, i.e. a unitary transformation followed by a translation, such that $T \circ \phi = \psi$.*

Here a map $\phi : U \to \mathbb{C}^r$ is said to be *full* if $\phi(U)$ is not contained in $\mathbb{C}^t$ with $t < r$.

## 3.3 H.i.i. in $(\mathcal{D}_N, \Omega^N_{hyp})$

Consider the open ball $\mathcal{D}_N$, endowed with the Kähler form

$$\Omega^N_{hyp} = \frac{i}{2\pi}\partial\bar{\partial}\log\frac{1}{1-\|z\|^2},\ \ \|z\|^2 = \sum_{j=1}^N |z_j|^2,$$

(see (2.3) and (2.4)). A Kähler potential, given by $\Phi_{\Omega^N_{hyp}}(z,\bar{z}) = \frac{1}{1-\|z\|^2}$, can be complex analytically continued to all $\mathcal{D}_N \times \mathcal{D}_N$ as

$$\Phi_{\Omega^N_{hyp}}(z,\bar{w}) = \log\frac{1}{1-z\cdot\bar{w}},\ \ z\cdot\bar{w} = \sum_{j=1}^N z_j\cdot\bar{w}_j.$$

Thus, by (3.2), the diastasis is given by:

$$D_{\Omega^N_{hyp}}(z,w) = -\log\frac{(1-\|z\|^2)(1-\|w\|^2)}{(1-z\cdot\bar{w})(1-\bar{z}\cdot w)}. \tag{3.8}$$

As for the flat case $D_{\Omega^N_{hyp}}$ is a globally defined real analytic non-negative function on $\mathcal{D}_N \times \mathcal{D}_N$. Hence, once again from 3.1.3, one obtains:

**Corollary 3.3.1** *If a Kähler manifold $(M,\omega)$ admits a h.i.i. in $(\mathcal{D}^N, \Omega^N_{hyp})$, then $D_\omega$ is a non-negative real analytic function on $M\times M$ and so $D_{\omega,q}$ is globally defined for all $q\in M$.*

Let $M$ be a complex manifold endowed with a real analytic Kähler form $\omega$. Suppose that a neighborhood of a point $p_0$ in $M$, say $U_{p_0}$, admits a h.i.i. $\phi$ in $(\mathcal{D}_N, \Omega^N_{hyp})$. Let $(x_1,\dots,x_n)$ be coordinates around the point $p_0$ and let $(\phi_1,\dots,\phi_N)$ be the components of $\phi$. From Proposition 3.1.3 it follows that:

$$D_\omega(p,p_0) = -\log(1-\sum_{\alpha=1}^N|\phi_\alpha(x(p))|^2).$$

Let

$$D^-(p,p_0) := 1-e^{-D_\omega(p,p_0)} = \sum_{\alpha=1}^N|\phi_\alpha(x(p))|^2) \tag{3.9}$$

and

$$\omega^- = \frac{i}{2\pi}\partial\bar{\partial}D^-(p,p_0).$$

By (3.9) and the results of Section 3.2, one deduces that $U_{p_0}$ admits a h.i.i. in $(\mathcal{D}_N, \Omega^N_{hyp})$ if and only if $\omega^-$ is resolvable of rank at most $N$. This means that if

$$D^-(p, p_0) = \sum_{j,k=0}^{+\infty} D^-_{jk} x(p)^{m_j} \overline{x(p)}^{m_k}$$

denotes the expansion in power series of $D^-(p, p_0)$ at $p_0$, then the matrix $D^-_{jk}$ is semi-positive definite and of rank at most $N$. The analogous of Theorem 3.2.3 holds:

**Theorem 3.3.2** *A simply connected complex manifold $M$ endowed with a real analytic Kähler form $\omega$ admits a (full) h.i.i. in $(\mathcal{D}_N, \Omega^N_{hyp})$ if and only if $\omega^-$ is resolvable of rank at most (precisely) $N$.*

## 3.4 H.i.i. in $(\mathbb{P}^N(\mathbb{C}), \Omega^N_{FS})$

The Fubini-Study form in $\mathbb{P}^N(\mathbb{C})$ can be written as:

$$\Omega^N_{FS} = \frac{i}{2\pi} \partial\bar{\partial} \log \|z\|^2$$

where $\|z\|^2 = \sum_{j=0}^N |z_j|^2$ (see (2.5) and (2.6)). In the chart $U_k = \{z_k \neq 0\}$, equipped with coordinates $u_j = \frac{z_j}{z_k}$, a Kähler potential is given by:

$$\Phi_{\Omega^N_{FS}}(u, \bar{u}) = \log(1 + \sum_{j\neq k} |u_j|^2) = \log(1 + \|u\|^2).$$

In this chart the diastasis has the following expression:

$$D_{\Omega^N_{FS}}(u, v) = \log(1 + \|u\|^2) + \log(1 + \|v\|^2) - \log(1 + \langle u, v\rangle) - \log(1 + \langle v, u\rangle).$$

This can be written in homogeneous coordinates as

$$D_{\Omega^N_{FS}}([z], [w]) = \log \frac{\|z\|^2\|w\|^2}{|\langle z, w\rangle|^2}. \tag{3.10}$$

In particular $D_{\Omega^N_{FS}} > 0$ unless $[z] = [w]$. For $[w] \in \mathbb{P}^N(\mathbb{C})$ the function $D_{\Omega^N_{FS},[w]}$ is everywhere defined apart from the set

$$\mathcal{P}^{\Omega^N_{FS}}_{[w]} = \{[z] \in \mathbb{P}^N(\mathbb{C})|\ \langle z, w\rangle = 0\}, \tag{3.11}$$

where it takes infinity value. From now on we will say that a Kähler form on a complex manifold $M$ is *projectively induced* if there exists a holomorphic and isometric immersion $\phi$ of $(M, \omega)$ into some complex projective space $(\mathbb{P}^N(\mathbb{C}), \Omega_{FS}^N)$.

Suppose that $\omega$ is projectively induced via a map $\phi : M \to \mathbb{P}^N(\mathbb{C})$. Consider the set

$$\mathcal{P}_q^\omega = \phi^{-1}(\phi(M) \cap \mathcal{P}_{\phi(q)}^{\Omega_{FS}^N}), \tag{3.12}$$

for $q \in M$. This set is called the *polar variety of $M$ w.r.t. the point $q$.*

**Remark 3.4.1** *$\mathcal{P}_q^\omega$ is a $(n-1)$-dimensional complex variety (may be singular) (see [9, pp. 3-4]). In the case of hermitian symmetric manifolds of compact type endowed with the standard homogeneous form this set is shown to be the cut locus of the point $q$ (see [2]).*

From 3.1.3 and 3.10 one easily obtains:

**Corollary 3.4.2** *Let $(M, \omega)$ be a complex manifold. Suppose that there exists a h.i.i. in $(P^N(\mathbb{C}), \Omega_{FS}^N)$. Then $D_\omega$ is a non-negative real analytic function and $\mathcal{M}_q^\omega = M \backslash \mathcal{P}_q^\omega$ for every $q \in M$, and so the function $D_{\omega,q}$ is everywhere defined apart from the set $\mathcal{P}_q^\omega$.*

To go deeper in our analysis we need, as in Section 3.2, to focus our attention in the power series expansion of the diastasis . Suppose that a neighborhood of a point $p_0$, say $U_{p_0}$, admits a h.i.i. in $(\mathbb{P}^N(\mathbb{C}), \Omega_{FS}^N)$. If $x = (x_1, \dots, x_n)$ denote complex local coordinates around $p_0$, $(\phi_0, \dots, \phi_N)$ the components of the immersion, then, from Proposition 3.1.3, it follows that:

$$D_\omega(p, p_0) = \log(1 + \sum_{\alpha=0}^{N} |\phi_\alpha(x(p))|^2).$$

Consider the new diastasis function

$$D^+(p, p_0) = e^{D_\omega(p,p_0)} - 1 = \sum_{\alpha=0}^{N} |\phi_\alpha(x(p))|^2 \tag{3.13}$$

associated to the Kähler form $\omega^+ = \frac{i}{2\pi}\partial\bar{\partial}D^+(p, p_0)$. By (3.13) and the results of Section 3.2, one can deduce that a necessary and sufficient condition for $\omega|_{U_{p_0}}$ to be projectively induced in $\mathbb{P}^N(\mathbb{C})$, is that $\omega^+$ is resolvable of rank at most $N$ at $p_0$. This means that if

$$D^+(p,p_0) = \sum_{j,k=0}^{+\infty} D^+_{jk} x(p)^{m_j} \overline{x(p)}^{m_k} \tag{3.14}$$

denotes the power series expansion of $D^+(p,p_0)$ at $p_0$, then the matrix $D^+_{jk}$ is semi-positive definite and of rank at most $N$.

Moreover, Calabi shows the global character of resolvability of $\omega^+$ and the projective version of the Local Rigidity Theorem which reads:

**Theorem 3.4.3** *Let $U \subset M$ be an open set of a complex manifold $M$, and $\phi : U \to \mathbb{P}^r(\mathbb{C})$ and $\psi : U \to \mathbb{P}^s(\mathbb{C})$ full holomorphic maps such that $\phi^*(\Omega^r_{FS}) = \psi^*(\Omega^s_{FS})$ is a Kähler form on $U$. Then $r = s = N$ and there exists $U \in PU(N+1)$ such that $U \circ \phi = \psi$.*

Here a map $\phi : U \to \mathbb{P}^r(\mathbb{C})$ is said to be *full* if $\phi(U)$ is not contained in $\mathbb{P}^t(\mathbb{C})$ with $t < r$. The analogous of 3.2.3 holds:

**Theorem 3.4.4** *A simply connected Kähler manifold $M$ endowed with a real analytic Kähler form $\omega$ admits a h.i.i. in $(\mathbb{P}^N(\mathbb{C}), \Omega^N_{FS})$ if and only if $\omega^+$ is resolvable of rank at most $N$. The immersion is full if the rank is precisely $N$.*

## 3.5 Examples and remarks

**Example 3.5.1** *Let $N$ be a natural number and $\Omega^N_0$ the flat Kähler form on $\mathbb{C}^N$ (see 2.0.1). Let $w_0$ be the origin of $\mathbb{C}^N$. By (3.13)*

$$\begin{aligned} D^+(z,w_0) = e^{D^N_{\Omega_0}(z,w_0)} - 1 &= e^{\pi \sum_{\alpha=1}^N |z_\alpha|^2} - 1 \\ &= \prod_{\alpha=1}^N e^{\pi |z_\alpha|^2} - 1 \\ &= \sum_{j_1,\dots,j_N=0}^{+\infty} \frac{\pi^{j_1+\cdots+j_N}}{j_1!\cdots j_N!} |z_1|^{2j_1} \cdots |z_N|^{2j_N} - 1 \\ &= \sum_{j,k=0}^{+\infty} \frac{\delta jk \pi^{|m_j|}}{m_j!} z^{m_j} \bar{z}^{m_k} - 1, \end{aligned}$$

*where $m_j! = m_{1,j}! \cdots m_{N,j}!$. Then*

$$D^+_{jk} = \frac{\delta_{jk}\pi^{|m_j|}}{m_j!}, \;\; j,k > 1.$$

*This implies that*

$$\omega^+ = \frac{i}{2\pi}\partial\bar{\partial}D^+(z, w_0)$$

*is resolvable of infinite rank. From Theorem 3.4.4,* $(\mathbb{C}^N, \Omega_0^N)$ *admits a h.i.i. in the infinite dimensional projective space* $(\mathbb{P}^\infty(\mathbb{C}), \Omega_{FS}^\infty)$*. This is given by:*

$$(z_1, \ldots, z_N) \to (\ldots, \sqrt{\frac{\pi^{|m_j|}}{m_j!}}z^{m_j}, \ldots) \tag{3.15}$$

*In fact, by (2.6),*

$$\phi^*\Omega_{FS}^\infty = \frac{i}{2\pi}\partial\bar{\partial}\log\sum_{j=0}^{+\infty}\frac{\pi^{|m_j|}}{m_j!}|z|^{2m_j} = \frac{i}{2\pi}\partial\bar{\partial}\log e^{\pi\|z\|^2} = \Omega_0^N.$$

**Example 3.5.2** *Let* $N$ *be a natural number and* $\Omega_{hyp}^N$ *the hyperbolic form on* $\mathcal{D}_N$*. Let* $w_0$ *denote the origin of* $\mathbb{C}^N$*. It follows, by (2.3), that:*

$$\begin{aligned} D_{\Omega_{hyp}^N}(z, w_0) &= -\log(1 - \|z\|^2) \\ &= \textstyle\sum_{j_1,\ldots,j_N=1}^{+\infty}\frac{(j_1+\cdots+j_N-1)!}{j_1!\ldots j_N!}|z_1|^{2j_1}\cdots|z_N|^{2j_N} \\ &= \textstyle\sum_{j,k=1}^{+\infty}\frac{\delta_{jk}(|m_j|-1)!}{m_j!}z^{m_j}\bar{z}^{m_k}. \end{aligned}$$

*Hence* $\Omega_{hyp}^N$ *is resolvable of infinite rank and, from 3.2.3,* $(\mathcal{D}_N, \Omega_{hyp}^N)$ *admits a h.i.i. in* $(\mathbb{C}^\infty, \Omega_0^\infty)$*. This is given explicitly by:*

$$(z_1, \ldots, z_N) \to (\ldots, \sqrt{\frac{(|m_j| - 1)!}{m_j!}}z^{m_j}, \ldots), \tag{3.16}$$

*as one can easily verify. Furthermore,*

$$\begin{aligned} D^+(z, w_0) = e^{D_{\Omega_{hyp}^N}(z, w_0)} - 1 &= \tfrac{1}{1-\|z\|^2} - 1 \\ &= \textstyle\sum_{j_1,\ldots,j_N=0}^{+\infty}\frac{(j_1+\cdots+j_N)!}{j_1!\ldots j_N!}z_1^{j_1}\cdots z_N^{j_N} - 1. \end{aligned}$$

*We deduce that* $\omega^+$ *has infinite rank and so, from 3.3.2,* $(\mathcal{D}_N, \Omega_{hyp}^N)$ *admits a h.i.i. in* $(\mathbb{P}^\infty(\mathbb{C}), \Omega_{FS}^\infty)$*, This is given by:*

$$(z_1, \ldots, z_N) \to (\ldots, \sqrt{\frac{|m_j|!}{m_j!}}z^{m_j}, \ldots), j = 0, 1, \ldots. \tag{3.17}$$

**Example 3.5.3** *Let* $N$ *be a natural number and* $\Omega_{FS}^N$ *the Fubini-Study form on* $\mathbb{P}^N(\mathbb{C})$*. From 3.2.3 and 3.3.2, the existence of the polar variety implies that* $(\mathbb{P}^N(\mathbb{C}), \Omega_{FS}^N)$

*cannot be h.i.i. in any flat or hyperbolic space. Here we want to describe a h.i.i. of* $(\mathbb{P}^N(\mathbb{C}), k\Omega^N_{FS})$ *in* $(\mathbb{P}^{N(k)}(\mathbb{C}), \Omega^{N(k)}_{FS})$*, where* $N(k) = \binom{N+k}{N}$ *is the complex dimension of the space of homogeneous polynomials of degree* $k$*. Let* $\phi : \mathbb{P}^N(\mathbb{C}) \to \mathbb{P}^{N(k)}(\mathbb{C})$ *be the map defined as follows:*

$$[(z_0, \dots, z_n)] \to [(\dots, \sqrt{\frac{k!}{j_0! \dots j_N!}} z_0^{j_0} \cdots z_N^{j_N}, \dots)], \tag{3.18}$$

*where* $j_0 + \cdots + j_N = k$*. The map (3.18) is holomorphic and*

$$\begin{aligned} \phi^* \Omega^{N(k)}_{FS} &= \tfrac{i}{2\pi} \partial\bar{\partial} \log \textstyle\sum_{j_0 + \cdots + j_N = k} \frac{k!}{j_0! \cdots j_N!} |z_0|^{2j_0} \cdots |z_N|^{2j_N} \\ &= \tfrac{i}{2\pi} \partial\bar{\partial} \log(|z_0|^2 + \cdots + |z_N|^2)^k = k\Omega^N_{FS}. \end{aligned}$$

**Remark 3.5.4** *Examples 3.5.1, 3.5.2 and 3.5.3 show that a* finite *dimensional complex space form cannot be holomorphically and isometrically immersed in a complex space form of different type (see [9, p. 22]).*

This fact has been generalized in [29] as follows:

**Theorem 3.5.5** *Any Kähler manifold which can be h.i.i. in a finite dimensional flat space cannot be h.i.i. in a finite dimensional hyperbolic or elliptic space. Any Kähler manifold h.i.i. in a finite complex hyperbolic space cannot be h.i.i. in a finite dimensional flat or elliptic space.*

Notice that $(\mathbb{C}^N, \Omega^N_0)$ and $(\mathcal{D}_N, \Omega^N_{hyp})$ are examples of non compact homogeneous Kähler manifolds, and so the fact that $\Omega^N_0$ and $\Omega^N_{hyp}$ are not projectively induced (in a finite dimensional complex projective space) could also be deduced from a Theorem in [27], which asserts that *a homogeneous Kähler manifold* $(M, \omega)$ *which admits a h.i.i. in a finite dimensional complex projective space has to be compact.*

Every homogeneous Kähler manifold, not necessarily compact, admits a h.i.i. in some complex projective space via the coherent states map, as we will see in the next chapter. For the case of compact homogeneous Kähler manifolds, we refer to [27], where one can find a description of their immersions in complex projective spaces in terms of Dynkin diagrams.

**Proposition 3.5.6** *Let $(M, \omega)$ be either the $N$-dimensional complex torus $(V/\Lambda, \Omega_0^N)$ or the Riemann surface $(\Sigma_{g\geq 2}, \omega_{hyp})$ (see 2.1.1 and 2.1.2). Then, $(M, \omega)$ cannot holomorphically and isometrically immersed in any complex projective space of any dimension.*

**Proof:** Suppose that this is not the case. Then there exists a natural number $N$ and a h.i.i. $\phi$ of $(M, \omega)$ in $(\mathbb{P}^N(\mathbb{C}), \Omega_{FS}^N)$. Then $\phi \circ p : (\tilde{M}, \tilde{\omega}) \to (\mathbb{P}^N(\mathbb{C}), \Omega_{FS}^N)$ is a h.i.i., where $p : (\tilde{M}, \tilde{\omega}) \to (M, \omega)$ is the universal covering map. Since $\tilde{\omega}$, is either the flat form $\Omega_0^N$ on $\mathbb{C}^N$ or the hyperbolic form $\omega_{hyp}$ on the unit disk $\mathbb{D}$, this is impossible by Remark 3.5.4. □

**Proposition 3.5.7** *The regularized Kepler manifold $X$, endowed with the Kähler form*

$$\Omega = 2i\partial\bar{\partial}\sqrt{z \cdot \bar{z}},$$

*cannot be holomorphically and isometrically immersed in any finite dimensional complex projective space endowed with the Fubini-Study metric, (see 2.0.3).*

**Proof:** Assume the contrary, i.e. there exists a natural number $N$ and a h.i.i. $\phi : (X, \Omega) \to (\mathbb{P}^N(\mathbb{C}), \Omega_{FS}^N)$. Let $U \subset \mathbb{C}^*$ be a simply connected open subset of $\mathbb{C}^*$ endowed with the Kähler form

$$\omega = \frac{i}{2}\partial\bar{\partial}|z|.$$

Take the holomorphic embedding:

$$j : (U, \omega) \to (X, \Omega)$$

defined by

$$j(z) = (\frac{z}{4\sqrt{2}}, i\frac{z}{4\sqrt{2}}, 0, \ldots, 0).$$

It is immediate to verify that $j^*\Omega = \omega$, and hence $\phi \circ j$ is a h.i.i. from $(U, \omega)$ into a finite dimensional complex projective space. On the other hand, the map

$$\psi : (U, \omega) \to (\mathbb{C}, \Omega_0^1) : z \mapsto \sqrt{z}$$

is a h.i.i. from $(U, \omega)$ into $(\mathbb{C}, \Omega_0^1)$, i.e. $\psi^*\Omega_0^1 = \omega$ (the map $\psi$ can be defined since on $U$ one can choose a single branch of the logarithm). This is a contradiction in view of Theorem 3.5.5. □

**Remark 3.5.8** *We do not know if $(X, \Omega)$ admits a h.i.i. in an infinite dimensional complex projective space, since we are not able to describe explicitly the $\infty \times \infty$ matrix $D^+_{jk}$, obtained by the power series expansion of the diastasis (see (3.13) and (3.14)).*

Sometimes a complex manifold admits a Kähler form given by the solution of complicated partial differential equations, therefore the diastasis and even the Kähler form cannot be given explicitly. A typical example is the Kähler-Einstein metric (see the definition below) given by the solution of the Calabi's conjecture (see [1]).

Given a Kähler form one can associate the *Ricci form* $\rho_\omega$ defined, in a complex coordinate system $z_j$, by

$$\rho_\omega = -\frac{i}{2\pi}\partial\bar{\partial}\log\det g_{j\bar{k}},$$

where the $g_{j\bar{k}}$'s are given in (1.10).

**Definition 3.5.9** *A Kähler form $\omega$ on a complex manifold $M$ is said to be Kähler-Einstein if*

$$\rho_\omega = \lambda\omega, \tag{3.19}$$

*where $\lambda$ is a constant called the scalar curvature.*

The first result about h.i.i. of complex manifolds endowed with a Kähler-Einstein form in complex projective spaces can be found in [10] and it is expressed by the following:

**Theorem 3.5.10** *Let $M$ be an hypersurface in $\mathbb{P}^N(\mathbb{C})$. Suppose that the restriction of $\Omega^N_{FS}$ to $M$ is Kähler-Einstein. Then $M$ is either a complex projective space endowed with the Fubini–Study form, or the hyperquadric*

$$Q_{N-1}(\mathbb{C}) = \{(z_0, \dots, z_N) \in \mathbb{P}^N(\mathbb{C}) \mid \sum_{j=0}^{N} |z_j|^2 = 0\},$$

*endowed with the standard Kähler form (see also [20, p. 278]).*

In [28] Theorem 3.5.10 has been generalized to the codimension two case. The case of general codimension is still an open problem. The h.i.i. of Kähler-Einstein manifolds in finite dimensional hyperbolic and flat spaces is treated in [29]:

**Theorem 3.5.11** *Let $M$ be a complex manifold. Suppose that there exists a natural number $N$ and a holomorphic immersion $\phi : M \to \mathbb{C}^N$ (resp. $\phi : M \to \mathcal{D}_N$) such that $\phi^*\Omega_0^N$ (resp. $\phi^*\Omega_{hyp}^N$) is Kähler-Einstein. Then $(M, \omega)$ is totally geodesic in $(\mathbb{C}^N, \Omega_0^N)$ (resp. in $(\mathcal{D}_N, \Omega_{hyp}^N)$).*

## 3.6 Applications to Hartogs domains in $\mathbb{C}^2$

Let $F : [0, x_0) \to (0, +\infty]$ be a non increasing lower semicontinuous function from the interval $[0, x_0) \subset \mathbb{R}$ to the extended positive reals $(0, +\infty]$ (the case $x_0 = +\infty$ is not excluded). Consider the following domain:

$$D_F = \{(z_1, z_2) \in \mathbb{C}^2 \mid \; |z_1|^2 < x_0, |z_2|^2 < F(|z_1|^2)\} \tag{3.20}$$

This is called the *Hartogs domain* corresponding to the function $F$. The lower semicontinuity of $F$ assures us that $D_F$ is an open set. In the hypothesis that $F(0) < \infty$, one can define a real valued function on $D_F$ by

$$\log \frac{1}{H(z)},$$

where $z = (z_1, z_2) \in D_F$ and $H(z) = F(|z_1|^2) - |z_2|^2$. Suppose, furthermore, that $F$ is $C^2$ in $[0, x_0)$ and let $\omega_F$ be the real 2-form on $D_F$ defined by

$$\omega_F := \frac{i}{2\pi} \partial\bar{\partial} \log \frac{1}{F(|z_1|^2) - |z_2|^2}. \tag{3.21}$$

In this Section we give necessary and sufficient conditions for $(D_F, \omega_F)$ to admit a h.i.i. in a complex space form. Moreover, we describe such immersions explicitly.

**Proposition 3.6.1** *Suppose that $F$ is $C^2$ in $[0, x_0)$. Then $\omega_F$ is a Kähler form if and only if*

$$\left(\frac{xF'}{F}\right)' < 0, \; \forall x \in [0, x_0), \tag{3.22}$$

*where the prime denotes the derivative w.r.t. the variable $x$.*

**Proof:** Let $\omega_F = \frac{i}{2\pi}\sum_{j,k=1}^{2} g_{j\bar{k}} dz_j \wedge d\bar{z}_k$ be the local expression of $\omega_F$ in the coordinates $(z_1, z_2)$. A calculation (see [11, pp. 440-441]) shows that

$$\begin{aligned} g_{1\bar{1}} &= \tfrac{-HF' - H|z_1|^2 F'' + |z_1|^2 F'^2}{H^2} \ |_{x=|z_1|^2} , \\ \bar{g}_{1\bar{2}} = g_{2\bar{1}} &= \tfrac{-F'}{H^2} z_1 \bar{z}_2 \ |_{x=|z_1|^2} , \\ g_{2\bar{2}} &= \tfrac{F}{H^2} \ |_{x=|z_1|^2} . \end{aligned}$$

It follows that:

$$\det g_{j\bar{k}} = g_{1\bar{1}} g_{2\bar{2}} - |g_{1\bar{2}}|^2 = -\frac{F^2}{H^3}\left(\frac{xF'}{F}\right)' |_{x=|z_1|^2} . \tag{3.23}$$

The form $\omega_F$ is Kähler if and only if the matrix $g_{j\bar{k}}$ is positive definite and, since $F(0) > 0$ and $g_{2\bar{2}} > 0$, this is the case if and only if $\det g_{j\bar{k}} > 0$. By (3.23) this condition turns out to be equivalent to (3.22). □

Suppose that $\omega_F$ is a Kähler form on $D_F$. Consider the function $\tilde{F} : D_{\sqrt{x_0}} \to \mathbb{R}^+$ defined by

$$\tilde{F}(z_1) := F(|z_1|^2) \tag{3.24}$$

on

$$D_{\sqrt{x_0}} = \{z_1 \in \mathbb{C} \mid \ |z_1| < \sqrt{x_0}\}. \tag{3.25}$$

Suppose that $(D_F, \omega_F)$ admits a h.i.i. in a complex space form. From Theorem 3.1.2, $\omega_F$ has to be real analytic and, by (3.21), this is equivalent to the requirement that $\tilde{F}$ is real analytic in $D_{\sqrt{x_0}}$. Under this hypothesis, a Kähler potential, globally defined in $D_F$, is given by:

$$\Phi_{\omega_F}(z, \bar{z}) = \log \frac{1}{F(|z_1|^2) - |z_2|^2}, \ \forall z = (z_1, z_2) \in D_F.$$

It can be complex analytically extended to $D_F \times D_F$ as

$$\Phi_{\omega_F}(z, \bar{w}) = \log \frac{1}{F(z_1\bar{w}_1) - z_2\bar{w}_2}, \ \forall z = (z_1, z_2), w = (w_1, w_2) \in D_F,$$

where $F(z_1\bar{w}_1) := \tilde{F}(z_1, \bar{w}_1)$ and $\tilde{F}(z_1, \bar{z}_1) = \tilde{F}(z_1)$. The diastasis has the form

$$D_{\omega_F}(z, w) = \log \frac{|F(z_1\bar{w}_1) - z_2\bar{w}_2|^2}{(F(|z_1|^2) - |z_2|^2)(F(|w_1|^2) - |w_2|^2)}. \tag{3.26}$$

Notice that the polar variety $\mathcal{P}_w^{\omega_F}$ reduces to the empty set for every $w \in D_F$. From Corollaries 3.2.1, 3.3.1 and 3.4.2, a necessary condition for $(D_F, \omega_F)$ to admit a h.i.i. in some complex space form, is that $D_{\omega_F}(z, w) \geq 0,\ \forall z, w \in D_F$. By (3.26), this condition is equivalent to

$$\frac{(F(|z_1|^2) - |z_2|^2)(F(|w_1|^2) - |w_2|^2)}{|F(z_1\bar{w}_1) - z_2\bar{w}_2|^2} \leq 1,\ \forall z = (z_1, z_2), w = (w_1, w_2) \in D_F. \quad (3.27)$$

Condition (3.27) has been studied in [11] for a different purpose, and it turns out to be equivalent to the fact that $F(|z_1|) \leq |F(z_1)|$, $\forall z_1 \in \mathcal{D}_{x_0} = \{z \in \mathbb{C} \mid\ |z| < x_0\}$. In Proposition 3.6.3 below we shall give a proof of this fact for completeness. First we need the following (see [11, p. 437 and p. 444]):

**Lemma 3.6.2** *Let $z_1$ and $w_1$ be in $\mathcal{D}_{\sqrt{x_0}}$. Define $u = \log |z_1|^2$ and $v = \log |w_1|^2$. Take the real valued function*

$$g : [-\infty, \log x_0) \to \mathbb{R}$$

*defined by $g(u) = \log F(e^u)$. Then the following conditions are equivalent:*

*(i)* $\dfrac{F(|z_1|^2)F(|w_1|^2)}{F(|z_1 w_1|)^2} \leq 1,\ \forall z_1, w_1 \in \mathcal{D}_{\sqrt{x_0}}$;

*(ii) $g$ is strictly concave;*

*(iii) $\omega_F$ is a Kähler form.*

**Proof:** To be strictly concave is equivalent to $g(u) + g(v) - 2g(\frac{u+v}{2}) \leq 0$ with equality if and only if $u = v$. Thus

$$\frac{F(|z_1|^2)F(|w_1|^2)}{(F(|z_1 w_1|))^2} = \frac{F(e^u)F(e^v)}{(F(\frac{e^{u+v}}{2}))^2} \leq 1.$$

This proves that (i) is equivalent to (ii).

The function $g$ is strictly concave if and only if its second derivative is strictly less than zero. A simple calculation gives

$$\frac{d^2 g(u)}{d^2 u} = \frac{d}{du}\frac{e^u F'(e^u)}{F(e^u)} = e^u \left(\frac{xF'(x)}{F(x)}\right)'_{x=e^u},$$

which, by Proposition 3.6.1, shows that (ii) is equivalent to (iii) and therefore our claim.

□

**Proposition 3.6.3** *Suppose that $\omega_F$ is a Kähler form on $D_F$ and $\tilde{F}$, given by (3.24), is real analytic in $\mathcal{D}_{\sqrt{x_0}}$. Then the following conditions are equivalent:*

*(i)* $\frac{(F(|z_1|^2)-|z_2|^2)(F(|w_1|^2)-|w_2|^2)}{|F(z_1\bar{w}_1)-z_2\bar{w}_2|^2}\leq 1,\ \forall z=(z_1,z_2), w=(w_1,w_2)\in D_F$;

*(ii)* $\frac{F(|z_1|^2)F(|w_1|^2)}{|F(z_1\bar{w}_1)|^2}\leq 1,\ \forall z_1,w_1\in\mathcal{D}_{\sqrt{x_0}}$;

*(iii)* $F(|z_1|)\leq|F(z_1)|,\ \forall z_1\in\mathcal{D}_{x_0}$.

**Proof:**

(i) $\Rightarrow$ (ii)

Immediately if one takes $z=(z_1,0)$ and $w=(w_1,0)$ in $D_F$.

(ii) $\Rightarrow$ (i)

Define $s(z,\bar{w})=\frac{z_2\bar{w}_2}{F(z_1\bar{w}_1)}$. Then

$$\frac{(F(|z_1|^2)-|z_2|^2)(F(|w_1|^2)-|w_2|^2)}{|F(z_1\bar{w}_1)-z_2\bar{w}_2|^2}=\frac{F(|z_1|^2)F(|w_1|^2)}{|F(z_1\bar{w}_1)|^2}\frac{(1-s(z,\bar{z}))(1-s(w,\bar{w}))}{|1-s(z,\bar{w})|^2}.$$

Thus, if (ii) is satisfied, we are left to show that

$$\frac{(1-s(z,\bar{z}))(1-s(w,\bar{w}))}{|1-s(z,\bar{w})|^2}\leq 1. \tag{3.28}$$

By (ii) it follows that

$$|s(z,\bar{w})|^2=\frac{|z_2|^2|w_2|^2}{|F(z_1\bar{w}_1)|^2}\leq\frac{|z_2|^2}{F(|z_1|^2)}\frac{|w_2|^2}{F(|w_1|^2)}.$$

Consequently

$$|1-s(z,\bar{w})|\geq 1-|s(z,\bar{w})|\geq 1-\sqrt{s(z,\bar{z})s(w,\bar{w})}.$$

On the other hand, from the definition of the domain $D_F$, $0\leq s(z,\bar{z})<1,\ \forall z\in D_F$ and so

$$|1-s(z,\bar{w})|^2\geq(1-\sqrt{s(z,\bar{z})s(w,\bar{w})})^2.$$

Hence

$$\begin{aligned}\frac{(1-s(z,\bar{z}))(1-s(w,\bar{w}))}{|1-s(z,\bar{w})|^2} &\leq \frac{(1-s(z,\bar{z}))(1-s(w,\bar{w}))}{(1-\sqrt{(s(z,\bar{z})(s(w,\bar{w}))^2}}\\ &= 1-\left(\frac{\sqrt{s(z,\bar{z})-s(w,\bar{w})}}{(1-\sqrt{s(z,\bar{z})s(w,\bar{w})}}\right)^2\leq 1,\end{aligned}$$

which proves (3.28).

(ii) $\Rightarrow$ (iii)

Let $z_1 \in \mathcal{D}_{x_0}$ different from zero. Define $\eta_1 = |z_1|^{\frac{1}{2}}$ and $\xi_1 = z_1|z_1|^{-\frac{1}{2}}$. The complex numbers $\eta_1$ and $\xi_1$ belong to $\mathcal{D}_{\sqrt{x_0}}$ and, by (ii),

$$\frac{F(|z_1|)^2}{|F(z_1)|^2} \leq 1.$$

(iii) $\Rightarrow$ (ii)

Let $z_1, w_1$ be in $D_{\sqrt{x_0}}$. Then

$$\frac{F(|z_1|^2)F(|w_1|^2)}{|F(z_1\bar{w}_1)|^2} = \frac{F(|z_1w_1|)^2}{|F(z_1\bar{w}_1)|^2}\frac{F(|z_1|^2)F(|w_1|^2)}{F(|z_1w_1|)^2} \leq \frac{F(|z_1|^2)F(|w_1|^2)}{F(|z_1w_1|)^2} \leq 1.$$

We apply (iii) to $z_1w_1 \in \mathcal{D}_{x_0}$ and the last inequality follows from Lemma 3.6.2. □

Our results so far can be summarised as

**Proposition 3.6.4** *If $(D_F, \omega_F)$ admits a h.i.i. in a complex space form then:*

*(i) The function $\tilde{F}$, given by (3.24), is analytic in $D_{\sqrt{x_0}}$;*

*(ii) $F(|z_1|) \leq |F(z_1)|$, $\forall z_1 \in \mathcal{D}_{x_0}$.*

### 3.6.1 H.i.i. of $(D_F, \omega_F)$ in $(\mathbb{C}^N, \Omega_0^N)$.

Suppose that $(D_F, \omega_F)$ admits a h.i.i. in $(\mathbb{C}^N, \Omega_0^N)$. It follows that the condition $(i)$ in 3.6.4 is satisfied. Then

$$D_{\omega_F}(z, w_0) = \log\frac{F(0)}{F(|z_1|^2) - |z_2|^2},$$

where $w_0$ is the origin of $\mathbb{C}^2$. Let $\rho_j = |z_j|^2, j = 1, 2$, and define

$$C(\rho_1, \rho_2) = \log\frac{F(0)}{F(\rho_1) - \rho_2}. \tag{3.29}$$

Since $F$ is a real analytic function it follows that $C$ is real analytic in the open set $\{(\rho_1, \rho_2) \in \mathbb{R}^2 \mid \ \rho_1 < \sqrt{x_0}, \rho_2 < \sqrt{F(\rho_1)}\}$. Hence, (3.29) can be expanded in power series as

$$C(\rho_1, \rho_2) = \sum_{j,k=0}^{+\infty} c_{jk}\rho_1^j\rho_2^k,$$

where

$$c_{jk} = \frac{\partial^{j+k} C}{\partial \rho_1^j \rho_2^k}(w_0). \tag{3.30}$$

Therefore,

$$D_{\omega_F}(z, w_0) = C(|z_1|^2, |z_2|^2) = \sum_{j,k=0}^{+\infty} c_{jk} |z_1|^{2j} |z_2|^{2k} = \sum_{j,k=0}^{+\infty} \delta_{jk} c_{m_j} z^{m_j} \bar{z}^{m^k},$$

where $c_{m_j} = c_{m_{1j}} c_{m_{2j}}$. Consequently $D_{jk} = \delta_{jk} c_{m_j}$.

**Theorem 3.6.5** *$(D_F, \omega_F)$ admits a h.i.i. in some flat space if and only if all the $c_{jk}$'s given by (3.30) are greater or equal than zero. Under this hypothesis, the number $N$ of the $c_{jk}$'s strictly greater than zero is the complex dimension of the flat space in which $(D_F, \omega_F)$ admits a full holomorphic isometric immersion. The h.i.i. in $(\mathbb{C}^N, \Omega_0^N)$ is given by:*

$$\phi(z_1, z_2) = (1, \ldots, \sqrt{c_{m_j}} z^{m_j}, \ldots), j = 1, \ldots, N$$

**Proof:** The stated conditions are equivalent to the resolvability of rank $N$ of $\omega_F$. Thus the result follows from 3.2.3 since $D_F$ is simply connected (even contractible). The last assertion is immediate. In fact

$$\phi^* \Omega_0^N = \frac{i}{2\pi} \partial \bar{\partial} \sum_{j=0}^{+\infty} c_{m_j} |z|^{2m_j} = \omega_F.$$

□

**Corollary 3.6.6** *$(D_F, \omega_F)$ cannot be h.i.i. in any finite dimensional flat space.*

**Proof:** Suppose that the contrary holds, that is $D_{\omega_F}$ is resolvable of finite rank. From Theorem 3.6.5 only a finite number of the $c_{jk}$'s are strictly greater than zero. On the other hand, It is not difficult to see that

$$c_{0k} = \frac{\partial^k C}{\partial \rho_2^k}(w_0) = (F(0))^{-k} > 0 \ \forall k,$$

which gives the desired contradiction. □

Theorem 3.6.5 gives an infinite number of conditions which involve the derivatives of all orders of the function $F$ at $w_0$. For example $c_{10} \geq 0$ is equivalent to $\frac{\partial C}{\partial \rho_1}(w_0) =$

$-\frac{F'(0)}{F(0)}$, which is automatically satisfied since $F(0) > 0$ and $F$ is a non increasing function. The first non trivial condition comes from $c_{20} \geq 0$. In fact

$$c_{20} = \frac{\partial^2 C}{\partial \rho_1^2}(w_0) = \frac{(F'(0))^2 - F''(0)F(0)}{F(0)^2} \geq 0,$$

i.e.

$$F''(0) \leq \frac{(F'(0))^2}{F(0)}. \tag{3.31}$$

**Example 3.6.7** *Let $F(x) = e^{-x}, x \in [0, +\infty)$. The function $F$ is non-increasing and $\left(\frac{xF'}{F}\right)' = -1 < 0$. Hence, condition (3.22) is satisfied and*

$$\omega_{e^{-x}} = \frac{i}{2\pi}\partial\bar{\partial}\log\frac{1}{e^{-|z_1|^2} - |z_2|^2}$$

*is a Kähler form on $D_{e^{-x}}$. This domain is considered also in [11, p. 451] and it is called the* Spring domain. *One can construct a h.i.i. of $(D_{e^{-x}}, \omega_{e^{-x}})$ into the infinite dimensional flat space $(\mathbb{C}^\infty, \Omega_0^\infty)$ as follows. First of all, consider*

$$\log\frac{1}{e^{-|z_1|^2} - |z_2|^2}. \tag{3.32}$$

*Since*

$$-\log(1-x) = \sum_{k=1}^{+\infty} k^{-1}x^k, 0 < x < 1,$$

*(3.32) can be expanded in power series as*

$$\begin{aligned}
\log\tfrac{1}{e^{-|z_1|^2} - |z_2|^2} &= |z_1|^2 - \log(1 - e^{|z_1|^2}|z_2|^2) \\
&= |z_1|^2 + \textstyle\sum_{k=1}^{+\infty} k^{-1}e^{k|z_1|^2}|z_2|^{2k} \\
&= |z_1|^2 + \textstyle\sum_{j=0}^{+\infty}\sum_{k=1}^{+\infty}\frac{k^{j-1}}{j!}|z_1|^{2j}|z_2|^{2k}.
\end{aligned}$$

*The h.i.i. in $(\mathbb{C}^\infty, \Omega_0^\infty)$ is then given by*

$$(z_1, z_2) \to (z_1, \ldots, \sqrt{\frac{k^{j-1}}{j!}}z_1^j z_2^k, \ldots) j = 0, 1 \ldots, k = 1, 2, \ldots.$$

**Example 3.6.8** *Consider the function $F(x) = e^{-x} + 2$ , $x \in [0, 1)$. Since*

$$\left(\frac{xF'}{F}\right)' = -\frac{1 + 2e^x(1-x)}{(1+2e^x)^2} < 0, \ \forall x \in [0, 1),$$

*it follows from 3.6.1 that $\omega_{e^{-x}+2}$ is a Kähler form on $D_{e^{-x}+2}$. Furthermore, since*

$$F(|z|) = e^{-|z|} + 2 \leq |e^{-z} + 2|,$$

*the conditions (i) and (ii) of Proposition 3.6.4 are satisfied. On the other hand,*

$$F''(0) = 1 > \frac{1}{3} = \frac{(F'(0))^2}{F(0)}.$$

*Therefore condition (3.31) is not satisfied, and so $(D_{e^{-x}+2}, \omega_{e^{-x}+2})$ cannot be h.i.i. in any flat space.*

### 3.6.2 H.i.i. of $(D_F, \omega_F)$ in $(\mathcal{D}_N, \Omega^N_{hyp})$

Suppose that $F$ is a real analytic function on $(-x_0, x_0)$ (see condition (i) in 3.6.4) and let $F(x) = \sum_{j=0}^{+\infty} F_j x^j$ be its power series expansion at the origin, where $F_j = \frac{\partial^j F}{\partial x^j}(0)$, $\forall j$.

**Theorem 3.6.9** *$(D_F, \omega_F)$ admits a h.i.i. in $(\mathcal{D}_N, \Omega^N_{hyp})$ if and only if $F_j \leq 0$, $\forall j \geq 1$. Under this hypothesis, let $N$ be the number of the $F_j$'s strictly less than zero. A full h.i.i. in $(\mathcal{D}_N, \Omega^N_{hyp})$ is given by:*

$$\phi(z_1, z_2) = \left(\frac{z_2}{\sqrt{F(0)}}, 1, \dots, \frac{\sqrt{-F_j}}{\sqrt{F(0)}} z_1^j, \dots\right), \ j = 1, \dots, N$$

**Proof:**

$$1 - e^{-D_{\omega_F}}(z, 0) = 1 + (|z_2|^2 - F(|z_1|^2))F(0)^{-1} = 1 + (|z_2|^2 - \sum_{j=0}^{+\infty} F_j)F(0)^{-1}$$

and so the conclusion follows easily from Theorem 3.3.2. □

Since the second derivative of $e^{-x}$ at 0 is positive it follows, from 3.6.9, that

**Corollary 3.6.10** *The Spring domain $(D_{e^{-x}}, \omega_{e^{-x}})$ cannot be holomorphically and isometrically immersed in any hyperbolic space of any dimension.*

**Remark 3.6.11** *Notice that, there is no upper bound on the dimension of the target hyperbolic space in which $(D_F, \omega_F)$ admits a h.i.i. (cf. Corollaries 3.6.6 and 3.6.13).*

### 3.6.3 H.i.i. of $(D_F, \omega_F)$ in $(\mathbb{P}^N(\mathbb{C}), \Omega^N_{FS})$

Suppose that condition (i) in Proposition 3.6.4 is satisfied. Let $w_0$ be the origin of $\mathbb{C}^2$. Define

$$G(z_1, z_2) = e^{D_{\omega_F}(z,w_0)} - 1 = \frac{F(0)}{F(|z_1|^2) - |z_2|^2} - 1.$$

Let $\rho_i = |z_i|^2, i = 1, 2$. The function $G(\rho_1, \rho_2)$ is a real analytic function on the open set $\{\rho_1^2 < x_0, \rho_2^2 < F(\rho_1)\}$. Take its power series expansion

$$G(\rho_1, \rho_2) = \sum_{j,k=0}^{+\infty} a_{jk}\rho_1^j \rho_2^k,$$

where $a_{jk} = \frac{\partial G^{j+k}}{\partial \rho_1^j \rho_2^k}(w_0)$. It follows that

$$G(z_1, z_2) = \sum_{j,k=0}^{+\infty} a_{jk} z_1^j z_2^k \bar{z}_1{}^j \bar{z}_2{}^k = \sum_{j=0}^{+\infty} a_{m_j} z^{m_j} \bar{z}^{m_j} = \sum_{j,k=0}^{+\infty} \delta_{jk} a_{m_j} z^{m_j} \bar{z}^{m_k},$$

where $a_{m_j} = a_{m_{1j} m_{2j}}$. Therefore

$$D^+_{jk} = \delta_{jk} a_{m_j} - 1,$$

and $D^+_{jk}$ is semipositive definite if and only if $a_{jk} \geq 0,\ \forall j, k = 0, 1, \ldots$ From Theorem 3.4.4 one then deduces:

**Theorem 3.6.12** *The form $\omega_F$ is projectively induced if and only if $F$ is analytic and $a_{jk} \geq 0, \forall j, k = 0, 1, \ldots$ Under this hypothesis, the number $N$ of the $a_{jk}$'s strictly greater than zero is the dimension of the complex projective space in which $(D_F, \omega_F)$ admits a full h.i.i.*

**Corollary 3.6.13** *There is not any h.i.i. of $(D_F, \omega_F)$ in a finite dimensional complex projective space.*

**Proof:** Suppose the contrary. From the previous theorem, only a finite number of $a_{jk}$'s are strictly greater than zero. On the other hand,

$$a_{0k} = \frac{\partial^k G}{\partial \rho_2^k}(w_0) = k! F(0)^{-k} > 0,\ \forall k,$$

hence the desired contradiction. □

Notice that, once that all the $a_{jk}$'s are known to be non-negative, then we can construct a h.i.i. in $(\mathbb{P}^\infty(\mathbb{C}), \Omega^\infty_{FS})$ explicitly. In fact, the map

$$\phi : (D_F, \omega_F) \to (\mathbb{P}^\infty(\mathbb{C}), \Omega^\infty_{FS})$$

given by

$$\phi(z_1, z_2) = [(\dots, \sqrt{a_{jk}} z_1^j z_2^k, \dots)], \; j, k = 1, \dots, N$$

is holomorphic and

$$\phi^* \Omega^\infty_{FS} = \frac{i}{2\pi} \partial\bar\partial \log \sum_{j,k=0}^{+\infty} a_{jk} |z_1|^{2j} |z_2|^{2k} = \frac{i}{2\pi} \partial\bar\partial \log \frac{1}{H} = \omega_F.$$

**Example 3.6.14** *(see Example 3.6.7)*

*Let $F(x) = e^{-x}, x \in [0, +\infty)$. One can construct a h.i.i. of $(D_{e^{-x}}, \omega_{e^{-x}})$ in the infinite dimensional projective space as follows. First of all, consider the expression*

$$\frac{1}{e^{-|z_1|^2} - |z_2|^2} = \frac{e^{|z_1|^2}}{1 - e^{|z_1|^2}|z_2|^2} = e^{|z_1|^2} \sum_{k=0}^{+\infty} \left(e^{|z_1|^2}|z_2|^2\right)^k.$$

*This can be further developed as*

$$\sum_{k=0}^{+\infty} e^{k+1|z_1|^2}|z_2|^{2k} = \sum_{j,k=0}^{+\infty} \frac{|z_1|^{2j}|z_2|^{2k}(k+1)^j}{j!} = \sum_{j,k=0}^{+\infty} \frac{(k+1)^j}{j!}|z_1|^{2j}|z_2|^{2k}.$$

*So $a_{jk} = \frac{(k+1)^j}{j!} \geq 0$ and, from 3.6.12, $(D_{e^{-x}}, \omega_{e^{-x}})$ admits a h.i.i. in $(\mathbb{P}^\infty(\mathbb{C}), \Omega^\infty_{FS})$ via the map*

$$(z_1, z_2) \to (\dots, \sqrt{\frac{(k+1)^j}{j!}} z_1^j z_2^k, \dots) \; j, k = 0, 1, \dots.$$

**Example 3.6.15** *(cf. [11, p. 450])*

*Let $F(x) = (1-x)^p, x \in [0, 1)$.*

$$F'(x) = -p(1-x)^{p-1}, \left(\frac{xF'(x)}{F(x)}\right)' = \frac{-p}{(1-x)^2} < 0.$$

*It follows from 3.6.1 that $\omega_{(1-x)^p}$ is a Kähler form on*

$$D_F = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 < 1, |z_2|^2 < (1 - |z_1|^2)^p\}.$$

*Consider the following development*

$$\frac{1}{(1-|z_1|^2)^p - |z_2|^2} = \frac{1}{(1-|z_1|^2)^p} \sum_{k=0}^{+\infty} \frac{|z_2|^{2k}}{(1-|z_1|^2)^{pk}}.$$

*Since*

$$\frac{1}{(1-x)^{a+1}} = \sum_{j=0}^{+\infty} \binom{j+a}{j} x^j, \ 0 < x < 1,$$

*the previous expression can be further developed as*

$$\sum_{k=0}^{+\infty} \frac{|z_2|^{2k}}{(1-|z_1|^2)^{p(k+1)}} = \sum_{j,k=0}^{+\infty} \binom{p(k+1)+j-1}{j} |z_1|^{2j}|z_2|^{2k}.$$

*From 3.6.12,* $(D_{(1-x)^p}, \omega_{(1-x)^p})$ *admits a h.i.i. in* $(\mathbb{P}^\infty(\mathbb{C}), \Omega^\infty_{FS})$ *given by:*

$$(z_1, z_2) \to (\dots, (\binom{p(k+1)+j-1}{j})^{\frac{1}{2}} z_1^j z_2^k, \dots), j, k = 1, 2, \dots$$

### 3.6.4 Further results about Hartogs domains

Let $F : [0, x_0) \to \mathbb{R}^+$ be a non increasing function such that $\omega_F$ defines a Kähler form on $D_F$.

In this Section we prove that, under suitable conditions, $(D_F, \omega_F)$ is holomorphically isometric (up to a homothety) to the hyperbolic two ball in $\mathbb{C}^2$. We start with an elementary fact:

**Lemma 3.6.16** *Let* $\phi$ *be a holomorphic function on a open set* $U \subset \mathbb{C}$ *containing the origin. Suppose that there exists a real analytic function* $f : (-x_0, x_0) \to \mathbb{R}$ *such that* $|\phi(z)|^2 = f(|z|^2)$*. Then* $\phi(z)$ *reduces to the constant* $\phi(0)$*.*

**Proof:** Let $\phi(z) = \sum_{j=0}^{+\infty} a_j z^j$ be the power series expansion of $\phi$ at the origin, and $f(x) = \sum_{l=0}^{+\infty} b_l x^l$ the Taylor expansion of $f$ at the origin. By hypothesis,

$$\sum_{j,k=0}^{+\infty} a_j \bar{a}_k z^j \bar{z}^k = \sum_{l=0}^{+\infty} b_l |z|^{2l},$$

which implies that all the terms of the form $a_j \bar{a}_k z^j \bar{z}^k$ with $j \neq k$, are zero. It follows that $a_j = 0$ for $j > 0$, and so the result. □

**Theorem 3.6.17** *Let* $F : [0, x_0) \to \mathbb{R}^+$ *be a non-increasing function such that the corresponding form* $\omega_F$ *is Kähler-Einstein (see 3.22). Suppose that* $F$ *can be extended to a real analytic function in* $(-x_0, x_0)$*. Then* $(D_F, 3\omega_F)$ *is holomorphically isometric to* $(\mathcal{D}_2, \Omega^2_{hyp})$*.*

**Proof:** If $\omega_F$ is Kähler-Einstein, then, by (3.19) and (3.21),

$$\rho_{\omega_F} = -\frac{i}{2\pi}\partial\bar{\partial}\log\det g_{j\bar{k}} = \lambda\omega_F = \lambda\frac{i}{2\pi}\partial\bar{\partial}\log\frac{1}{H} = -\frac{i}{2\pi}\partial\bar{\partial}\log H^{\lambda}, \tag{3.33}$$

where $\lambda$ is the scalar curvature and $\rho_{\omega_F}$ is the Ricci form. From (3.33) we deduce that

$$\partial\bar{\partial}\log H^{-\lambda}\det g_{j\bar{k}} = 0.$$

Since the domain $D_F$ is simply connected, it follows from Lemma 1.4.9, that there exists a holomorphic function $\phi$ on $D_F$ such that

$$H^{-\lambda}\det g_{j\bar{k}} = |\phi|^2.$$

Therefore, by (3.23),

$$H^{-\lambda}\det g_{j\bar{k}} = -\frac{F^2}{H^{\lambda+3}}\left(\frac{xF'}{F}\right)'|_{x=|z_1|^2} = -\frac{(F'+|z_1|^2F'')F-|z_1|^2F'^2}{H^{\lambda+3}}|_{x=|z_1|^2} = |\phi|^2.$$

By hypothesis, $F$ is analytic in $(-x_0, x_0)$, and it is not hard to see, using Lemma 3.6.16, that the function $\phi$ equals a constant, say $C$. Hence

$$\frac{(F'+|z_1|^2F'')F-|z_1|^2F'^2}{H^{\lambda+3}} = -C^2. \tag{3.34}$$

The numerator of (3.34) depends only on $|z_1|^2$, while the denominator depends also on $|z_2|^2$. Therefore $\lambda = -3$ and

$$(F'+xF'')F - xF'^2 = -C^2,\ \forall x \in (-x_0, x_0). \tag{3.35}$$

Taking the first derivative of (3.35) at zero one gets:

$$2F(0)F''(0) = 0.$$

Since $F(0) \neq 0$, it follows that $F''(0) = 0$. Taking the higher order derivatives of (3.35) at zero it is not difficult to see that:

$$0 = \frac{\partial^k (F'+xF'')F - xF'^2}{\partial x^k}(0) = (k+1)F(0)\frac{\partial^k F}{\partial x^k}(0),\ k \geq 1,$$

and so $\frac{\partial^k F}{\partial x^k}(0) = 0$. Since $F$ is analytic in $(-x_0, x_0)$, it follows that $F(x) = \alpha - \beta x$, where $\alpha$ and $\beta$ are positive constants. Consider now the holomorphic map

$$\phi : D_{\alpha-\beta x} \to \mathcal{D}_2 : (z_1, z_2) \mapsto (\sqrt{\frac{\beta}{\alpha}}z_1, \sqrt{\frac{1}{\alpha}}z_2).$$

It is easy to check that

$$\phi^* \Omega^2_{hyp} = 3\omega_{\alpha - \beta x},$$

hence the conclusion. □

**Corollary 3.6.18** *If* $(D_F, \omega_F)$ *admits a h.i.i. in a complex space form and* $\omega_F$ *is Kähler-Einstein, then* $(D_F, 3\omega_F)$ *is holomorphically isometric to the hyperbolic two ball.*

**Proof:** In the hypothesis of Corollary 3.6.18, condition (i) in 3.6.4 is satisfied and therefore $F$ is a real analytic function in $(-x_0, x_0)$. Hence the conclusion follows from Theorem 3.6.17. □

# Chapter 4

# The function epsilon

## 4.1 Definition and elementary properties

Let $(L, h) \stackrel{\pi}{\to} (M, \omega)$ be a geometric quantization of a Kähler manifold $(M, \omega)$. Consider the space $\mathcal{H}_h \subset H^0(L)$ consisting of global holomorphic sections $s$ of $L$, which are bounded with respect to

$$\langle s, s\rangle_h = \|s\|_h^2 = \int_M h(s(x), s(x))\frac{\omega^n(x)}{n!}.$$

On can show that $(\mathcal{H}_h, \langle\cdot,\cdot\rangle_h)$ is a separable complex Hilbert space (see [5]). Let $x \in M$ and $q \in L^+$ such that $\pi(q) = x$. If one evaluates $s \in \mathcal{H}_h$ at $x$, one gets a multiple $\delta_q(s)$ of $q$, i.e. $s(x) = \delta_q(s)q$. It can be shown that $\delta_q : \mathcal{H}_h \to \mathbb{C}$ is a linear continuous functional of $s$ (see [5]) thus, from Riesz theorem, there exists a unique $e_q \in \mathcal{H}_h$ such that $\delta_q(s) = \langle s, e_q\rangle_h$, i.e.

$$s(x) = \langle s, e_q\rangle_h q. \tag{4.1}$$

**Definition 4.1.1** *The holomorphic section $e_q \in \mathcal{H}_h$ is called the* coherent state*, relative to the point $q \in L^+$.*

It follows, by (4.1), that

$$e_{cq} = \bar{c}^{-1}e_q,\ \forall c \in \mathbb{C}^*.$$

Then, one can define a real valued function on $M$ by the formula

$$\epsilon_{(L,h)}(x) := h(q, q)\|e_q\|_h^2, \tag{4.2}$$

where $q \in L^+$ is any point on the fibre of $x$. We will speak about the *function epsilon* when the pair $(L,h)$ is clear from the context. Let $s_j$, $j = 0, \dots, N$ ($N \leq \infty$), be a unitary basis for $(\mathcal{H}_h, \langle \cdot, \cdot \rangle_h)$. Take $\lambda_j \in \mathbb{C}$ such that $s_j(x) = \lambda_j q$, $j = 0, \dots, N$. Then

$$s(x) = \sum_{j=0}^{N} \langle s, s_j \rangle_h s_j(x) = \sum_{j=0}^{N} \langle s, s_j \rangle_h \lambda_j q = \langle s, \sum_{j=0}^{N} \bar{\lambda}_j s_j \rangle_h q.$$

By (4.1) it follows that

$$e_q = \sum_{j=0}^{N} \bar{\lambda}_j s_j, \tag{4.3}$$

and

$$\epsilon_{(L,h)}(x) = h(q,q) \|e_q\|_h^2 = \sum_{j=0}^{N} h(s_j(x), s_j(x)). \tag{4.4}$$

When $M$ is compact, the dimension of $\mathcal{H}_h = H^0(L)$ is finite and, by (4.4), one obtains:

$$\int_M \epsilon_{(L,h)}(x) \frac{\omega^n(x)}{n!} = \dim \mathcal{H}_h. \tag{4.5}$$

In order to write down the local expression of the function epsilon, let $\sigma : U \to L^+$ be a trivialising section on a open set $U \subset M$. Once again by (4.4) one obtains

$$\epsilon_{(L,h)}(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x)) = h(\sigma(x), \sigma(x)) \sum_{j=0}^{N} \frac{s_j(x)}{\sigma(x)} \overline{\frac{s_j(x)}{\sigma(x)}}, \ \forall x \in U.$$

In a possibly smaller open set $V \subset U$ one can write

$$\epsilon_{(L,h)}(x) = e^{-\Phi_\omega(x)} \sum_{j=0}^{N} \frac{s_j(x)}{\sigma(x)} \overline{\frac{s_j(x)}{\sigma(x)}}, \ \forall x \in V, \tag{4.6}$$

where $\Phi_\omega$ is a Kähler potential for $\omega$.

We conclude this Section by describing how the function $\epsilon_{(L,h)}$ varies with the pair $(L,h)$ in $\mathcal{L}_{hol}(M,\omega)$. By its very definition, it follows that $\epsilon_{(L,h)}$ does not depend on the representative $(L,h)$ in the class $[(L,h)]_{hol} \in \mathcal{L}_{hol}(M,\omega)$. When $M$ is simply connected, $\mathcal{L}_{hol}(M,\omega)$ consists of a single equivalence class (see 1.4.6). Therefore, in this case, the function epsilon depends only on the Kähler form $\omega$.

**Remark 4.1.2** *When $(M,\omega)$ is a quantizable simply connected Kähler manifold we will often write $\epsilon_\omega$ instead of $\epsilon_{(L,h)}$.*

Another important fact, for the proof of which we refer to [5], is the following:

**Proposition 4.1.3** *The function $\epsilon_{(L,h)}$ is invariant under the group $D_{[(L,h)]}(M)$, i.e. $F^*(\epsilon_{(L,h)}) = \epsilon_{(L,h)}$, for every $F \in D_{[(L,h)]}(M)$ (see Section 1.5).*

**Corollary 4.1.4** *Let $(L, h)$ be a quantization of a simply connected homogeneous Kähler manifold $(M, \omega)$. Then the function $\epsilon_{(L,h)}$ is constant.*

**Proof:** Since the manifold is simply connected, it follows from 1.5.1, that the group $D_{[(L,h)]}(M)$ equals $\mathrm{Aut}(M) \cap \mathrm{Isom}(M, \omega)$. Since $(M, \omega)$ is homogeneous, the latter acts transitively on $M$ and so, from Proposition 4.1.3, $\epsilon_\omega$ turns out to be constant . □

## 4.2 The coherent states map

Let $(L, h)$ be a geometric quantization of a Kähler manifold $(M, \omega)$. In analogy with the compact case we say that $(L, h)$ is *base point free* if for all $x \in M$ there exists $s \in \mathcal{H}_h$ such that $s(x)$ is different from zero (see Section 2.0.4). In this hypothesis, one can define a map

$$\tilde{\phi}_{(L,h)} : L^+ \to \mathcal{H}_h^* \setminus \{0\} : q \mapsto \langle \cdot, e_q \rangle_h,$$

where $\mathcal{H}_h^*$ is the dual of $\mathcal{H}_h$. Consider the holomorphic map $\phi_{(L,h)} : M \to \mathbb{P}(\mathcal{H}_h^*)$ which makes the following diagram commutative:

$$\begin{array}{ccc} L^+ & \stackrel{\tilde{\phi}_{(L,h)}}{\longrightarrow} & \mathcal{H}_h^* \setminus \{0\} \\ \downarrow & & \downarrow \\ M & \stackrel{\phi_{(L,h)}}{\longrightarrow} & \mathbb{P}(\mathcal{H}_h^*) \end{array}$$

**Definition 4.2.1** *The map $\phi_{(L,h)}$ is called the coherent states map.*

The following theorem can be found in [26] and [5]. Here we give a different proof.

**Theorem 4.2.2** *Suppose that $(L, h)$ is base point free. Then $\phi_{(L,h)}$ is a full holomorphic map from $M$ to $\mathbb{P}(\mathcal{H}_h^*)$. Moreover, if $\Omega_{\mathcal{H}_h^*}$ denotes the Fubini-Study form on $\mathbb{P}(\mathcal{H}_h^*)$, then*

$$\phi_{(L,h)}^* \Omega_{\mathcal{H}_h^*} = \omega + \frac{i}{2\pi} \partial\bar{\partial} \log \epsilon_{(L,h)}, \tag{4.7}$$

*(see (2.7)).*

**Proof:** Let $s_j$, $j = 0, 1, \ldots, N$ ($N \leq \infty$) be a unitary basis for $(\mathcal{H}_h, \langle \cdot\,, \cdot \rangle_h)$ and $\sigma : U \to L^+$ a trivialising holomorphic section on a open set $U \subset M$. Define the holomorphic map

$$\phi_\sigma : U \to \mathbb{C}^{N+1} : x \mapsto \left(\ldots, \frac{s_j(x)}{\sigma(x)}, \ldots\right).$$

As in the compact case, this map can be extended to a holomorphic map

$$\phi : M \to \mathbb{P}^N(\mathbb{C}) : x \mapsto [(\ldots, s_j(x), \ldots)]\,, \tag{4.8}$$

(see (2.8) and (2.9)). Let $b : \mathcal{H}_h^* \setminus \{0\} \to \mathbb{C}^\infty \setminus \{0\}$ defined by: $b(s^*) = (\ldots, s^*(s_j), \ldots)$. Denote with the same symbol $b$ the induced map in the complex projective spaces $b : \mathbb{P}(\mathcal{H}_h^*) \to \mathbb{P}^N(\mathbb{C})$. It follows immediately, by the very definition of the coherent states map, that

$$\phi = b \circ \phi_{(L,h)}.$$

In fact, by (4.8),

$$b \circ \phi_{(L,h)}(x) = [(\ldots, \langle s_j, e_q\rangle_h, \ldots)] = [(\ldots, \frac{s_j(x)}{q}, \ldots)] = \phi(x). \tag{4.9}$$

Formula (4.9) shows that the coherent states map is full, since it is constructed by the evaluation of a basis of $\mathcal{H}_h$. In order to prove the second part of the theorem, let $U \subset M$ be an open set where a Kähler potential $\Phi_\omega$ can be defined. Then

$$\begin{aligned}
\phi^*_{(L,h)}\Omega_{\mathcal{H}_h^*} &= (b \circ \phi_{(L,h)})^*(\Omega^N_{FS}) = \phi^*\Omega^N_{FS} \\
&= \tfrac{i}{2\pi}\partial\bar\partial \log \textstyle\sum_{j=0}^N \frac{s_j(x)}{\sigma(x)}\overline{\frac{s_j(x)}{\sigma(x)}} \\
&= \tfrac{i}{2\pi}\partial\bar\partial \log \textstyle\sum_{j=0}^N \frac{s_j(x)}{\sigma(x)}\overline{\frac{s_j(x)}{\sigma(x)}} + \omega - \omega \\
&= \tfrac{i}{2\pi}\partial\bar\partial \log \textstyle\sum_{j=0}^N \frac{s_j(x)}{\sigma(x)}\overline{\frac{s_j(x)}{\sigma(x)}} + \omega + \tfrac{i}{2\pi}\partial\bar\partial\log(e^{-\Phi_\omega}) \\
&= \omega + \tfrac{i}{2\pi}\partial\bar\partial \log(e^{-\Phi_\omega} \textstyle\sum_{j=0}^N \frac{s_j(x)}{\sigma(x)}\overline{\frac{s_j(x)}{\sigma(x)}}) \\
&= \omega + \tfrac{i}{2\pi}\partial\bar\partial \log \epsilon_{(L,h)},
\end{aligned}$$

where the last equality follows by (4.6). □

**Corollary 4.2.3** *Let $(L, h)$ be a quantization of a Kähler manifold $(M, \omega)$. If $\epsilon_{(L,h)}$ is a constant different from zero, then $\omega$ is projectively induced via the coherent states map.*

**Proof:** If $\epsilon_{(L,h)}$ equals a positive constant then it follows, by (4.4), that $(L,h)$ is base point free and consequently the coherent states map can be defined. Thus the conclusion follows by (4.7). □

From Lemma 1.4.8 and formula (4.7) one deduces:

**Corollary 4.2.4** *Let $(L,h)$ be a quantization of a* compact *Kähler manifold $(M,\omega)$. Then the function $\epsilon_{(L,h)}$ is equal to a constant if and only if the coherent states map is a full h.i.i. in $(\mathbb{P}(\mathcal{H}_h^*), \Omega_{\mathcal{H}_h^*})$.*

## 4.3 On the constancy of the function epsilon

The following notion can be found in [5]:

**Definition 4.3.1** *A hermitian holomorphic line bundle $(L,h)$ over a Kähler manifold $(M,\omega)$ such that curv$(L,h) = -2\pi i\omega$ is called regular if the function $\epsilon_{(L,h)}$ is constant.*

One can calculate the function $\epsilon_{(L^k,h^k)}$ for every natural number $k$. Namely, one considers the Kähler form $k\omega$ on $M$ and $(L^k, h^k)$ the quantum line bundle for $(M, k\omega)$, where $L^k$ is the $k$-tensor power of $L$ and $h^k := h \otimes \ldots \otimes h$, $k$-times.

**Definition 4.3.2** *A quantization $(L,h)$ of a Kähler manifold $(M,\omega)$ is called regular if, for any natural number $k$, $(L^k, h^k)$ is regular.*

In the programme of quantization of Kähler manifolds carried out in [5], [6], [7], [8] and [26], the Kähler manifolds $(M,\omega)$ which admit a regular quantization play a prominent role. In fact, under this hypothesis, it is possible to apply a procedure called *quantization by deformation* (we refer to the above mentioned papers for details).

By Corollary 4.1.4 a homogeneous simply connected Kähler manifold admits a regular quantization. Not all the homogeneous manifolds admit a regular quantization. An example is given by the complex torus $V/\Lambda$, equipped with the flat form $\Omega_0^N$. In fact, suppose that $(L,h)$ is a regular quantization of $(V/\Lambda, \Omega_0^N)$, then, from Corollary 4.2.3, $\Omega_0^N$ would be projectively induced, contradicting 3.5.6. An example of regular quantization over a *non homogeneous* Kähler manifold can be constructed as follows.

**Example 4.3.3** *Let $(\mathcal{O}_1(1), h)$ be the quantization of $(\mathbb{P}^1(\mathbb{C}), \Omega^1_{FS})$ described in 2.0.4. Since $(\mathbb{P}^1(\mathbb{C}), \Omega^1_{FS})$ is a simply connected Kähler manifold, then, from 4.1.4, the quantization is regular, i.e. $\epsilon_{k\Omega^1_{FS}}$ is constant for every non negative integer $k$. Take the chart*

$$U_0 = \{[(z_0, z_1)] |\ z_0 \neq 0\} \subset \mathbb{P}^1(\mathbb{C})$$

*endowed with coordinate $z = \frac{z_1}{z_0}$. The restriction of $\Omega^1_{FS}$ to $U_0$ can be expressed as*

$$\omega := \Omega^1_{FS} |_{U_0} = \frac{i}{2\pi}\partial\bar{\partial}\log(1 + |z|^2) = \frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$

*It is immediate to verify that the trivial bundle $L_0 := U_0 \times \mathbb{C} \to U_0$ over $U_0$, endowed with the hermitian structure*

$$h(z, t) := \frac{|t|^2}{(1 + |z|^2)},\ \forall z \in U_0, \forall t \in \mathbb{C},$$

*is a geometric quantization of $(U_0, \omega)$. The Hilbert space $(\mathcal{H}_h, \langle \cdot\ , \ \cdot \rangle_h)$ consists of holomorphic functions on $\mathbb{C}$ bounded w.r.t.*

$$\|f\|_h = \langle f, f\rangle_h = \int_{\mathbb{C}} \frac{|f(z)|^2}{(1 + |z|^2)^3}\frac{i}{2}dz \wedge d\bar{z}.$$

*It is easy to check that $(\mathcal{H}_h, \langle \cdot\ , \ \cdot \rangle_h)$ consists of polynomials of degree 1 in $z$, i.e. $\alpha - \beta z$ with $\alpha$ and $\beta$ in $\mathbb{C}$. Furthermore, one can show that $(1, z)$ is a unitary basis for $(\mathcal{H}_h, \langle \cdot\ , \ \cdot \rangle_h)$ and so, by (4.4), $\epsilon_\omega = 1$. A similar argument shows that $\epsilon_{k\omega}$ is constant for every natural number $k$, and hence the quantization $(L_0, h)$ of $(U_0, \omega)$ is regular.*

Notice that the quantization $(L_0, h)$ of $(U_0, \omega)$, described in the previous example, is just the restriction of the quantization $(\mathcal{O}_1(1), h)$ of $(\mathbb{P}^1(\mathbb{C}), \Omega^1_{FS})$ to $(U_0, \omega)$. We believe that this is a very special case

**Conjecture 1** *Let $(M, \omega)$ be a complete Kähler manifold which admits a regular quantization. Then $(M, \omega)$ is simply connected and homogeneous.*

The first explicit calculation of the function epsilon in the case when it is not constant is given in [26] for the regularized Kepler manifold $(X, \Omega)$ (see 2.0.3 and 3.5.7).

In Section 4.6 and 4.7 we shall calculate the function $\epsilon_{(L^k, h^k)}$ for the complex tori and the Riemann surfaces.

## 4.4 Some examples

**Example 4.4.1** *Let $k$ and $N$ be natural numbers and*

$$\omega = k\Omega_0^N = \frac{ki}{2}\sum_{j=1}^{N} dz^j \wedge d\bar{z}^j$$

*the flat Kähler form on $\mathbb{C}^N$. The trivial bundle on $\mathbb{C}^N$, endowed with the hermitian structure*

$$h^k(z,w) = e^{-k\pi\|z\|^2}|t|^2,\ \forall z \in \mathbb{C}^N, \forall t \in \mathbb{C},$$

*defines a geometric quantization of $(\mathbb{C}^N, k\Omega_0^N)$. Let $(\mathcal{H}_{h^k}, \langle\,\cdot\,,\,\cdot\,\rangle_{h^k})$ be the complex Hilbert space consisting of holomorphic functions $s$ on $\mathbb{C}^N$ which are bounded w.r.t.*

$$\|s\|^2_{h^k} = \langle s, s\rangle_{h^k} = \int_{\mathbb{C}^N} e^{-k\pi\|z\|^2} s(z)\bar{s}(z)\frac{\omega^N}{N!}.$$

*It is well-known and not hard to verify that $s_j := z_1^{j_i}\cdots z_N^{j_N}$ is a orthogonal basis for $(\mathcal{H}_{h^k}, \langle\,\cdot\,,\,\cdot\,\rangle_{h^k})$. Furthermore,*

$$\begin{aligned}\|s_j\|^2_{h^k} &= \int_{\mathbb{C}^N} e^{-k\pi\|z\|^2}|z_1|^{2j_i}\ldots|z_N|^{2j_N}\frac{\omega^N}{N!}\\ &= \frac{k^N}{N!}\prod_{l=1}^{N}\int_{\mathbb{C}} e^{-k\pi|z_l|^2}|z_l|^{2j_l}\frac{i^N}{2^N}dz_l\wedge d\bar{z}_l.\end{aligned}$$

*The last expression can be written in polar coordinates $(r_l, \theta_l)$ as*

$$\begin{aligned}\frac{k^N}{N!}\prod_{l=1}^{N}\int_{r_l=0}^{+\infty} e^{-k\pi r_l^2} r_l^{2j_l} 2r_l dr_l &= \frac{k^N}{N!}\prod_{l=1}^{N}\int_{\rho_l=0}^{+\infty} e^{-k\pi\rho_l}\rho_l{}^{j_l}d\rho_l\\ &= \frac{k^N}{N!}\prod_{l=1}^{N}\frac{j_l!}{(k\pi)^{j_l+1}} = \frac{k^N}{N!}\prod_{l=1}^{N}\frac{j_l!}{(k\pi)^{j_l}}.\end{aligned}$$

*Therefore*

$$\|s_j\|^2_{h^k} = \frac{k^N}{N!}\prod_{l=1}^{N}\frac{j_l!}{(k\pi)^{j_l}}.$$

*By (4.4) one obtains:*

$$\epsilon_{k\Omega_0^N}(z) = \frac{N!}{K^N}e^{-k\pi\|z\|^2}\prod_{l=1}^{N}\sum_{j_l=0}^{+\infty}\frac{(k\pi)^{j_l}}{j_l!}|z_l|^{2j_l} = \frac{N!}{K^N}.$$

*It is not difficult to see that the coherent states map is given by*

$$(z_1,\ldots,z_N) \to [(\ldots,\sqrt{\frac{k\pi^{|m_j|}}{m_j!}}z^{m_j},\ldots)] \tag{4.10}$$

*(see (3.15)).*

**Example 4.4.2** *This Example is taken from [7] and it will be used later to calculate the function epsilon for the Riemann surfaces. Let $k$ be a natural number and let $k\omega_{hyp}$ be the Kähler form on the unit disk $\mathbb{D}$ (see 2.0.2). The trivial line bundle $\mathbb{D} \times \mathbb{C} \to \mathbb{D}$ on $\mathbb{D}$, endowed with the hermitian structure*

$$h^k(z, t) = (1 - |z|^2)^{2k}|t|^2, \ \forall z \in \mathbb{D}, \forall t \in \mathbb{C},$$

*defines a geometric quantization of $(\mathbb{D}, k\omega_{hyp})$. Here we want to calculate the function $\epsilon_{k\omega_{hyp}}$ which, from 4.1.4, it is known to be a constant. Let $\sigma : \mathbb{D} \to \mathbb{D} \times \mathbb{C}^*$ be the holomorphic section of the trivial bundle on $\mathbb{D}$ which maps $z$ to $(z, 1)$. Let $w \in \mathbb{D}$ and $e^k_{\sigma(w)}$ the coherent state relative to the point $\sigma(w)$, i.e. the holomorphic function on $\mathbb{D}$ such that*

$$s(w) = \langle s, e^k_{\sigma(w)} \rangle_{h^k} \sigma(w), \ \forall s \in \mathcal{H}_{h^k}. \tag{4.11}$$

*By (4.2) and (4.11),*

$$\epsilon_{k\omega_{hyp}} = \|e^k_{\sigma(w)}\|^2_{h^k} h^k(\sigma(w), \sigma(w)) = \|e^k_{\sigma(w)}\|^2_{h^k}(1 - |w|^2)^{2k}.$$

*By extending the previous expression holomorphically in $z$ and antiholomorphically in $w$, one obtains:*

$$\langle e^k_{\sigma(w)}, e^k_{\sigma(z)} \rangle_{h^k} = \epsilon_{k\omega_{hyp}}(1 - z\bar{w})^{-2k}.$$

*By (4.11),*

$$e^k_{\sigma(w)}(z) = \langle e^k_{\sigma(w)}, e^k_{\sigma(z)} \rangle_{h^k} \sigma(z) = \epsilon_{k\omega_{hyp}}(1 - z\bar{w})^{-2k}\sigma(z). \tag{4.12}$$

*For $w = 0$ the previous expression becomes*

$$e^k_{\sigma(0)}(z) = \epsilon_{k\omega_{hyp}}\sigma(z), \ \forall z \in \mathbb{D}.$$

*Once again, by (4.11), $\sigma(0) = \langle \sigma, e^k_{\sigma(0)} \rangle_{h^k} \sigma(0)$. Therefore*

$$\begin{aligned} 1 &= \langle \sigma, e^k_{\sigma(0)} \rangle_{h^k} = \int_{\mathbb{D}} \epsilon_{k\omega_{hyp}} h^k(\sigma(z), \sigma(z)) \frac{ik}{\pi} \frac{dz \wedge d\bar{z}}{(1 - |z|^2)^2} \\ &= \frac{2k}{\pi} \epsilon_{k\omega_{hyp}} \int_{\mathbb{D}} (1 - |z|^2)^{2k-2} \frac{i}{2} dz \wedge d\bar{z}. \end{aligned}$$

*The last expression can be written in polar coordinates $(r, \theta)$ as follows:*

$$1 = 2k\epsilon_{k\omega_{hyp}} \int_0^1 (1-r^2)^{2k-2} 2r dr = 2k\epsilon_{k\omega_{hyp}} \int_0^1 (1-\rho)^{2k-2} \rho d\rho = \frac{2k}{2k-1}\epsilon_{k\omega_{hyp}}.$$

*Therefore*

$$\epsilon_{k\omega_{hyp}} = \frac{2k-1}{2k},$$

*and from (4.12) one concludes*

$$e^k_{\sigma(w)}(z) = \frac{2k-1}{2k}(1-z\bar{w})^{-2k}\sigma(z). \tag{4.13}$$

**Example 4.4.3** *Let $k$ and $N$ be natural numbers and $\omega = k\Omega^N_{FS}$ $k$-times the Fubini–Study form on $\mathbb{P}^N(\mathbb{C})$. The $k$-th tensor power $\mathcal{O}_N(k)$ of the hyperplane bundle $\mathcal{O}_N(1)$, equipped with the hermitian structure $h^k$ described in 2.0.4, defines a geometric quantization of $(\mathbb{P}^N(\mathbb{C}), \omega)$ The quantization is regular in this case as well, by 4.1.4. By (4.5), we get*

$$\epsilon_\omega = dim\, H^0(\mathcal{O}_N(k))(vol\mathbb{P}^N(\mathbb{C}))^{-1} = \binom{N+k}{N}(vol\mathbb{P}^N(\mathbb{C}))^{-1},$$

*where*

$$vol(\mathbb{P}^N(\mathbb{C})) = k^N \int_{\mathbb{P}^N(\mathbb{C})} \frac{\omega^N}{N!}.$$

*Furthermore, by 4.2.3 the coherent states map is given by (3.18).*

**Example 4.4.4** *Let $D_F$ be an Hartogs domain in $\mathbb{C}^2$, equipped with the Kähler form $\omega_F$ (see 3.6). The trivial line bundle $D_F \times \mathbb{C} \to D_F$ over $D_F$, endowed with the hermitian structure*

$$h(z,t) = (F(|z_1|^2) - |z_2|^2)|t|^2,\ \forall z = (z_1, z_2) \in D_F, \forall t \in \mathbb{C},$$

*is a geometric quantization of $(D_F, \omega_F)$. In [11, p. 448] the function $\epsilon_{\omega_F}$ is calculated in terms of the function $F$. The only known cases when $\epsilon_{\omega_F}$ is constant are obtained for $F(x) = 1 - x$ and thus $(D_{1-x}, \omega_{1-x})$ is a homogeneous Kähler manifold in accord with Conjecture 1 (see also question at the end of p. 477 in [11]).*

## 4.5 The function epsilon for $(\mathbb{C}^*, \omega^*)$

Let $exp : \mathbb{C} \to \mathbb{C}^*$ be the map defined by $exp(z) := e^z$ and $\omega^*$ the Kähler form on $\mathbb{C}^*$ which satisfies

$$exp^*\omega^* = \Omega_0^1 = \frac{i}{2} dz \wedge d\bar{z}.$$

It follows easily that

$$\omega^* = \frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2}.$$

In this Section we describe explicitly the set $\mathcal{L}_{hol}(\mathbb{C}^*, \omega^*)$ and how the function $\epsilon_{(L,h)}$ varies with $(L, h)$ in $\mathcal{L}_{hol}(\mathbb{C}^*, \omega^*)$. From Theorem 1.4.6, the group

$$\mathcal{L}_{hol}(\mathbb{C}^*, 0) \cong \mathrm{Hom}(\pi_1(\mathbb{C}^*), S^1) = \mathrm{Hom}(S^1, S^1) \cong S^1$$

acts simply transitively on $\mathcal{L}_{hol}(\mathbb{C}^*, \omega^*)$. On the other hand, any holomorphic line bundle $L$ over $\mathbb{C}^*$ is holomorphically trivial. Therefore, any pair $(L, h)$ in $\mathcal{L}_{hol}(\mathbb{C}^*, \omega^*)$ can be seen as the trivial line bundle $L_0 := \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^*$, endowed with a hermitian structure $h$ satisfying

$$\mathrm{curv}(L_0, h) = -2\pi i \omega^*.$$

Define an action of $\mathbb{R}$ on $\mathcal{L}_{hol}(\mathbb{C}^*, \omega^*)$ by

$$\lambda \cdot (L_0, h) = (L_0, h_\lambda), \ \forall \lambda \in \mathbb{R}, \tag{4.14}$$

where

$$h_\lambda(z, t) = |z|^{2\lambda} h(z, t), \ \forall z \in \mathbb{C}^*, \forall t \in \mathbb{C}. \tag{4.15}$$

Let $\mu$ be a real number such that $\lambda - \mu \in \mathbb{Z}$. It is easy to see that the map

$$\psi : (L_0, h_\mu) \to (L_0, h_\lambda) : (z, t) \mapsto (z, z^{\nu - \lambda} t)$$

is a holomorphic automorphism of the trivial bundle and $\psi^*(h_\lambda) = h_\nu$, i.e.

$$[(L_0, h_\mu)]_{hol} = [(L_0, h_\lambda)]_{hol}.$$

Furthermore, if $\lambda - \mu \notin \mathbb{Z}$ then $[(L, h_\lambda)]_{hol} \neq [(L, h_\mu)]_{hol}$. To see this, suppose that $\psi : L_0 \to L_0$ is a holomorphic automorphism of the trivial bundle, such that $\psi^* h_\lambda = h_\mu$.

It follows that $\psi(z,t) = (z, f(z)t)$, where $f$ is a holomorphic function on $\mathbb{C}^*$, satisfying $|f(z)|^2 = |z|^{2(\mu-\lambda)}$. This is impossible unless $\lambda - \mu$ is an integer.

Let fix a reference pair $(L_0, h_0) \in \mathcal{L}_{hol}(\mathbb{C}^*, \omega^*)$, where

$$h_0(z,t) = e^{\frac{-\pi}{2}\log^2|z|^2}|t|^2,\ \forall z \in \mathbb{C}^*, \forall t \in \mathbb{C}.$$

Our result can be then summarized as follows:

**Proposition 4.5.1** *There exists a bijection between the set of equivalence classes in* $\mathcal{L}_{hol}(\mathbb{C}^*, \omega^*)$ *and* $S^1 = \mathbb{R}/\mathbb{Z}$. *Moreover, every class in* $\mathcal{L}_{hol}(\mathbb{C}^*, \omega^*)$ *can be represented by a pair* $(L_0, h_{0\lambda})$, *where* $L_0$ *is the trivial line bundle of* $\mathbb{C}^*$ *and*

$$h_{0\lambda}(z,t) := e^{\frac{-\pi}{2}\log^2|z|^2}|z|^{2\lambda}|t|^2,\ \forall z \in \mathbb{C}^*, \forall t \in \mathbb{C}. \tag{4.16}$$

Let $k$ be a natural number and consider the Kähler form $k\omega^*$ on $\mathbb{C}^*$. The trivial line bundle $L_0$ on $\mathbb{C}^*$, endowed with the hermitian structure

$$h_0^k(z,t) = e^{\frac{-k\pi}{2}\log^2|z|^2}|t|^2,\ \forall z \in \mathbb{C}^*, \forall t \in \mathbb{C},$$

defines a geometric quantization of $(\mathbb{C}^*, k\omega^*)$. As before, one can deduce that there exists a bijection between $\mathcal{L}_{hol}(\mathbb{C}^*, k\omega^*)$ and $S^1 = \mathbb{R}/\mathbb{Z}$ and each class in $\mathcal{L}_{hol}(\mathbb{C}^*, k\omega^*)$ can be represented by a pair $(L_0, h_{0\lambda}^k)$, where

$$h_{0\lambda}^k(z,t) := e^{\frac{-k\pi}{2}\log^2|z|^2}|z|^{2\lambda}|t|^2,\ \forall z \in \mathbb{C}^*, \forall t \in \mathbb{C}, \tag{4.17}$$

for $\lambda \in \mathbb{R}$.

Let $\mathcal{H}_{h_{0\lambda}^k}$ be the space of holomorphic functions $f$ in $\mathbb{C}^*$ such that

$$\|f\|^2_{h_{0\lambda}^k} = \langle f, f\rangle_{h_{0\lambda}^k} = \int_{\mathbb{C}^*} e^{\frac{-k\pi}{2}\log^2|z|^2}|z|^{2\lambda}|f(z)|^2 k\frac{i}{2}\frac{dz\wedge d\bar{z}}{|z|^2} < +\infty.$$

One can check that the functions $z^j$, with $j \in \mathbb{Z}$, form an orthogonal system for $(\mathcal{H}_{h_{0\lambda}^k}, \langle\cdot,\cdot\rangle_{h_{0\lambda}^k})$. Since every holomorphic function in $\mathbb{C}^*$ can be expanded in Laurent series, it follows that $z^j$ are in fact an orthogonal basis for $(\mathcal{H}_\lambda^k, \langle\cdot,\cdot\rangle_{h_{0\lambda}^k})$. Their norms are given by

$$\begin{aligned} \|z^j\|^2_{h_{0\lambda}^k} &= k\int_{\mathbb{C}^*} e^{\frac{-k\pi}{2}\log^2|z|^2}|z|^{2(j+\lambda)}\frac{i}{2}\frac{dz\wedge d\bar{z}}{|z|^2} \\ &= k\pi\int_0^{+\infty} e^{\frac{-k\pi}{2}\log^2 r^2} r^{2(j+\lambda)}\frac{2r}{r^2}dr. \end{aligned}$$

By the change of variable $e^\rho = r^2$ one obtains:

$$\begin{aligned}\|z^j\|^2_{h^k_{0\lambda}} &= k\pi\int_{-\infty}^{+\infty} e^{\frac{-k\pi}{2}\rho^2}e^{(j+\lambda)\rho}d\rho = k\pi e^{\frac{(j+\lambda)^2}{2k\pi}}\int_{-\infty}^{+\infty} e^{-\left(\sqrt{\frac{k\pi}{2}}\rho-\sqrt{\frac{1}{2k\pi}}(j+\lambda)\right)^2}\\ &= k\pi e^{\frac{(j+\lambda)^2}{2k\pi}}\sqrt{\frac{2}{k\pi}}\int_{-\infty}^{+\infty}e^{-t^2}dt = \sqrt{2k}\pi e^{\frac{(j+\lambda)^2}{2k\pi}},\end{aligned}$$

where we have used the well-known Gauss integral $\int_{-\infty}^{+\infty} e^{-t^2}dt = \sqrt{\pi}$. Hence, by (4.4),

$$\epsilon_{(L^k,h^k_{0\lambda})}(z) = \frac{e^{\frac{-k\pi}{2}\log^2|z|^2}|z|^{2\lambda}}{\sqrt{2k}\pi}\sum_{j=-\infty}^{+\infty} e^{-\frac{(j+\lambda)^2}{2k\pi}}|z|^{2j}. \tag{4.18}$$

One can immediately check that (4.18) depends, in fact, on the class $[\lambda]\in S^1$, that is if $\mu\in\mathbb{R}$ is such that $\lambda-\mu\in\mathbb{Z}$, then $\epsilon_{(L^k,h^k_{0\lambda})}(z) = \epsilon_{(L^k,h^k_{0\mu})}(z)$.

## 4.6 The function epsilon for Complex tori

We refer to 2.1.1 for the material contained in this Section. Let

$$\Lambda = \{p+iq \mid\ p,q\in\mathbb{Z}\}$$

be the lattice in $\mathbb{C}$ generated by $(1,0)$ and $(0,1)$ and $\mathbb{C}/\Lambda$ be the 1-dimensional complex torus. Let $H(z,w) = z\bar{w}$ be the standard hermitian form on $\mathbb{C}$ and

$$\Omega^1_0 = \frac{i}{2}\partial\bar{\partial}|z|^2 = \frac{i}{2}dz\wedge d\bar{z}$$

the flat Kähler form on $\mathbb{C}/\Lambda$. A simple calculation shows that

$$ImH(\lambda,\mu) = mq-pn,\ \forall\lambda = p+iq, \mu = m+in,$$

i.e. $H$ is integral on the lattice. Let $\chi:\Lambda\to S^1$ be defined by

$$\chi(\lambda) = e^{i\pi pq},\ \forall\lambda = p+iq\in\Lambda.$$

It is immediate to verify that

$$\chi(\lambda_1+\lambda_2) = \chi(\lambda_1)\chi(\lambda_2)e^{i\pi ImH(\lambda_1,\lambda_2)},\ \forall\lambda_1,\lambda_2\in\Lambda.$$

Therefore $\chi$ is a semicharacter associated to $H$. Consider the factor of automorphy

$$A(\lambda,z) = \chi(\lambda)e^{\pi z\bar{\lambda}+\frac{\pi}{2}|\lambda|^2}$$

and the holomorphic line bundle $L = L(H, \chi)$, endowed with the hermitian structure

$$h(\theta(z), \theta(z)) = e^{-\pi|z|^2}|\theta(z)|^2,$$

where $\theta$ is a global holomorphic section of $L$. The space $H^0(L)$ of global holomorphic sections of $L$ can be realized as the space of holomorphic functions on $\mathbb{C}$ satisfying the functional equation

$$\theta(z+\lambda) = A(\lambda, z)\theta(z) = \chi(\lambda)e^{\pi z\bar{\lambda}+\frac{\pi}{2}|\lambda|^2}\theta(z), \ \forall\lambda \in \Lambda.$$

The Riemann-Roch theorem tells us that the space $H^0(L)$ is 1-dimensional (see [12]). Furthermore, the function

$$\theta(z) = \sum_{m\in\mathbb{Z}} e^{\frac{\pi}{2}z^2}e^{(-\pi m^2+2\pi imz)}$$

is a generator of $H^0(L)$ (see [24, p.1] for a proof). One can check (see the calculation below) that

$$\|\theta\|_h^2 = \langle\theta, \theta\rangle_h = \int e^{-\pi|z|^2}|\theta|^2\frac{i}{2}dz \wedge d\bar{z} = \frac{1}{\sqrt{2}}.$$

Thus, by (4.4), the function epsilon becomes

$$\epsilon_{(L,h)}(z) = \sqrt{2}e^{-\pi|z|^2}|\theta(z)|^2.$$

For every natural number $k$ consider $(\mathbb{C}/\Lambda, k\Omega_0^1)$ and the factor of automorphy

$$A^k(\lambda, z) = e^{k\pi pqi}e^{k\pi z\bar{\lambda}+\frac{k\pi}{2}|\lambda|^2}.$$

The global holomorphic sections of $L^k$, can be seen as the holomorphic functions $\theta$ on $\mathbb{C}$ satisfying

$$\theta(z+\lambda) = A^k(\lambda, z)\theta(z), \ \forall\lambda \in \Lambda. \tag{4.19}$$

Define the hermitian structure $h^k$ on $L^k$ by

$$h^k(\theta(z), \theta(z)) = e^{-k\pi|z|^2}|\theta(z)|^2, \forall\theta \in H^0(L^k).$$

Since $\mathrm{curv}(L^k, h^k) = -2\pi ki\Omega_0^1$, it follows that the couple $(L^k, h^k)$ is a geometric quantization of $(\mathbb{C}/\Lambda, k\Omega_0^1)$. By the Riemann-Roch theorem $H^0(L^k) = \mathcal{H}_{h^k}$ is $k$-dimensional.

For $j = 0, \ldots, k-1$ consider

$$\theta_j(z) = e^{k\frac{\pi}{2}z^2} \sum_{m\in\mathbb{Z}} e^{\frac{-\pi}{k}(km+j)^2 + 2\pi i(km+j)z}$$

It is easy to check that the functions $\theta_j$'s satisfy the functional equation (4.19). Futhermore

**Proposition 4.6.1** $\left((\frac{2}{k})^{\frac{1}{4}}\theta_0, \ldots, (\frac{2}{k})^{\frac{1}{4}}\theta_{k-1}\right)$ *form a unitary basis for* $(\mathcal{H}_{h^k}, \langle\,\cdot\,,\,\cdot\,\rangle_{h^k})$.

**Proof:** For $a, b = 0, 1, \ldots, k-1$

$$\begin{aligned}
&\langle\theta_a, \theta_b\rangle_{h^k} \\
&\quad = \sum_{m,p\in\mathbb{Z}} e^{\frac{-\pi}{k}((km+a)^2+(kp+b)^2)} \int_{\mathbb{C}/\Lambda} e^{-k\pi|z|^2} e^{\frac{k\pi}{2}(z^2+\bar{z}^2)} e^{2\pi i(km+a)z} e^{-2\pi i(km+a)\bar{z}} k\Omega_0^1.
\end{aligned}$$

If $z = x+iy$, the previous integral can be written as

$$\sum_{m,p\in\mathbb{Z}} e^{\frac{-\pi}{k}((km+a)^2+(kp+b)^2)} \int_0^1\int_0^1 e^{-2k\pi y^2} e^{2\pi i(k(m-p)+(a-b))x} e^{-2\pi(k(m+p)+(a+b))y} k dx\wedge dy.$$

Integrating with respect to $x$ we obtain

$$\int_0^1 e^{2\pi i(k(m-p)+(a-b))x} dx = \delta_{0k(m-p)+b-a} = \delta_{mp}\delta_{ab},$$

where the last equality follows from the fact that $b-a$ is divisible by $k$ if and only if $b = a$. Thus,

$$\langle\theta_a, \theta_b\rangle_{h^k} = k\delta_{ab} \sum_{m\in\mathbb{Z}} e^{\frac{-\pi}{k}((km+a)^2+(km+b)^2)} \int_0^1 e^{-2k\pi y^2} e^{-4\pi(km+\frac{a+b}{2})y} dy.$$

Therefore the $\theta_j$'s form an orthogonal basis for $(\mathcal{H}_{h^k}, \langle\,\cdot\,,\,\cdot\,\rangle_{h^k})$. For $a = b = j$ one gets:

$$\begin{aligned}
\|\theta_j\|^2_{h^k} &= k\int_0^1 e^{-2k\pi y^2} \sum_{m\in\mathbb{Z}} e^{\frac{-2\pi}{k}(km+j)^2} e^{-4\pi(km+j)y} dy \\
&= k\sum_{m\in\mathbb{Z}} \int_0^1 e^{-2k\pi(y+m+\frac{j}{k})^2} dy.
\end{aligned}$$

By the change of variable $t = y+m+\frac{j}{k}$ one obtains:

$$\|\theta_j\|^2_{h^k} = k\int_{-\infty}^{+\infty} e^{-2k\pi t^2} dt = \sqrt{\frac{k}{2}}.$$

□

By (4.4) and 4.6.1, the function epsilon can be calculated as

$$\epsilon_{(L^k,h^k)}(z) = e^{-k\pi|z|^2}\sqrt{\frac{2}{k}}\sum_{j=0}^{k-1}|\theta_j(z)|^2.$$

**Remark 4.6.2** *The previous calculation can be generalized to a $N$-dimensional complex abelian variety $(V/\Lambda, \Omega_0^N)$ (see [15, pp. 40-43]). Similar calculations can be found in ([4, Appendix b]), where the authors claim that $\epsilon_{(L^k,h^k)}$ is constant for all natural numbers $k$ ([4, p. 229]). This is clearly wrong, since $k\Omega_0^N$ cannot be projectively induced (see Section 4.3).*

## 4.7 The function epsilon for Riemann surfaces

Let $(\Sigma_g, \omega_{hyp})$ be a Riemann surface of genus $g$ greater or equal to 2. Let $(K, h)$ be the geometric quantization of $(\Sigma_g, \omega_{hyp})$ described in Section (2.1.2). For a natural number k, let $K^k$ be the $k$-th tensor power of $K$. A global holomorphic section of $K^k$ can be seen as a holomorphic function $s$ on $\mathbb{D}$ satisfying the functional equation

$$s(\gamma(z)) = (\gamma'(z))^{-k}s(z), \tag{4.20}$$

(see 2.14). Let $h^k$ be the hermitian structure on $K^k$ defined by

$$h^k(s(z), s(z)) = (1-|z|^2)^{2k}|s(z)|^2, \ \forall s \in H^0(L^k) = \mathcal{H}_{h^k}.$$

By (1.12), $\mathrm{curv}(L^k, h^k) = -2\pi i k\omega_{hyp}$, and so $(K^k, h^k)$ is a geometric quantization for $(\Sigma_g, k\omega_{hyp})$. In order to calculate the coherent states for the Hilbert space $(\mathcal{H}_{h^k}, \langle\cdot,\cdot\rangle_{h^k})$, let $w \in \mathbb{D}$ and

$$e^k_{\sigma(w)}(z) = \frac{2k-1}{2k}(1-z\bar{w})^{-2k}\sigma(z)$$

the coherent states for the unit disk, where $\sigma(z) = (z, 1)$, (see 4.13). Consider the series

$$\sum_{\gamma\in\Gamma}(1-\gamma(z)\bar{w})^{-2k}(\gamma'(z))^k. \tag{4.21}$$

Classical theorems going back to Poincare' (see [21, pp. 101-104]) assert that the series (4.21) converges almost uniformly for all $z \in \mathbb{D}$. Consider the holomorphic function on $\mathbb{D}$ defined by:

$$\tilde{e}^k_{\sigma(w)}(z) := \sum_{\gamma \in \Gamma} e^k_{\sigma(w)}(\gamma(z))(\gamma^{'}(z))^k. \tag{4.22}$$

It is easily seen that

$$\tilde{e}^k_{\sigma(w)}(\gamma(z)) = (\gamma^{'}(z))^{-k} \tilde{e}^k_{\sigma(w)}(z), \ \forall \gamma \in \Gamma.$$

This means that $\tilde{e}^k_{\sigma(w)}(z)$ can be identified with a global holomorphic section of $K^k$. Let $U$ be a fundamental domain in $\mathbb{D}$ for the action of $\Gamma$ and $s$ a holomorphic section for $K^k$. By (2.15), (4.11), (4.13) and (4.22) one obtains

$$\begin{aligned}
\langle s, \tilde{e}^k_{\sigma(w)} \rangle_{h^k} &= \int_{\Sigma_g} s(z) \overline{\tilde{e}^k_{\sigma(w)}}(z)(1-|z|^2)^{2k} k\omega_{hyp}(z) \\
&= \sum_{\gamma \in \Gamma} \int_U s(z) \overline{e^k_{\sigma(w)}}(\gamma(z))(\bar{\gamma}^{'}(z))^k (1-|z|^2)^{2k} k\omega_{hyp}(z) \\
&= \sum_{\gamma \in \Gamma} \int_U s(\gamma(z)) \overline{e^k_{\sigma(w)}}(\gamma(z))(1-|\gamma(z)|^2)^{2k} k\omega_{hyp}(z) \\
&= \int_{\mathbb{D}} s(z) \overline{e^k_{\sigma(w)}}(z)(1-|z|^2)^{2k} k\omega_{hyp}(z) = s(w),
\end{aligned}$$

i.e. $\tilde{e}^k_{\sigma(w)}$ is the coherent state relative to $\sigma(w)$. Therefore, by (4.2), it follows that:

$$\epsilon_{(K^k,h^k)}(z) = \|\tilde{e}^k_{\sigma(z)}\|^2_{h^k} h^k(\sigma(z), \sigma(z)) = \frac{2k-1}{2k}(1-|z|^2)^{2k} \sum_{\gamma \in \Gamma}(1-\gamma(z)\bar{z})^{-2k}(\gamma^{'}(z))^k.$$

## 4.8 The function epsilon and bounded domains

### 4.8.1 The Bergman metric

Let $M$ be a $n$-dimensional complex manifold and $K$ its canonical bundle, i.e. the holomorphic line bundle whose global holomorphic sections are the holomorphic $n$-forms on $M$. If $\alpha$ belongs to $H^0(K)$, then in a complex coordinate system $U$, endowed with local coordinates $(z_1, \ldots, z_n)$, there exists a holomorphic function $f_\alpha$ such that

$$\alpha(z) = f_\alpha(z) dz_1 \wedge \ldots \wedge dz_n, \ \forall z \in U. \tag{4.23}$$

Let $(w_1, \ldots, w_n)$ be a complex local coordinate system in an open set $V$ and $g_\alpha$ be holomorphic function on $V$ such that

$$\alpha(z) = g_\alpha(z) dw_1 \wedge \ldots \wedge dw_n.$$

On $U \cap V$

$$g_\alpha(z) = \det(\frac{\partial w_i}{\partial z_j}(z)) f_\alpha(z), \ \forall z \in V. \tag{4.24}$$

Consider the space $\mathcal{F}$ consisting of holomorphic $n$-forms $\alpha$ bounded with respect to

$$\|\alpha\|^2 = (\alpha, \alpha) := \frac{i^n}{2^n} \int_M \alpha \wedge \bar{\alpha} < \infty.$$

The space $\mathcal{F}$ turns out to be a separable complex Hilbert space with respect to the scalar product

$$(\alpha, \beta) = \frac{i^n}{2^n} \int_M \alpha \wedge \bar{\beta}, \tag{4.25}$$

(see [18]). Let $\alpha_j, j = 0, 1, \ldots N (N \leq \infty)$ be an unitary basis for $(\mathcal{F}, (\cdot, \cdot))$. Define a $2n$-form $K$ on $M \times M$ by

$$K(z, \bar{w}) = \frac{i^n}{2^n} \sum_{j=0}^{N} \alpha_j(z) \wedge \bar{\alpha}_j(w) = \frac{i^n}{2^n} \sum_{j=0}^{N} f_{\alpha_j}(z) \bar{f}_{\alpha_j}(w) d\mu(z, \bar{w}), \tag{4.26}$$

where

$$d\mu(z, \bar{w}) = dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{w}_1 \wedge \ldots \wedge d\bar{w}_n. \tag{4.27}$$

It is immediate to check that this is independent of the choice of the unitary basis. We call $K$ *the Kernel form* of $M$. Define the $2n$-form on $M$

$$K(z, \bar{z}) = \frac{i^n}{2^n} \sum_{j=0}^{N} \alpha_j(z) \wedge \bar{\alpha}_j(z) = \frac{i^n}{2^n} K^*(z, \bar{z}) d\mu(z, \bar{z}), \tag{4.28}$$

where $K^*$ is the real analytic function on $U$ given by

$$K^*(z, \bar{z}) = \sum_{j=0}^{N} f_{\alpha_j}(z) \bar{f}_{\alpha_j}(z). \tag{4.29}$$

Suppose now that the following conditions are satisfied:

(A.1) for every $x \in M$ there exists $\alpha \in \mathcal{F}$ such that $\alpha(x) \neq 0$;

(A.2) for every $x \in M$ and every tangent vector

$$Z = \sum_{k=1}^{n} a_k \frac{\partial}{\partial z_k} \mid_x, \; a_k \in \mathbb{C}$$

at $x$, there exists $\alpha \in \mathcal{F}$ such that $\alpha(x) = 0$ and $Z(f_\alpha) \neq 0$.

If $(A.1)$ is satisfied, then $K^*$ is a real analytic positive function. One can then consider the form on $M$ defined by:

$$\omega_B = \frac{i}{2\pi} \partial\bar{\partial} \log K^*.$$

One can show (see [18, Theorem 3.1]) that $(A.1)$ implies that $\omega_B$ is semipositive definite. Furthermore $(A.2)$ turns out to be equivalent to the fact that $\omega_B$ is a Kähler form on $M$. This form is called the *Bergman metric.*

**Example 4.8.1** *Let $D \subset \mathbb{C}^N$ be a bounded domain. The canonical bundle over $D$ is trivial and the space $(\mathcal{F}, (\cdot, \cdot))$ can be identified with the space of holomorphic functions on $M$ bounded w.r.t. to the Lebesgue measure. Hence, in this case, the expression (4.26) equals the reproducing kernel for $(\mathcal{F}, (\cdot, \cdot))$ (we refer to [14, p. 364] for details).*

**Example 4.8.2** *Let $M$ be a compact hypersurface of degree $d > n + 2$ in $\mathbb{P}^N(\mathbb{C})$. In this case conditions $(A.1)$ and $(A.2)$ are satisfied. (see [18, pp. 287-288] for details).*

Suppose now that the group $\mathrm{Aut}(M)$, acts transitively on $M$. Let $f$ be the function on $M$ such that $\omega_B^n = fK$. Since both $\omega_B$ and $K$ are invariant by $\mathrm{Aut}(M)$ (see [18] for a proof), it follows that $f$ is an invariant under the action of $\mathrm{Aut}(M)$, and hence it is a constant, say $\lambda$. Let

$$\omega_B = \frac{i}{2\pi} \sum_{j,\bar{k}=1}^{n} g_{j\bar{k}} dz_j \wedge d\bar{z_{\bar{k}}}$$

be the expression of the Bergman metric in local coordinates $(z_1, \ldots, z_n)$. It follows that

$$\omega_B^n = \frac{i^n}{(2\pi)} \det(g_{j\bar{k}}) d\mu, \qquad (4.30)$$

where

$$d\mu = dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z_1} \wedge \ldots \wedge d\bar{z_n}.$$

By (4.26) one obtains

$$\det(g_{j\bar{k}}) = \pi^n \lambda K^*. \tag{4.31}$$

In particular,

$$\rho_{\omega_B} = -\frac{i}{2\pi}\partial\bar{\partial}\log\det(g_{j\bar{k}}) = -\frac{i}{2\pi}\partial\bar{\partial}\log K^* = -\omega_B, \tag{4.32}$$

where $\rho_{\omega_B}$ is the Ricci form (see 3.5.9). Therefore (see [18, p. 274]):

**Theorem 4.8.3** *Let $M$ be a complex manifold such that $(A.1)$ and $(A.2)$ are satisfied. Suppose that $Aut(M)$ acts transitively on $M$, i.e. $(M, \omega_B)$ is a homogeneous Kähler manifold. Then $\omega_B$ is Kähler-Einstein with scalar curvature $\lambda = -1$.*

### 4.8.2 H.i.i. in complex projective spaces

Here we want to prove that the conditions $(A.1)$ and $(A.2)$ of the previous Section are equivalent to the fact that $\omega_B$ is projectively induced. This fact is stated in [16, p. 283] without a proof. We shall give a proof of this fact for later use. If $(A.1)$ holds then, by (4.24), the map

$$j : M \to \mathbb{P}(\mathcal{F}^*)$$

given by

$$j(x)(\alpha) := f_\alpha(x), \ \forall \alpha \in \mathcal{F} \tag{4.33}$$

is well defined.

**Proposition 4.8.4** *The map $j$ is a full holomorphic immersion if and only if condition $(A.2)$ holds. Furthermore, under this hypothesis, $j^*(\Omega_{\mathcal{F}^*}) = \omega_B$, where $\Omega_{\mathcal{F}^*}$ is the Fubini-Study form on $\mathbb{P}(\mathcal{F}^*)$.*

**Proof:** Let $U \subset M$ be an open set of $M$ and $\tilde{\jmath} : U \to \mathbb{P}(\mathcal{F}^*)$ a lift of $j$, i.e. a holomorphic map such that $p \circ \tilde{\jmath} = j$, where $p : \mathcal{F}^* \backslash 0 \to \mathbb{P}(\mathcal{F}^*)$ denotes the standard projection map. The tangent space of $\mathcal{F}^* \backslash 0$ at a point $\alpha^*$ can be naturally identified with $\mathcal{F}^*$. Moreover, the tangent space of $\mathbb{P}(\mathcal{F}^*)$ at a point $[\alpha^*]$ can be seen as the subspace $\mathcal{F}^*_{\alpha^*}$ of $\mathcal{F}^*$ consisting of those $\beta^*$ orthogonal to $\alpha^*$ (see [18, pp. 280-281]). As for the finite dimensional case, the differential of the map $p$ at a point $\alpha^*$, $dp_{\alpha^*} : \mathcal{F} \to T_{p(\alpha^*)}\mathbb{P}(\mathcal{F}^*) \cong \mathcal{F}^*_{\alpha^*}$, is given by the projection on $\mathcal{F}^*_{\alpha^*}$, i.e.

$$dp_{\alpha^*}(\beta^*) = \beta^* - (\beta^*, \alpha^*)^* \frac{\alpha^*}{\|\alpha^*\|^2}, \ \forall \beta^* \in \mathcal{F}^*, \tag{4.34}$$

where $\|\alpha^*\|^2 = (\alpha^*, \alpha^*)^*$ and $(\,\cdot\,,\,\cdot\,)^*$ is the natural scalar product in $\mathcal{F}^*$ induced by $(\,\cdot\,,\,\cdot\,)$. In order to calculate the differential $dj_x(Z)$ of the map $j$ at a point $x \in M$ applied to a tangent vector $Z$ at $x$, we need to specify its value at $\alpha \in \mathcal{F}$. We claim that:

$$dj_x(Z)(\alpha) = Z(f_\alpha) - \frac{(d\tilde{\jmath}_x(Z), \tilde{\jmath}(x))^*}{\|\tilde{\jmath}(x)\|^2} f_\alpha(x). \tag{4.35}$$

It is not hard to see, using formula (4.35), that $j$ is a holomorphic immersion if and only if $(A.2)$ is satisfied. This proves the first part of the Proposition.

In order to prove (4.35), let $\gamma(t)$ be a curve in $U$ with $\gamma(0) = x$ and $\frac{d\gamma}{dt}\mid_0 = Z$. By (4.33)

$$d\tilde{\jmath}_x(Z)(\alpha) = \frac{d}{dt}(\tilde{\jmath}(\gamma(t))\alpha)\mid_{t=0} = \frac{d}{dt} f_\alpha(\gamma(t))\mid_{t=0} = Z(f_\alpha).$$

Therefore, by (4.34),

$$dj_x(Z)(\alpha) = dp_{\tilde{\jmath}(x)} d\tilde{\jmath}_x(Z)(\alpha) = Z(f_\alpha) - \frac{(d\tilde{\jmath}_x(Z), \tilde{\jmath}(x))^*}{\|\tilde{\jmath}(x)\|^2} \tilde{\jmath}(x)(\alpha),$$

and so the claim.

In order to prove the second part of the Proposition, let

$$\tilde{b} : \mathbb{P}(\mathcal{F}^*) \to \mathbb{P}^N(\mathbb{C}) : \alpha^* \mapsto [(\ldots, \alpha^*(\alpha_j), \ldots)], j = 0, \ldots N,$$

where $N + 1$ is the complex dimension of $\mathcal{F}$. By (4.33),

$$(\tilde{b} \circ j)(x) = [(\ldots, f_{\alpha_j}(x), \ldots)], j = 0, 1, \ldots N, \tag{4.36}$$

which establishes that $j$ is a full holomorphic map. Furthermore, by the very definition of the Bergman metric

$$j^*\Omega_{\mathcal{F}^*} = j^*(\tilde{b}^*\Omega^N_{FS}) = (\tilde{b} \circ j)^*\Omega^N_{FS} = \frac{i}{2\pi}\partial\bar{\partial} log \sum_{j=0}^N |f_{\alpha_j}|^2 = \omega_B,$$

and this concludes the proof of the proposition. □

### 4.8.3 Geometric quantization

Let $M$ be a complex manifold such that $(A.1)$ and $(A.2)$ are satisfied. By (4.24) one can define a hermitian structure $h$ on $K$ by

$$h(\alpha, \alpha) = \frac{|f_\alpha|^2}{K^*}, \ \forall \alpha \in H^0(K). \tag{4.37}$$

By (1.12), one obtains:

$$\mathrm{curv}(K, h) = -\partial\bar{\partial} log \frac{1}{K^*} = -2\pi i \omega_B,$$

which shows that the pair $(K, h)$ is a geometric quantization for $(M, \omega_B)$.

Consider the complex Hilbert space $\mathcal{H}_h$ consisting of holomorphic $n$-forms bounded with respect to

$$\langle s, s \rangle_h = \|s\|_h^2 = \int_M h(s(x), s(x)) \frac{\omega_B^n(x)}{n!},$$

(see Section 4.1). It follows, by (4.4), that

$$\epsilon_{(K,h)} = \sum_{j=0}^{N} \frac{|f_{s_j}|^2}{K^*}, \tag{4.38}$$

where $s_j = f_{s_j} dz_1 \wedge \ldots \wedge dz_n$ is a unitary basis for $(\mathcal{H}_h, \langle \cdot, \cdot \rangle_h)$.

In this Section we want to compare the two Hilbert spaces $(\mathcal{F}, (\cdot, \cdot))$ and $(\mathcal{H}_h, \langle \cdot, \cdot \rangle_h)$ with the help of the function $\epsilon_{(K,h)}$.

**Theorem 4.8.5** *Let $M$ be a $n$-dimensional complex manifold such that $(A.1)$ and $(A.2)$ hold. Suppose that the following conditions are satisfied:*

*(i) the complex dimension of $\mathcal{F}$ is equal to the complex dimension of $\mathcal{H}_h$;*

*(ii) there exists a positive constant $\lambda$ such that $\frac{\omega_B^n}{n!} = \lambda K$.*

*Then $\epsilon_{(K,h)} = \frac{1}{\lambda}$.*

**Proof:** If (ii) holds, then, by (4.26),

$$h(s, s) \frac{\omega_B^n}{n!} = \frac{|f_s|^2 \lambda K}{K^*} d\mu = \lambda |f_s|^2 \frac{i^n}{2^n} d\mu, \ \forall s \in H^0(K).$$

Thus

$$\langle s, s\rangle_h = \int_M h(s(x), s(x)) \frac{\omega_B^n(x)}{n!} = \lambda \int_M |f_s|^2 \frac{i^n}{2^n} d\mu = \lambda(s, s), \ \forall s \in H^0(K). \qquad (4.39)$$

Suppose that (i) holds. Let $N + 1 = \dim \mathcal{H}_h = \dim \mathcal{F}$ and $\alpha_j, j = 0, 1, \dots N$ a unitary basis for $(\mathcal{F}, (\cdot, \cdot))$. From (4.39) it follows that

$$C f_{s_j} = f_{\alpha_j},$$

where $|C|^2 = \lambda$. Hence, by (4.38),

$$\epsilon_{(K,h)} = \frac{\sum_{j=0}^N |f_{s_j}|^2}{K^*} = \frac{\sum_{j=0}^N |f_{s_j}|^2}{\sum_{j=0}^N |f_{\alpha_j}|^2} = \frac{1}{\lambda}.$$

□

**Remark 4.8.6** *It is not clear to us if the converse of Theorem 4.8.5 is true, since we do not know whether* $\langle \cdot\, , \cdot \rangle_h = \lambda(\cdot, \cdot)$ *implies* $\frac{\omega_B^n}{n!} = \lambda K$.

**Remark 4.8.7** *According with conjecture 1, we believe that if* $(M, \omega_B)$ *is a complete Kähler manifold and conditions (i) and (ii) of 4.8.5 are satisfied, then* $(M, \omega_B)$ *is a homogeneous Kähler manifold.*

By (4.32) we know that, if condition (ii) in 4.8.5 is satisfied, then $\omega_B$ is Kähler-Einstein. When $M$ is compact, it is not hard to see, using Lemma (1.4.8), that the converse also holds. Therefore, by the previous theorem, it follows:

**Corollary 4.8.8** *Let* $M$ *be a compact complex manifold such that* $(A.1)$ *and* $(A.2)$ *are satisfied. Suppose that (i) of 4.8.5 holds and* $\omega_B$ *is Kähler-Einstein. Then* $\epsilon_{(k,h)}$ *is a constant.*

**Theorem 4.8.9** *Let* $M$ *be a complex manifold such that* $(A.1)$ *and* $(A.2)$ *are satisfied. Then* $\epsilon_{(K,h)}$ *equals the constant* $\frac{1}{\lambda}$ *if and only if the complex dimensions of* $\mathcal{H}_h$ *and* $\mathcal{F}$ *are the same and* $\langle \cdot\, , \cdot \rangle_h = \lambda(\cdot, \cdot)$.

**Proof:** Suppose that $\dim \mathcal{H}_h = \dim \mathcal{F}$ and $\langle \cdot , \cdot \rangle_h = \lambda(\cdot,\cdot)$. Then, from the proof of Theorem 4.8.5, $\epsilon_{(K,h)} = \frac{1}{\lambda}$ (see (4.39)). Conversely, suppose that $\epsilon_{(K,h)} = \frac{1}{\lambda}$. From 4.2.2 and (4.9) the coherent states map

$$b \circ \phi_{(L,h)} : (M, \omega_B) \to (\mathbb{P}^{N'}(\mathbb{C}), \Omega_{FS}^{N'}) : x \mapsto [(\ldots, f_{s_k}, \ldots)], k = 0, 1, \ldots N',$$

where $N' + 1 = \dim \mathcal{H}_h$, is a full h.i.i. in $(\mathbb{P}^{N'}(\mathbb{C}), \Omega_{FS}^{N'})$. On the other hand, from 4.8.4

$$\tilde{b} \circ j : (M, \omega_B) \to (\mathbb{P}^{N}(\mathbb{C}), \Omega_{FS}^{N}) : x \mapsto [(\ldots, f_{\alpha_j}(x), \ldots)],\ j = 0, 1, \ldots N,$$

where $N + 1 = \dim \mathcal{F}$, is a full h.i.i. in $(\mathbb{P}^{N}(\mathbb{C}), \Omega_{FS}^{N})$ (see formula (4.36)). It follows from the Calabi's Rigidity Theorem 3.4.3 that $N = N'$, i.e. $\dim \mathcal{H}_h = \dim \mathcal{F}$ and there exist a $N + 1 \times N + 1$ unitary matrix $u_{jk}$ and a complex number $C$ such that

$$C \sum_{k=0}^{N} u_{jk} f_{s_k} = f_{\alpha_j}.$$

Thus, $\langle \cdot , \cdot \rangle_h = |C|^2(\cdot,\cdot)$. and, from the proof of the first part, $|C|^2 = \lambda$. □

## 4.9 Fixing the cohomology class

Let $(L_0, h_0)$ be a quantization of a compact Kähler manifold $(M, \omega_0)$. Suppose that $M$ is simply connected, so that the epsilon function $\epsilon_{\omega_0}$ depends only on the Kähler form $\omega_0$ and not on the pair $(L_0, h_0) \in \mathcal{L}_{hol}(M, \omega_0)$ chosen (cf. remark 4.1.2). Consider the space

$$\mathcal{C}_{\omega_0} = \{\omega \sim \omega_0 \mid \omega \text{ is Kähler}\}$$

of Kähler forms on $M$, cohomologous to $\omega_0$.

**Definition 4.9.1** *We say that a Kähler form $\omega$ on $M$ is N-projectively induced if there exists a full holomorphic immersion $\phi : M \to \mathbb{P}^N(\mathbb{C})$ such that $\phi^*(\Omega_{FS}^N) = \omega$.*

We denote by

$$\mathcal{P}_{\omega_0} = \{\omega \in \mathcal{C}_{\omega_0} \mid \omega \text{ is N-projectively induced }\}$$

the set of all $N$-projectively induced Kähler forms $\omega$ cohomologous to $\omega_0$.

**Definition 4.9.2** *We say that $A$ and $B$ in $PGL(N+1,\mathbb{C})$, the group of automorphism of $\mathbb{P}^N(\mathbb{C})$, are equivalent, and we write $A \sim B$, if and only if there exists $U \in PU(N+1)$ such that $A = UB$.*

Let $\mathrm{PGL}(N+1,\mathbb{C})/\!\sim$ be the corresponding quotient space. The following proposition shows that $\mathcal{P}_{\omega_0}$ is a "very small" subset of $\mathcal{C}_{\omega_0}$. We borrow a classical fact from algebraic geometry

**Lemma 4.9.3** *Let $M$ be a compact algebraic manifold and $\phi : M \to \mathbb{P}^r(\mathbb{C})$ a full holomorphic map. Then there exist $(s_0, \ldots, s_t)$, $t \leq r$, linearly independent holomorphic sections of $\phi^*(\mathcal{O}_r(1))$, such that*

$$\phi(x) = [(s_0(x), \ldots, s_r(x))],$$

*(see [12, pp. 176-177] and (2.9)).*

**Proposition 4.9.4** *Let $(M, \omega_0)$ be a simply connected Kähler manifold such that $\omega_0$ is $N$-projectively induced via a map $\phi_0$. Then there exists a bijection between $PGL(N+1,\mathbb{C})/\!\sim$ and $\mathcal{P}_{\omega_0}$.*

**Proof:** Let

$$\Phi : \mathrm{PGL}(N+1,\mathbb{C}) \to \mathcal{P}_{\omega_0}$$

be defined by $\Phi(A) = (A \circ \phi_0)^* \Omega_{FS}^N$ for $A \in \mathrm{PGL}(N+1,\mathbb{C})$. Since $\Omega_{FS}^N$ is invariant under the group $\mathrm{PU}(N+1)$, it follows that the map $\Phi$ gives rise to a map, denoted by the same symbol,

$$\Phi : \mathrm{PGL}(N+1,\mathbb{C})/\!\sim\, \to \mathcal{P}_{\omega_0}.$$

Let $\phi$ and $\psi$ be two full holomorphic maps such that $\phi^*\Omega_{FS}^N = \psi^*\Omega_{FS}^N = \omega$. From the Calabi's Rigidity Theorem 3.4.3, there exists $U \in \mathrm{PU}(N+1)$ such that $U \circ \phi = \psi$, which implies that $\Phi$ is injective. In order to prove the surjectivity of $\Phi$, let $\omega \in \mathcal{P}_{\omega_0}$ and $\phi : M \to \mathbb{P}^N(\mathbb{C})$ a full holomorphic map such that $\phi^*\Omega_{FS}^N = \omega$. Consider $L = \phi^*(\mathcal{O}_N(1))$ the pull-back via $\phi$ of the hyperplane bundle on $\mathbb{P}^N(\mathbb{C})$. Since $\omega_0$ and $\omega$ are cohomologous, it follows that $L$ and $L_0$ have the same first Chern class, i.e. $L$ and $L_0$ are holomorphically equivalent (see 1.1.2). Therefore, from 4.9.3, there exist $\{s_j\}$ and $\{t_j\}$ bases of $H^0(L_0)$ such that $\phi(x) = [(t_0(x), \ldots, t_N(x))]$ and $\phi_0(x) = [(s_0(x), \ldots, s_N(x))]$. Consider the $N+1 \times N+1$ invertible matrix $A = \{a_{jk}\}$ such that $t_j = \sum_{k=0}^{N} a_{jk} s_k$. Define $A : \mathbb{P}^N(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C})$ by the action of the matrix $A$ on the homogeneous coordinates of $\mathbb{P}^N(\mathbb{C})$. Therefore $\phi = A \circ \phi_0$, which shows that the map $\Phi$ is surjective. $\square$

Consider the set

$$\mathcal{E}_{\omega_0} = \{\omega \in \mathcal{C}_{\omega_0} \cap H^2(M, \mathbb{Z}) \mid \epsilon_\omega \text{ is constant}\} \subset \mathcal{P}_{\omega_0},$$

of all integral Kähler forms $\omega$ cohomologous to $\omega_0$ with constant epsilon. It follows from (4.2.3) that

$$\mathcal{E}_{\omega_0} \subset \mathcal{P}_{\omega_0}.$$

The set $\mathcal{E}_{\omega_0}$ could be empty, so suppose that $(M, \omega_0)$ is a homogeneous Kähler manifold. By 4.1.4 $\epsilon_{\omega_0}$ is constant and by (4.2.2) the map $\phi_0$ equals the coherent states map. In the sequel the coherent states map associated to a Kähler form $\omega$ is denoted by $\phi_\omega$ (cf. Remark 4.1.2). Consider the orbit of $\omega_0$ under the action of $\mathrm{Aut}(M)$, i.e. the set

$$[\omega_0]_{hom} = \{\omega = f^*(\omega_0) \mid \ f \in \mathrm{Aut}(M)\}. \tag{4.40}$$

Notice that the pair $(M, f^*\omega_0)$ is a homogeneous Kähler manifold with respect to the group $f \circ (\mathrm{Aut}(M) \cap \mathrm{Isom}(M, \omega_0)) \circ f^{-1}$. Therefore, by (4.1.4), $\epsilon_\omega$ is constant, $\forall \omega \in [\omega_0]_{hom}$. On the other hand, an element of $[\omega_0]_{hom}$, could belong to a cohomology class different from that of $\omega_0$, depending on the connectedness of $\mathrm{Aut}(M)$. Thus, consider the set

$$\mathcal{C}_{\omega_0} \cap [\omega_0]_{hom} \subset \mathcal{E}_{\omega_0}.$$

In accordance with Conjecture 1 we state,

**Conjecture 2**

$$\mathcal{C}_{\omega_0} \cap [\omega_0]_{hom} = \mathcal{E}_{\omega_0},$$

*i.e. the integral Kähler forms $\omega$ cohomologous to $\omega_0$ such that $\epsilon_\omega$ is constant are precisely the homogeneous one.*

We want to restate Conjecture 2, in terms of coherent states maps. We need the following:

**Definition 4.9.5** *We say that* $\phi$ *and* $\phi_0$ *two full holomorphic immersions from* $M$ *to* $\mathbb{P}^N(\mathbb{C})$ *are equivalent if there exists* $f \in Aut(M)$ *and* $U \in PU(N+1)$ *such that* $U \circ \phi = \phi_0 \circ f$.

It follows, by the previous definition, that $\omega$ belongs to $\mathcal{C}_{\omega_0} \cap [\omega_0]_{hom}$ if and only if $\phi_\omega$ is equivalent to $\phi_{\omega_0}$. Therefore, Conjecture 2 can be stated as follows:

*Let* $(M, \omega_0)$ *be a compact simply connected homogeneous Kähler manifold. If* $\omega$ *belongs to* $\mathcal{E}_{\omega_0}$*, then* $\phi_\omega$ *is equivalent to* $\phi_{\omega_0}$.

**Example 4.9.6** *Let* $M = \mathbb{P}^N(\mathbb{C})$ *and* $\omega_0 = \Omega^N_{FS}$. *The quantum line bundle is given by the hyperplane bundle* $L_0 = \mathcal{O}_N(1)$ *and the space of holomorphic sections of* $L_0$ *has complex dimension* $N+1$. *Since* $H^2(M, \mathbb{Z}) = \mathbb{Z}$*, it follows that:*

$$[\omega_0]_{hom} \cap \mathcal{C}_{\omega_0} = [\omega_0]_{hom}.$$

*Moreover, in this case, the following equalities hold:*

$$[\omega_0]_{hom} = \mathcal{E}_{\omega_0} = \mathcal{P}_{\omega_0}, \tag{4.41}$$

*which, in particular, show that Conjecture 2 holds. The proof of 4.41 is immediate. In fact, let* $\omega \in \mathcal{P}_{\omega_0}$ *and* $\phi : \mathbb{P}^N(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C})$ *be a full h.i.i. such that* $\phi^*\Omega^N_{FS} = \omega$. *This implies that* $\phi$ *belongs to* $Aut(\mathbb{P}^N(\mathbb{C})) = PGL(N+1, \mathbb{C})$*, and thus* $\omega \in [\omega_0]_{hom}$.

Consider the case $M = \mathbb{P}^1(\mathbb{C})$, endowed with the Kähler form $\omega_0 = 2\Omega^1_{FS}$. The quantum line bundle $L_0$ is given by $\mathcal{O}_1(2)$ and the space of holomorphic sections of $L_0$ is 3-dimensional (see 2.0.4). Let $z_0$ and $z_1$ be homogeneous coordinates in $\mathbb{P}^1(\mathbb{C})$. The global holomorphic sections of $\mathcal{O}_1(2)$ can be identified to the homogeneous polynomials of degree two in the variables $z_0$ and $z_1$. The coherent states map $\phi_{\omega_0}$ is given, by the *Veronese map*

$$V_2 : (\mathbb{P}^1(\mathbb{C}), 2\Omega^1_{FS}) \to (\mathbb{P}^2(\mathbb{C}), \Omega^2_{FS}) : [(z_0, z_1)] \to [(z_0^2, \alpha z_0 z_1, z_1^2)], \tag{4.42}$$

with $|\alpha|^2 = 2$. In fact, it follows immediately that $V_2^* \Omega_{FS}^2 = 2\Omega_{FS}^1$. Therefore, in this case, Conjecture 2 reads as follows:

*Let $\omega = \phi^*(\Omega_{FS}^2)$ be an integral Kähler form on $\mathbb{P}^1(\mathbb{C})$ such that $\epsilon_\omega$ is constant. Then there exists $f \in PGL(2, \mathbb{C})$ and $U \in PU(3)$ such that*

$$\phi_\omega \circ f = U \circ V_2. \tag{4.43}$$

Even if this case is extremely elementary, we are able to attack Conjecture 2 only when the map $\phi_\omega$ has a diagonal form, (see Theorem 4.9.8 below). First we need the following:

**Lemma 4.9.7** *Let $A$ be a positive real number and $n$ a natural number. Let*

$$I_n = \int_{\rho=0}^{+\infty} \frac{1}{(1 + A\rho + \rho^2)^n} d\rho.$$

*Then, the following equalities hold:*

$$\begin{aligned}
\int_{\rho=0}^{+\infty} \frac{\rho}{(1 + A\rho + \rho^2)^3} d\rho &= \frac{1}{4} - \frac{A}{2} I_3; \\
\int_{\rho=0}^{+\infty} \frac{\rho^2}{(1 + A\rho + \rho^2)^3} d\rho &= \frac{1}{4} I_2 + \frac{A^2}{4} I_3 - \frac{A}{8}; \\
\int_{\rho=0}^{+\infty} \frac{\rho^3}{(1 + A\rho + \rho^2)^3} d\rho &= \frac{1}{4} + \frac{A^2}{16} - \frac{3A}{8} I_2 - \frac{A^3}{8} I_3; \\
\int_{\rho=0}^{+\infty} \frac{\rho^4}{(1 + A\rho + \rho^2)^3} d\rho &= \frac{3}{8} I_1 + \frac{3A^2}{8} I_2 + \frac{A^4}{16} I_3 - \frac{5A}{16} - \frac{A^3}{32}.
\end{aligned}$$

**Proof:**

$$\begin{aligned}
\int_{\rho=0}^{+\infty} \frac{\rho}{(1 + A\rho + \rho^2)^3} d\rho &= \frac{1}{2} \int_{\rho=0}^{+\infty} \frac{(2\rho + A) - A}{(1 + A\rho + \rho^2)^3} d\rho \\
&= \frac{-1}{4} (1 + A\rho + \rho^2)^{-2} |_0^{+\infty} - \frac{A}{2} I_3 = \frac{1}{4} - \frac{A}{2} I_3.
\end{aligned}$$

$$\begin{aligned}
\int_{\rho=0}^{+\infty} \frac{\rho^2}{(1 + A\rho + \rho^2)^3} d\rho &= \frac{1}{2} \int_{\rho=0}^{+\infty} \frac{\rho(2\rho + A) - A\rho}{(1 + A\rho + \rho^2)^3} d\rho \\
&= \frac{-\rho}{4} (1 + A\rho + \rho^2)^{-2} |_0^{+\infty} + \frac{1}{4} I_2 - \frac{A}{2} \int_{\rho=0}^{+\infty} \frac{\rho}{(1 + A\rho + \rho^2)^3} d\rho \\
&= \frac{1}{4} I_2 + \frac{A^2}{4} I_3 - \frac{A}{8}.
\end{aligned}$$

$$\begin{aligned}
\int_{\rho=0}^{+\infty} \frac{\rho^3}{(1+A\rho+\rho^2)^3}d\rho &= \frac{1}{2}\int_{\rho=0}^{+\infty} \frac{\rho^2(2\rho+A)-A\rho^2}{(1+A\rho+\rho^2)^3}d\rho \\
&= \frac{-\rho^2}{4}(1+A\rho+\rho^2)^{-2}|_0^{+\infty} + \frac{1}{2}\int_{\rho=0}^{+\infty} \frac{\rho}{(1+A\rho+\rho^2)^2}d\rho \\
&- \frac{A}{2}\int_{\rho=0}^{+\infty} \frac{\rho^2}{(1+A\rho+\rho^2)^3}d\rho \\
&= \frac{1}{4}\int_{\rho=0}^{+\infty} \frac{(2\rho+A)-A}{(1+A\rho+\rho^2)^2}d\rho - \frac{A}{2}(\frac{1}{4}I_2 + \frac{A^2}{4}I_3 - \frac{A}{8}) \\
&= -\frac{1}{4}(1+A\rho+\rho^2)^{-1}|_0^{+\infty} - \frac{A}{4}I_2 - \frac{A}{8}I_2 - \frac{A^3}{8}I_3 + \frac{A^2}{16} \\
&= \frac{1}{4} + \frac{A^2}{16} - \frac{3A}{8}I_2 - \frac{A^3}{8}I_3.
\end{aligned}$$

$$\begin{aligned}
\int_{\rho=0}^{+\infty} \frac{\rho^4}{(1+A\rho+\rho^2)^3}d\rho &= \frac{1}{2}\int_{\rho=0}^{+\infty} \frac{\rho^3(2\rho+A)-A\rho^3}{(1+A\rho+\rho^2)^3}d\rho \\
&= \frac{-\rho^3}{4}(1+A\rho+\rho^2)^{-2}|_0^{+\infty} + \frac{3}{4}\int_{\rho=0}^{+\infty} \frac{\rho^2}{(1+A\rho+\rho^2)^2}d\rho \\
&- \frac{A}{2}\int_{\rho=0}^{+\infty} \frac{\rho^3}{(1+A\rho+\rho^2)^3}d\rho \\
&= \frac{3}{8}\int_{\rho=0}^{+\infty} \frac{\rho(2\rho+A)-A\rho}{(1+A\rho+\rho^2)^2}d\rho - \frac{A}{2}(\frac{1}{4} + \frac{A^2}{16} - \frac{3A}{8}I_2 - \frac{A^3}{8}I_3) \\
&= -\frac{3}{8}\rho(1+A\rho+\rho^2)^{-1}|_0^{+\infty} + \frac{3}{8}I_1 - \frac{3A}{8}\int_{\rho=0}^{+\infty} \frac{\rho}{(1+A\rho+\rho^2)^2}d\rho \\
&- \frac{A}{8} - \frac{A^3}{32} + \frac{3A^2}{16}I_2 + \frac{A^4}{16}I_3 \\
&= \frac{3}{8}I_1 - \frac{3A}{8}[-\frac{1}{2}(1+A\rho+\rho^2)^{-1}|_0^{\infty} - \frac{A}{2}I_2] \\
&- \frac{A}{8} - \frac{A^3}{32} + \frac{3A^2}{16}I_2 + \frac{A^4}{16}I_3 \\
&= \frac{3}{8}I_1 + \frac{3A^2}{8}I_2 + \frac{A^4}{16}I_3 - \frac{5A}{16} - \frac{A^3}{32}.
\end{aligned}$$

□

We are now in the position to prove the main Theorem of this Section.

**Theorem 4.9.8** *Let* $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^2(\mathbb{C})$ *be given by*

$$\phi([(z_0, z_1)]) = [(az_0^2, bz_0z_1, cz_1^2)], \tag{4.44}$$

*where* $a, b$ *and* $c$ *are element in* $\mathbb{C}^*$. *Consider* $\omega = \phi^*(\Omega^2_{FS})$ *and suppose that* $\epsilon_\omega$ *is constant. Then,* $\phi$ *is equivalent to the Veronese map* $V_2$, *(see definition 4.9.5).*

**Proof:** Under the action of $f \in PGL(2,\mathbb{C})$, we can suppose that the map (4.44) is given by

$$\phi([(z_0,z_1)]) = [(z_0^2, \alpha z_0 z_1, z_1^2)],$$

for $\alpha \in \mathbb{C}^*$ (see (4.43). Namely, one defines $f([(z_0,z_1)]) = [(\frac{1}{\sqrt{a}}z_0, \frac{1}{\sqrt{c}}z_1)]$. In the chart $U_0 = \{z_0 \neq 0\}$, equipped with coordinate $z = \frac{z_1}{z_0}$, the map $\phi$ is given by

$$\phi : U_0 \to \mathbb{C}^2 : z \mapsto (1, \alpha z, z^2).$$

One can easily see that

$$\omega = \phi^*(\Omega_{FS}^2) = \frac{i}{2\pi}\partial\bar{\partial}\log(1 + A|z|^2 + |z|^4) = \frac{i}{2\pi}\frac{A + 4|z|^2 + A|z|^4}{(1 + A|z|^2 + |z|^4)^2}dz \wedge d\bar{z},$$

where $|\alpha|^2 = A$. If $P(z)$ and $Q(z)$ are polynomials of degree 2 in $z$, then the hermitian structure $h$ in $\mathcal{O}_1(2)$ satisfying $\mathrm{curv}(\mathcal{O}_1(2), h) = -2\pi i\omega$ is given on $U_0$ by:

$$h(P(z), Q(z)) = \frac{P(z)\bar{Q}(z)}{1 + A|z|^2 + |z|^4}.$$

Furthermore,

$$\langle P, Q\rangle_h = \int_{\mathbb{C}} \frac{(A + 4|z|^2 + A|z|^4)P\bar{Q}}{(1 + A|z|^2 + |z|^4)^3}\frac{i}{2\pi}dz \wedge d\bar{z}.$$

This can be written in polar coordinates $(r, \theta)$ as

$$\langle P, Q\rangle_h = \frac{1}{\pi}\int_{r=0}^{+\infty}\int_{\theta=0}^{2\pi} \frac{(A + 4r^2 + Ar^4)P\bar{Q}}{(1 + Ar^2 + r^4)^3} r dr d\theta.$$

By the change of variable $r^2 = \rho$, one obtains

$$\langle P, Q\rangle_h = \int_{\rho=0}^{+\infty} \frac{(A + 4\rho + A\rho^2)P\bar{Q}}{(1 + A\rho + \rho^2)^3}d\rho. \tag{4.45}$$

It follows immediately by (4.45) that $\{1, z, z^2\}$ is an orthogonal basis of $(\mathcal{H}_h, \langle\cdot\,,\,\cdot\rangle_h)$. Furthermore,

$$\begin{aligned}
\|1\|_h^2 &= \langle 1, 1\rangle_h = \int_{\rho=0}^{+\infty} \frac{(A + 4\rho + A\rho^2)}{(1 + A\rho + \rho^2)^3}d\rho, \\
A\|z\|_h^2 &= \langle \alpha z, \alpha z\rangle_h = A\int_{\rho=0}^{+\infty} \frac{(A\rho + 4\rho^2 + A\rho^3)}{(1 + A\rho + \rho^2)^3}d\rho, \\
\|z^2\|_h^2 &= \langle z^2, z^2\rangle_h = \int_{\rho=0}^{+\infty} \frac{(A\rho^2 + 4\rho^3 + A\rho^4)}{(1 + A\rho + \rho^2)^3}d\rho.
\end{aligned}$$

A direct calculation, using Lemma (4.9.7), gives:

$$\|1\|_h^2 = (\frac{A^3}{4} - A)I_3 + \frac{A}{4}I_2 + 1 - \frac{A^2}{8}, \tag{4.46}$$

$$\|\alpha z\|_h^2 = (\frac{A^3}{2} - \frac{A^5}{8})I_3 + (A - \frac{3A^3}{8})I_2 + \frac{A^4}{16}, \tag{4.47}$$

$$\|z^2\|_h^2 = (\frac{A^5}{16} - \frac{A^3}{4})I_3 + (\frac{3A^3}{8} - \frac{5A}{4})I_2 + \frac{3A}{8}I_1 + 1 - \frac{3A^2}{16} - \frac{A^4}{32}. \tag{4.48}$$

If $A = 2$, the case of the Veronese map, one has

$$I_1 = \int_0^{+\infty} \frac{1}{(1+\rho)^2} d\rho = 1,$$
$$I_2 = \int_0^{+\infty} \frac{1}{(1+\rho)^4} d\rho = \frac{1}{3},$$
$$I_3 = \int_0^{+\infty} \frac{1}{(1+\rho)^6} d\rho = \frac{1}{5},$$

and by (4.46),(4.47) and (4.48) one obtains: $\|1\|_h^2 = 2\|z\|_h^2 = \|z^2\|_h^2 = \frac{2}{3}$ in accordance with the fact that $V_2$ is the coherent states map.

In order to prove the theorem we need to show that if $A \neq 2$, then either $\|1\|_h^2 \neq A\|z\|_h^2$, or $\|1\|_h^2 \neq \|z^2\|_h^2$. Suppose, for example, that $\|1\|_h^2 = A\|z\|_h^2$. Then, by subtracting (4.46) from (4.47) one obtains:

$$-32 + 6A^2 + 3A^4 - 12AI_1 + (72A - 24A^3)I_2 + 6A^3(A^2 - 4)I_3 = 0. \tag{4.49}$$

We distinguish two cases: $0 < A < 2$ and $A > 2$.

- For $0 < A < 2$, we easily obtain:

$$\begin{aligned}
I_1 &= \frac{\pi}{\sqrt{4-A^2}} - \frac{2}{\sqrt{4-A^2}} arctan \frac{A}{\sqrt{4-A^2}}, \\
I_2 &= \frac{2\pi}{(\sqrt{4-A^2})^3} - \frac{A}{4-A^2} - \frac{4}{(\sqrt{4-A^2})^3} arctan \frac{A}{\sqrt{4-A^2}}, \\
I_3 &= \frac{6\pi}{(\sqrt{4-A^2})^5} + \frac{A^3 - 10A}{2(4-A^2)^2} - \frac{12}{(\sqrt{4-A^2})^5} arctan \frac{A}{\sqrt{4-A^2}}.
\end{aligned}$$

  By (4.49) one gets:

$$-(8 + A^2)\sqrt{4 - A^2} + 6a\pi - 12A arctan \frac{A}{\sqrt{4-A^2}} = 0,$$

  which can not hold for $0 < A < 2$.

- For $A > 2$, we get:

$$\begin{aligned} I_1 &= -\frac{1}{\sqrt{A^2-4}} log \frac{A-\sqrt{A^2-4}}{A+\sqrt{A^2-4}}, \\ I_2 &= \frac{A}{A^2-4} + \frac{2}{(\sqrt{A^2-4})^3} log \frac{A-\sqrt{A^2-4}}{A+\sqrt{A^2-4}}, \\ I_3 &= \frac{A^3-10A}{2(A^2-4)^2} - \frac{6}{(\sqrt{A^2-4})^5} log \frac{A-\sqrt{A^2-4}}{A+\sqrt{A^2-4}}. \end{aligned}$$

By (4.49) one gets:

$$(8+A^2)\sqrt{A^2-4} + 6A log \frac{A-\sqrt{A^2-4}}{A+\sqrt{A^2-4}},$$

which can not hold for $A > 2$. □

**Remark 4.9.9** *Theorem 4.9.8 provides us with a family of projectively induced Kähler forms $\omega$ with $\epsilon_\omega$ not constant. Hence, in the case $\omega_0 = 2\Omega^1_{FS}$, $\mathcal{E}_{\omega_0}$ is strictly contained in $\mathcal{P}_{\omega_0}$. For $A \neq 2$, using (4.4), the function epsilon can be calculated explicitly as*

$$\epsilon_{\omega_0}(z) = \frac{\|1\|_h^{-2} + A\|z\|_h^{-2}|z|^2 + \|z^2\|_h^{-2}|z|^4}{1 + A|z|^2 + |z|^4},$$

*where $\|1\|^2_h$, $\|z\|^2_h$ and $\|z^2\|^2_h$ are given by (4.46), (4.47) and (4.48), respectively.*

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