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The Kodaira Embedding Problem

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Contents

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Introduction

It is a well known result in differential geometry that every differentiable $n-$manifold M admits a differentiable embedding f into $\mathbb{R}^{2n+1}$. Moreover, the image of M, $f(M)$ can be realized as a real-analytic submanifold of $\mathbb{R}^{2n+1}$. This result was proved by Hassler Whitney in 1936. Whitney's theorem tells us that all differentiable manifolds (compact and non compact) can be considered as submanifolds of Euclidean space, such submanifolds having been the motivation for the definition and concept of manifold in general. However, when we turn to complex manifolds, things are completely different. Indeed, it is well known in the literature that not all complex manifolds admits an embedding into Euclidean space in contrast to the differentiable and real-analytic situations, and of course, there are many examples of such complex manifolds. To give an insight, because of the *maximum principle*, there are no compact submanifolds embedded into the Euclidean space $\mathbb{C}^n$. One *can* characterize the (necessarily non compact) complex manifolds which admits embedding into $\mathbb{C}^n$, and these are called *Stein manifolds*, which have an abstract definition and have been largely studied in the past thirty years. In this work, we want to develop the material necessary to provide a characterization of the compact complex manifolds which admit an embedding into projective spaces. This was firstly accomplished by Kodaira in 1954, and his theorem states that every compact Hodge manifold X is projective algebraic, i.e. there exist a non negative integer N and a holomorphic embedding of X into $\mathcal{P}_N(\mathbb{C})$. The material in the next chapters is developed with this characterization in mind. The entire work is subdivided into four chapters.
In the first chapter we give some preliminary notions of basic differential geometry such as complex manifolds, vector bundles, almost complex structures and at the end we will introduce the $\bar{\partial}-$operator. In this chapter, we omit many details, since we assume the reader already familiar with the mentioned material.
In the second chapter we will explain some construction behind sheaf theory, from the very basic notions to the sheaf cohomology. Sheaf theory will provide the necessary machinery for the proof of the Kodaira's embedding theorem.
In the third chapter, we will discuss Hermitian differential geometry by giv-

ing the notions of metrics, connections and curvature in the smooth and holomorphic case. A very important part of this chapter is dedicated to the definitions of Chern classes. Indeed, we will give a geometric and a topological definition, and by exploring their properties we will understand the importance and the geometric meaning of Chern classes. At the end we will focus our attention to the geometry of complex line bundles, that will be useful for the rest of the work.
In the fourth and last chapter we will focus our attention on introducing Hodge manifolds, namely the class of manifolds that are suitable to state the Kodaira's embedding theorem, we will talk about the Hopf's blow-up and we will finally state and prove the Kodaira's theorem.

Chapter 1

Some preliminary notions

In this chapter we recall in the first section some standard notions of the topology of manifolds, in the second section we recall some basic results of vector bundles and in the last section we introduce the basic notions of complex geometry.

1.1 S-structures, S-manifolds

Let $\mathbb{K}$ be a field, that could be $\mathbb{R}$ or $\mathbb{C}$. Consider the space $\mathbb{K}^n$ endowed with its standard topology. Let $U \subseteq \mathbb{K}^n$ be a open subset. We have the following class of functions:

for $\mathbb{K} = \mathbb{R}$ we have

- $\mathcal{E}(U) := \{f : U \to \mathbb{R} : f$ is infinite times differentiable$\}$
- $\mathcal{A}(U) := \{f : U \to \mathbb{R} : f$ is analytic$\}$, that is $f \in \mathcal{U}$ if and only if in the neighbourhood of every point of U, f can be written as Taylor expansion convergent to f.

For $\mathbb{K} = \mathbb{C}$ we have

- $\mathcal{O} := \{f : U \to \mathbb{C} : f$ is holomorphic$\}$ that is $f \in \mathcal{O}$ if and only if in the neighbourhood of every point of U, f can be written as a Laurent series convergent to f.

We observe from analysis that $\mathcal{E}(U)$ and $\mathcal{A}(U)$ are subcategories of $\mathcal{O}$, since a holomorphic function is both analytic and differentiable. Therefore, we have the following chain of inclusions:

$$\mathcal{E}(U) \subseteq \mathcal{A}(U) \subseteq \mathcal{O}(U)$$

For every open set $U \subseteq \mathbb{K}^n$.

We want to see the behavior of the above functions on mafifolds. Recall that a topological manifold M^n is a Hausdorff topological space for which there exists a countable basis and it is locally euclidean of dimension n, that is at every point $p \in M^n$ there exists a open neighbourhood $U \subset M^n$ of p and a homeomorphism $h : U \to h(U) \subseteq \mathbb{R}^n$. The natural number n is called topological dimension of the given manifold. Henceforth, denote by $\mathcal{S}$ one of the three class of functions previously mentioned.

Definition 1.1. *Let M be a topological manifold, a $\mathcal{S}$-structure on M consist of a family of function of M $\mathbb{K}-$valued, such that the following axioms hold:*

(a) $f \in \mathcal{S}$ is continuous and defined on open sets of M.

(b) For all $x \in M$ there exists a local chart (U, h) around x such that for every open subset $U' \subset U$, $f : U' \to \mathbb{K}$ is an element of $\mathcal{S}$ if and only if $f \circ h^{-1} \in \mathcal{S}(h(U))$.

(c) If $f : U \to \mathbb{K}$, where $U = \bigcup_i U_i$, $U_i \subset M$ open sets, then $f \in \mathcal{S}$ if and only if $f_{|U_i} \in \mathcal{S}$.

$(M, \mathcal{S}_M)$ is called $\mathcal{S}-$manifold, denote by $\dim_{\mathbb{K}} M = n$ the topological dimension of M.

From previous considerations, we deduce that:

1. $\mathcal{S} = \mathcal{E}$ then M is a *smooth manifold.*
2. $\mathcal{S} = \mathcal{A}$ then M is a *analytic manifold.*
3. $\mathcal{S} = \mathcal{O}$ then M is a *complex manifold.*

Therefore, $\mathcal{S}$ determines the *type* of maifolds.

Definition 1.2. *Let $(M, \mathcal{S}_M)$ and (N, S_N) be $\mathcal{S}$-manifolds.*

(a) A $\mathcal{S}$-morphism consist of a continuous mapping $F : M \to N$ such that for all $f \in \mathcal{S}_N$ we have $f \circ F \in \mathcal{S}_M$.

(b) A S-isomorphism $F : (M, \mathcal{S}_M) \to (N, \mathcal{S}_N)$ consist of a homeomorphism $F : M \to N$ and a $\mathcal{S}$-morphism given by $F^{-1} : (N, \mathcal{S}_N) \to (M, \mathcal{S}_M)$.

From previous definition, we have the following characterization:

$\mathcal{S}$-structure	**$\mathcal{S}$-morphism**	**$\mathcal{S}$-isomorphism**
$\mathcal{E}$	smooth functions	diffeomorphism
$\mathcal{A}$	analytic functions	bianalytic function
$\mathcal{O}$	holomorphic functions	biholomorphic function

Remark 1.1.1. *Let* $(M, \mathcal{S}_M)$ *be a* $\mathcal{S}$*-manifold. Let* $(U_1, h_1), (U_2, h2)$ *be local charts such that* $U_1 \cap U_2 \neq \emptyset$*. Then we have*

$$h_2 \circ h_1^{-1} : h_1(U_1 \cap U_2) \longrightarrow h_2(U_1 \cap U_2)$$

defines a $\mathcal{S}$*-isomorphism in* $(\mathbb{K}^n, \mathcal{S}_{\mathbb{K}})$*. Conversely, let* $\{U_\alpha\}_{\alpha \in A}$ *be a open cover of* M *such that* $\forall \alpha, \beta \in A$ $U_\alpha \cap U_\beta \neq \emptyset$ *and* $h_\beta \circ h_\alpha^{-1} \in \mathcal{S}_{\mathbb{K}}(h_\alpha(U_\alpha \cap U_\beta))$*. Then, by letting* $\mathcal{S}_M := \{f : U \to \mathbb{K} : U \subseteq M \text{ open}\}$*,*

$$f \circ h_\alpha^{-1} \in \mathcal{S}_{\mathbb{K}}(h_\alpha(U_\alpha \cap U_\beta))$$

$\forall \alpha \in A$ *we have a* $\mathcal{S}$*-structure on* M*. The family* $\{(U_\alpha, h_\alpha\}_{\alpha \in A}$ *definines a* $\mathcal{S}$*-structure on* M *and is called* $\mathcal{S}$-Atlas*.*

Definition 1.3. *Let* M *be a* $\mathcal{S}$*-manifold and* $N \subset M$ *be subset. A* $\mathcal{S}$*-function on* N *is a* $\mathcal{S}$*-morphism on* M *such that for every open set* $U \subset M$ *with* $N \subset U$ $f_{|U} \in \mathcal{S}_M$*. The space of* $\mathcal{S}$*-function on* N *will be denoted by* $\mathcal{S}_{N|M}$*.*

Definition 1.4. *Let* N *be a closed subset of a* $\mathcal{S}$*-manifold* M *of dimension* n*. We say that* N *is a* $\mathcal{S}$*-submanifold if at every point* $x_0 \in N$ *there exists a open chart* (U, h) *of* M *around* x_0 *such that* $h(U \cap N) = h(U) \cap \mathbb{K}^r$*, with* $\mathbb{K}^r \subset \mathbb{K}^n$ *as subspace. The natural number* r *is called dimension of* N*, and* $n - r$ *is called codimension. Moreover, on* N *there is the induced* $\mathcal{S}$*-structure* $\mathcal{S}_{M|N}$*.*

Example 1.1.1. (The euclidean space $\mathbb{K}^n$) $\mathbb{K}^n = \mathbb{R}^n$ is a real analytic manifold (hence also smooth) whose atlas consist of a unique chart $(\mathbb{R}^n, id_R^n)$. For $\mathbb{K}^n = \mathbb{C}^n$ we have a complex manifold.

Example 1.1.2. Let $(M, \mathcal{S}_M)$ be $\mathcal{S}$-manifold of dimension n. Let $U \subseteq M$ be a open set, we have a natural $\mathcal{S}$-structure $\mathcal{S}_U$, such that $(U, \mathcal{S}_U)$ is a open $\mathcal{S}$-submanifold that has the same dimension of M.

Example 1.1.3. (The projective space) Let V be a vector space over the field $\mathbb{K}$ of dimension n. Define the projective space as the totality of all lines passing through the origin of V. Namely,

$$\mathcal{P}(V) := \{W \leq V : \dim W = 1\}.$$

For $\mathbb{K} = \mathbb{R}$ and $V = \mathbb{R}^n$ consider $\mathcal{P}(\mathbb{R}^n) = \mathcal{P}_n(\mathbb{R})$, we want to show that it is a real-analytic manifold. The following considerations can be easily applied also to $\mathcal{P}_n(\mathbb{C})$, that will be a complex manifold. Observe that we have a natural surjective map, such that to a point of $\mathbb{R}^{n+1} \setminus \{0\}$ assigns the space generated by that point:

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathcal{P}_n(\mathbb{R})$$
$$x \mapsto \pi(x) =< x >= \{\lambda x : \lambda \in \mathbb{R}\}$$

The space $\mathbb{R}^{n+1} \setminus \{0\}$ can be equipped with the standard euclidean topology.Therefore, we can endow $\mathcal{P}_n(\mathbb{R})$ with the quotient topology, i.e. $U \subseteq \mathcal{P}_n(\mathbb{R})$ is open if and only if $\pi^{-1}(U) \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ is open. With that topology π becomes a continuous mapping. Moreover, because of the properties of quotient topology [13] becomes a second countable Hausdorff space. The images of the map π become equivalence classes in $\mathcal{P}_n(\mathbb{R})$, namely $\pi(x) = [x_0, ..., x_n]$. A representative of a equivalence class will be called set of homogeneous coordinates, in particular, two set of homogeneous coordinates $(x_0, ..., x_n), (x_0^{'}, ..., x_n^{'})$ define the same equivalence class if and only if their components differ by a constant $t \in \mathbb{R}_* = \mathbb{R} \setminus \{0\}$, i.e. $x_i^{'} = tx_i \; 0 \leq i \leq n$. Thus, $\pi(x) = \pi(tx)$. Now, we can define a family of open sets for $P_n(\mathbb{R})$ given by:

$$U_\alpha := \{[x_0, ..., x_\alpha, ..., x_n] \in P_n(\mathbb{R}) : \pi(x) = [x] \text{ e } x_\alpha \neq 0\},$$

with $\alpha = 0, ..., n$. On every of these open set, we can define a homeomorphism

$$\begin{aligned} h_\alpha : U_\alpha &\longrightarrow \mathbb{R}^n \\ [x_0, ..., x_n] &\mapsto (\frac{x_0}{x_\alpha}, ..., \frac{x_{\alpha-1}}{x_\alpha}, \frac{x_{\alpha+1}}{x_\alpha}, ..., \frac{x_n}{x_\alpha}) \end{aligned}$$

with inverse

$$\begin{aligned} h_\alpha^{-1} : \mathbb{R}^n &\longrightarrow U_\alpha \\ (w_0, ..., w_{n-1}) &\mapsto [w_0, ..., 1, ..., w_n]. \end{aligned}$$

The open sets U_α cover $P_n(\mathbb{R})$. Furthermore, since h_α is bianalytic, we deduce that $\{(U_\alpha, h_\alpha)\}$ is a $\mathcal{A}$-Atlas of $P_n(\mathbb{R})$. It follows that $P_n(\mathbb{R})$ is a real analytic manifold.

Example 1.1.4. (*The Grassmann spaces*) Let V be a $\mathbb{K}$- vector space of dimension n, the Grassmann space with respect to V is defined as:

$$G_k(V) := \{W \leq V : dim_{\mathbb{K}} W = k \leq n\}.$$

Namely, al vector subspaces of V of dimension lower or equal than n (k-planes). Observe that $G_k(V)$ generalizes the projective space. Indeed, for $k = 1$, it follows from the definition that $P(V) = G_1(V)$. Fix $\mathbb{K} = \mathbb{R}$ and choose some basis of V, namely by that we choose an isomorphism $\mathbb{R}^n \xrightarrow{\sim} V$. Therefore, we let $G_k(\mathbb{R}^n) = G_{k,n}(\mathbb{R})$. We want to show that $G_{k,n}(\mathbb{R})$ is a real analytic manifold. We have a natural projection

$$\pi : \mathcal{M}_{n\times k}(\mathbb{R}) \longrightarrow G_{k,n}(\mathbb{R})$$
$$A \mapsto \pi(A) =< \begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix} >$$

Namely, $\pi(A)$ is the vector space generated by the column

$$\begin{pmatrix} a_1 \\ \vdots \\ a_k \end{pmatrix}$$

of A. The coordinates in that column are elements of $\pi(A)$ and are called, homogeneous coordinates. In particular, two set of homogeneous coordinates for $\pi(A)$ are said to be equivalent if and only if they differir by a invertible matrix $g \in GL_k(\mathbb{R})$, that is $\pi(A) = \pi(gA)$. We can equipp $G_{k,n}(\mathbb{R})$ with the quotient topology like in the case of the projective space. Then, the map π becomes continuos and from the properties of quotient topology we deduce that $G_{k,n}$ is a topological manifold. The analytic structure is given along the lines of previous example. We leave further details to the reader.

Example 1.1.5. (*Algebraic submanifolds*) Consider $\mathcal{P}_n = \mathcal{P}_n(\mathbb{C})$, and let

$$H = \{[z_0, ..., z_n] \in \mathcal{P}_n : a_0 z_0 + ... + a_n z_n = 0\},$$

where $(a_0, ..., a_n) \in \mathbb{C}^{n+1} \setminus \{0\}$. Then H is called a projective hyperplane. We shall see that H is a submanifold of $\mathcal{P}_n$ of dimension $n-1$. Let U_α be the coordinate system of $\mathcal{P}_n$ as defined in the Example 1.1.3. Let us consider $U_0 \cap H$, and let $(\zeta_1, ..., \zeta_n)$ be coordinates in $\mathbb{C}^n$. Suppose that $[z_0, ..., z_n] \in H \cap U_0$; then, since $zeta_0 \neq 0$, we have

$$a_1 \frac{z_1}{z_0} + ... + a_n \frac{z_n}{z_0} = -a_0,$$

which implies that if $\zeta = (\zeta_1, ..., \zeta_n) = h_0([z_0, ..., z_n])$, then ζ satisfies

$$a_1\zeta_1 + ... + a_n\zeta_n = -a_0, \tag{1.1}$$

which is an affine linear subspace of $\mathbb{C}^n$, provided that at least one of $a_1, ..., a_n$ is not zero. If, however, $a_0 \neq 0$ and $a_1 = ... = a_n = 0$, then it is clear that there is no point $(\zeta_1, ..., \zeta_n) \in \mathbb{C}^n$ which satisfies (1.1), and hence in this case $U_0 \cap H = \emptyset$. It now follows easily that H is a submanifold of dimension $n-1$ of $\mathcal{P}_n$. More generally, one can consider

$$V = \{[z_0, ..., z_n] \in \mathcal{P}_n(\mathbb{C}) : p_1(z_0, ..., z_n) = ... = p_r(z_0, ..., z_n) = 0\},$$

where $p_1, ..., p_r$ are homogeneous polynomials of varying degrees. In local coordinates, one can find equations of the form

$$p_1\left(1, \frac{z_1}{z_0}, ..., \frac{z_n}{z_0}\right) = 0$$

$$p_r\left((1, \frac{z_1}{z_0}, ..., \frac{z_n}{z_0}\right) = 0,$$

and V will be a submanifold of $\mathcal{P}_n$ if the Jacobian matrix of these equations in the various coordinate systems has maximal rank. More generally, V is called a *projective algebraic variety*, and points where the Jacobian has less than maximal rank are called *singular points* of the variety

Definition 1.5. *Let (M, S_M) and (N, S_n) be two $S-$manifolds. A $S-$morphism*

$$F : (M, S_M) \to (N, S_N)$$

*is called $S-$*embedding *if*

(a) $M \xrightarrow{\sim} F(M)$ *is a homeomorphism,*

(b) $(M, S_M) \to (F(M), S_{N|f(M)})$ *is a $S-$isomorphism.*

We recall the following important result in differential topology for a full treatment see [5], [2]

Theorem 1.1.1. *(*Whitney*) Let M be a $\mathcal{A}-$manifold of dimension n. Then there exist a analytic embedding $F : M \to \mathbb{R}^{2n+1}$. In particular, $F(M)$ is a submanifold of* $\mathbb{R}^{2n+1}$.

The Withney's theorem tells that every real-analytic manifold, hence also smooth, can be seen as a subanifold of $\mathbb{R}^{2n+1}$. This is not, in general, true for complex manifolds. Indeed, not every complex manifold admits an embedding in $\mathbb{C}^n$ in particular compact complex manifolds, but this latter under some hypothesis admits an embedding in $\mathcal{P}_n(\mathbb{C})$.

Theorem 1.1.2. *Let X be a complex manifold compact and connected. Then, every holomorphic function $f \in \mathcal{O}(X)$ is a constant function.*

Proof. Let $f \in \mathcal{O}(X)$, then since X is compact, for the Weierstrass theorem $|f|$ has a maximal element in X, say $x_0 \in X$, i.e. $|f(x_0)| \geq |f(x)|, \forall x \in X$. Consider the following closed subset of X:

$$S = \{x \in X : f(x) = f(x_0)\}.$$

We show that S is open. Then, since X is connected, it will follows that $X = S$, therefore f is constant. Let $x \in S$, consider a local chart (U, h)

around x such that $h(x) = 0$, i.e. $h(x) = (z_1, ..., z_n) = z = 0$. Let $B(0)$ be an open ball centered in 0, such that $B(0) \simeq h(U)$. We let for $w \in B(0)$

$$g(\lambda) = (f \circ h^{-1})(\lambda w).$$

Then g is a complex variable function and it assumes its maximal value in $\lambda = 0$. Thus, for the principle of maximal value g is constant in $B(0)$. Therefore, we can cover S with such open charts. Hence, S is open. $\square$

It follows from the above theorem that

Corollary 1.1.1. *There are no compact submanifold in $\mathbb{C}^n$.*

The submanifolds of $\mathbb{C}^n$, necessarily non compact, are called *Stein manifolds.*

Definition 1.6. *A compact complex manifold X is called* projective algebraic *if it admits an embedding into some finite dimensional projective space.*

1.2 Vector bundles on S-manifolds

Definition 1.7. *A* $\mathbb{K}-$vector bundle *of rank r on a manifold X consist of*

a. A manifold E called total space

b. A continuous surjection $\pi : E \to X$

c. For every $x \in X$, $E_x := \pi^{-1}(x)$ has a $\mathcal{K}-$vector space structure of dimension r. E_x is called fiber *of E at the point x.*

b. For all $x \in X$ there exist a open neighbourhood $U \subset X$ and a homeomorphism

$$h : \ \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{K}^r$$

such that,

$$h^x = h_{|x} : E_x \to \{x\} \times \mathbb{K}^r$$

is a linear isomorphism.

The condition d of the above definition can be expressed by saying that E is *locally trivial.* The tuple (U, h) is called *local trivialization.* The manifold X is called *base space.* Directly from the definition we have the following examples

Example 1.2.1. (*Restricted vector bundle*). Let $\pi E \to X$ be a $\mathbb{K}-$vector bundle of rank r. Let $E_U = \pi^{-1}(U)$ with $U \subset X$ open. Then, $E_U \to U$ is a $\mathbb{K}-$vector bundle of rank r.

Example 1.2.2. (*Trivial vector bundle*) Let X be a manifold. Consider $\mathbb{K}^r$ as a $\mathbb{K}-$vector space. Then the product $X \times \mathbb{K}^r$ is a $\mathbb{K}-$vector bundle of rank r called the *trivial vector bundle*. Indeed, $X \times \mathbb{K}^r$ is a manifold and the continuous projection is given by the natural projection onto the first factor. By construction, we see that $X \times \mathbb{K}^r$ is globally trivial.

Remark 1.2.1. *From the definition of vector bundle and the two previous examples, we see that every vector bundle, locally, looks like the trivial bundle.*

Let $\pi : E \to X$ be a vector bundle of rank r. Let (U_α, h_α) and (U_β, h_β) be local trivializations such that $U_\alpha \cap U_\beta \neq \emptyset$. Then we have the following composition

$$h_\alpha \circ h_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{K}^r \xrightarrow{\sim} (U_\alpha \cap U_\beta) \times \mathbb{K}^r. \tag{1.2}$$

By definition, the above isomorphism, induces a linear isomorphism:

$$(h_\alpha \circ h_\beta^{-1})^x : \mathbb{K}^r \to \mathbb{K}^r,\ \forall x \in U_\alpha \cap U_\beta.$$

Write $g_{\alpha\beta}(x) = (h_\alpha \circ h_\beta^{-1})^x$. Therefore, that induces a map

$$\begin{aligned} g_{\alpha\beta} : U_\alpha \cap U_\beta &\to GL(r, \mathbb{K}) \\ x &\mapsto g_{\alpha\beta}(x) : \mathbb{K}^r \xrightarrow{\sim} \mathbb{K}^r. \end{aligned}$$

The functions $g_{\alpha\beta}$ are called *transition function*. Because of the above construction, for a given vector bundle $E \to X$ of rank r, to give a trivializing atlas is equivalent to determine he transition functions. Unravelling the definition of transition function, we see easily that they satisfy the so called *Čech cocycle* condition:

1. $g_{\alpha\alpha} = I_r$ in $U_\alpha \neq \emptyset$.
2. $g_{\alpha\beta} = (g_{\beta\alpha}^{-1})$ in $U_\alpha \cap U_\beta$.
3. $g_{\alpha\beta} \cdot g_{\alpha\beta} = g_{\alpha\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$.

Within the transition functions we are able to completely reconstruct a vector bundle, as the following result shows [1]

Proposition 1.2.1. *Let X be a manifold and $\{U_\alpha\}$ be a open cover of x. Suppose that for every index α, β such that $U_\alpha \cap U_\beta \neq \emptyset$, $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{K})$ are given and they satisfy the Čech cocycle conditions. Then, there exist a unique structure of vector bundle of rank r above X that has $g_{\alpha\beta}$ as transition functions.*

Example 1.2.3. (*Universal bundle*). Let $U_{r,n}$ be the disjoint union of $r-$planes in $\mathbb{K}^r$. Then, we have a natural projection given by

$$\pi : U_{r,n} \to G_{r,n}(\mathbb{K},$$

such that to a vector v that belongs to a $r-$plane S of $U_{r,n}$ assigns a plane in $G_{r,n}(\mathbb{K}$. In order to simplify the notations and the discussion, we study the case when $n = 1$, thus the projection becomes

$$\begin{aligned} \pi : U_{r,1} &\to \mathcal{P}_{n-1} \\ r &\mapsto \pi(v) = [x_0, ..., x_{n-1}]. \end{aligned}$$

The element v is represented as follows $v = (tx_0, ..., tx_{n-1}) = t(x_0, ..., x_n)$ where $t \in \mathbb{K}$ and $(x_0, ..., x_{n-1}) \in \mathbb{K}^n \setminus \{0\}$. Then

$$\pi(v) = \pi(t(x_0, ..., x_{n-1})) = \pi(x_0, .., x_{n-1}) = [x_0, ..., x_{n-1}].$$

Consider $U_\alpha \subseteq \mathcal{P}_{n-1}(\mathbb{K})$, then

$$\pi^{-1}(U_\alpha) := \{v = t(x_0, ..., x_{n-1}) : t \in \mathbb{R}, x_\alpha \neq 0\}$$

from here, we can define some bijections that preserves the fibers by letting

$$\begin{aligned} h_\alpha : &\pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^r \\ &v \mapsto h_\alpha(v) = ([x_0, ..., x_{n-1}], t_\alpha), \end{aligned}$$

where $t_\alpha = tx_\alpha$.

In the non empty overlapping $U_\alpha \cap U_\beta$, $v \in \pi^{-1}(U_\alpha \cap U_\beta)$ has local representations in $U_\alpha \times \mathbb{K}^r$ and $U_\beta \times \mathbb{K}^r$:

$$h_\alpha(v)([x_0, ..., x_{n-1}], t_\alpha),$$

$$h_\beta(v) = ([x_0, .., x_{n-1}], t_\beta)$$

respectively. Therefore, we have

$$t = \frac{t_\alpha}{t_\beta} = \frac{x_\alpha}{x_\beta}$$

. We let

$$g_{\alpha\beta} = \frac{x_\alpha}{x_\beta}$$

It is easy to verify that the above satisfy the Čech cocycle condition, therefore by Proposition 1.2.1 we conclude that $\pi : U_{r,1} \to \mathcal{P}_{n-1}$ is a vector bundle. In the spirit of the previous section, a $S-$vector bundle of rank r over a $S-$manifold X, consist of a

- $S-$manifold E,
- a surjective $S-$morphism $\pi : E \to X$,
- the transition functions are $S-$isomorphism.

Definition 1.8. *Let $\pi_E : E \to X$ and $\pi_F : F \to X$ be two $S-$vector bundles. Then*
(a) A homomorphism of $S-$vector bundles $f : E \to F$ consist of

a.1 a $S-$isomorphism f,

a.2 $forall x \in X$ $f_x : E_x \to E_{f(x)}$ is linear,

a.3 $\pi_F \circ f = \pi_E$.

(b) A isomorphism of $S-$vector bundles is a $S-$homomorphism f of vector bundles such that

b.1 f is a $S-$isomorphism,

b.2 $\forall x \in X$, f_x is a linear isomorphism.

(c) Two $S-$vector bundles over the same manifold X, are called equivalent *if there exists a $S-$isomorphism between them.*

1.2.1 Operations of Vector Bundles

Recall that, given two vector spaces A and B, we have the following constructions: $A \oplus B$, $A \otimes B$, $\mathrm{Hom}(A, B)$, $A^* = \mathrm{Hom}(A, \mathbb{K})$. The same constructions can be extended to vector bundles, the principle is very simple: natural operations with vector spaces, applied fiberwise, extend to vector bundles. Consider two vector bundles $\pi_E : E \to X$ and $\pi_F : F \to X$, and let $\{U_\alpha\}$ be a open cover of X.
Let us start with the *direct sum* operation. The direct sum $E \oplus F$ is another vector bundle over X, with fibers

$$(E \oplus F)_x := E_x \oplus F_x, \ \forall x \in X.$$

The transition functions with respect to the open cover $\{U_\alpha\}$ are given by

$$g^{E\oplus F}_{\alpha\beta}(x) = \mathrm{diag}(g^E_{\alpha\beta}, g^F_{\alpha\beta}) \tag{1.3}$$

It is easy to verify that (1.3) satisfy the Čech cocycle conditions, therefore by Proposition 1.2.1 $E \oplus F$ is a vector bundle, called *direct sum bundle*.
In the same spirit we can build the *tensor product* vector bundle By proceeding fiberwise, we have a collection of vector spaces

$$(E \oplus F)_x := E_x \oplus F_x, \ \forall x \in X.$$

The transition functions, with respect to the open cover $\{U_\alpha\}$ are given by

$$g_{\alpha\beta}^{E\otimes F}(x) = (g_{\alpha\beta}^{E} \otimes g_{\alpha\beta}^{F})(x) = g_{\alpha\beta}^{E}(x) \cdot g_{\alpha\beta}^{F}(x). \tag{1.4}$$

It is easy to verify that (1.4) satisfy the Čech cocycle conditions, therefore by Proposition 1.2.1 $E \otimes F$ is a vector bundle.

By applying the same principles we can define the so called Hom$-$bundle. By proceding fiberwise

$$\mathrm{Hom}(E, F)_x = \mathrm{Hom}(E_x, F_x).$$

Local trivializations of E and F induce a local trivialization of Hom(E, F), and we obtain a canonical $S-$structure. With the same principles all the natural operations with vector spaces extends to vector bundles, we leave further details to the reader. ♠

1.2.2 S-sections of S-vector bundles

Definition 1.9. *Let $\pi : E \to X$ be a $S-$vector bundle of rank r. A $S-$*section *of E is a $S-$morphism $s : X \to E$, such that $\pi \circ s = id_X$. A* local *$S-$section $s_U : U \to E_U$ is a $S-$morphism such that $\pi \circ s_U = id_U$. Denote the spaces of sections and local sections respectively by*

$$S(X, E) := \{s : X \to E : \ s \text{ is a section}\}$$
$$S(U, E) := \{s : U \to E_U : \ s \text{ is a local section}\}$$

Example 1.2.4. Let M be a smooth manifold, consider the trivial bundle $M \times \mathbb{R}^r$. Then, the sections $\mathcal{E}(M, M \times \mathbb{R}^r)$ are smooth vector valued functions, i.e. elements of $\mathcal{E}(M, \mathbb{R}^r)$. Indeed, if $s \in \mathcal{E}(M, M \times \mathbb{R}^r)$, then

$$s : \ M \to M \times \mathbb{R}^r \xrightarrow{proj_2} \mathbb{R}^r$$

and $\mathrm{proj}_2 \circ s \in \mathcal{E}(M, \mathbb{R}^r)$. Conversely, if $s \in \mathcal{E}(M, \mathbb{R}^r)$, then we have that the assignment $M \in p \mapsto (p, g(p)) \in M \times \mathbb{R}^r$, is a $S-$section of $M \times \mathbb{R}^r$.

Observe that, every vector bundle $\pi : E \to X$ has a *zero section*

$$O : X \to E$$

given by $O(x) = 0_x \in E_x, \forall x \in X$. That means, at every point of X the zero section assigns the origin of each fiber. Therefore, that *zero* could be formally identified with the base space. Thus, sections are copies of the base space X on the total space E, such that at every point $x \in X$, $s(x) \in E_x$, where s is a section.

Remark 1.2.2. *Let $\pi : E \to X$, $\pi' E' \to$ be S-vector bundles. Then*

$$\mathrm{Hom}(E, E') = S(X, \mathrm{Hom}(E, E')).$$

Such correspondence is given as follows: a section $s \in S(X, \mathrm{Hom}(E, E')$ picks out for each point $x \in X$ a linear map $s(x) : E_x \to E'_x$, and s is identified with $f_s : E \to E'$ which is defined by

$$f_s|_{E_{\pi(e)}} = s(\pi(e)), \text{ for } \; e \in E.$$

Remark 1.2.3. *If $E \to X$ is an $S-$bundle of rank r with transition functions $\{g_{\alpha\beta}\}$ associated with a trivializing cover $\{U_\alpha\}$, then let $f_\alpha : U_\alpha \to \mathbb{K}^r$ be $S-$morphisms satisfying the compatibility conditions*

$$f_\alpha = g_{\alpha\beta} f_\beta \text{ in } U_\alpha \cap U_\beta \neq \emptyset.$$

Here we are using matrix multiplication, considering f_α and f_β as column vectors. Then the collection $\{f_\alpha\}$ defines an $S-$section f of E, since each f_α gives a section of $U_\alpha \times \mathbb{K}^r$, and this pulls back by the trivialization to a section of $E_{|U_\alpha}$. These sections of $E_{|U_\alpha}$ agree on the overlap $U_\alpha \cap U_\beta$ by the compatibility conditions imposed on $\{f_\alpha\}$, and thus define a global section. Conversely, any $S-$section of E has this type of representation. We call each f_α a trivializing section *of the section f.*

Remark 1.2.4. *Let $\pi : E \to X$ be a vector bundle, the set of all sections $S(X, E)$ can be equipped with the structure of $\mathbb{K}-$vector space by defining $\forall s, s' \in S(X, E), \forall x \in X, \forall \lambda \in \mathbb{K}$, the operations*

$$(s + s')(x) := s(x) + s'(x),$$
$$(\lambda s)(x) := \lambda s(x).$$

Moreover, $S(X, E)$ can be endowed with the S_X-module structure, by defining $\forall f \in S_X, \forall s \in S(X, E)$

$$(f \cdot s) := f(x)s(x), \quad \forall x \in X.$$

♠

1.2.3 Differential forms on vector bundles

A standard result in basic differential geometry is that to a given manifold M one can associate formally two vector bundles: the *tangent bundle TM* and the *cotangent bundle T^*M*, for details see [1], [2]. By using these two vector bundles one can construct the *exterior bundles*, i.e.

$$\bigwedge TM = \bigoplus_{p=0}^{n} \wedge^p TM \tag{1.5}$$

$$\bigwedge T^*M = \bigoplus_{p=0}^{n} \wedge^p T^*M. \tag{1.6}$$

With these bundles, it is possible to talk about differential forms, namely sections of (1.6)

$$\mathcal{E}^p(U) := \mathcal{E}(U, \wedge^p T^*M).$$

With this construction, we can define as usual the exterior derivative or de Rham operator

$$d: \ \mathcal{E}^p(U) \to \mathcal{E}^{p+1}(U)$$

for the construction of the de Rham operator the reader may see [5] and [15]. ♠

1.2.4 The pullback bundle

We begin by generalizing the notions of homomorphism of S-vector bundles.

Definition 1.10. *A $S-$morphism of $S-$vector bundles $\pi_E : E \to X$, $\pi_F : F \to Y$ consist of*

1. *a $S-$morphism,*
2. *for all $x \in X$, $f_x : E_x \xrightarrow{\simeq} F_x$ is $\mathbb{K}$-linear,*
3. *The following diagram is commutative*

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ X & \xrightarrow{\tilde{f}} & Y \end{array}$$

Proposition 1.2.2. *Let $f : X \to Y$ be a $S-$morphism and let $\pi : E \to Y$ be a $S-$bundles. Then there exist a unique (up to equivalences) vector bundle $\pi' : E' \to X$ such that the following diagram commutes*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\tilde{f}} & Y \end{array}$$

Such a vector bundle is called pullback bundle*, and as it is customary in the literature we denote E' by f^*E.*

For a proof of the above proposition see [6].
If $\pi : E \to X$ and $\pi' E' \to Y$ are two $S-$bunldes and $f : E \to E'$ is a $S-$morphism, then f can be decomposed as the composition of a $S-$homomorphism and a $S-$morphism of vector bundles. Indeed, by Proposition 1.2.2 we have a pushforward diagram

$$\begin{tikzcd}
E \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\pi"'] & & \\
& f^*E \arrow[r, "\hat{f}"] \arrow[d, "\hat{\pi}"'] & E' \arrow[d, "\pi'"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Where the dashed arrow is given by

$$h(e) = (\pi(e), f(e).$$

Thus, it follows that $f = \hat{f} \circ h$ as wanted. ♠

1.3 Almost complex manifolds and the $\bar{\partial}-$operator

1.3.1 Linear intermezzo: complex structures

Definition 1.11. *Let V be a real vector space. A complex structure on V is an element $J \in \mathrm{Aut}(V) = GL(V)$ such that $J^2 = -I$. The tuple (V, J) is called linear complex structure.*

In (V, J) is a linear complex structure then it induces on V a $\mathbb{C}-$vector space structure, indeed the exterior binary operation is given by

$$(\alpha + i\beta)v := \alpha v + \beta J(v), \ \forall \alpha, \beta \in \mathbb{R}, \ \forall v \in V.$$

Conversely, let V be a $\mathbb{C}-$vector space, then the multiplication by the imaginary unit i induces a linear complex structure for the underlying real vector space associated to V, i.e. $J(v) = iv$.

Example 1.3.1. Consider the usual euclidean space $\mathbb{C}^n$ of $n-$tuples of complex numbers $\{z_1, ..., z_n\}$and let $z_j = x_j + iy_j, j = 1, ..., n$ be the real and imaginary parts. Then $\mathbb{C}^n$ can be identified with $\mathbb{R}^{2n}$. The scalar multiplication by i in $\mathbb{C}^n$ induces a mapping

$$J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

given by

$$J(x_1, y_1, ..., x_n, y_n) = (-y_1, x_n, ..., -y_n, x_n).$$

Moreover $J^2 = -1$. This is the *standard complex structure* on $\mathbb{R}^{2n}$.

Example 1.3.2. (*The complex tangent space*) Let X be a complex manifold. As is known from analysis, every complex manifold admits a underlying smooth structure that will be called henceforth X_0. Let $x \in X$, denote by T_xX the complex tangent space and by T_xX_0 the complex space at x in X_0. We want to show that $T_xX \simeq T_xX_0$ canonically, and that T_xX induces a linear complex structure

$$J_x : T_xX_0 \to T_xX_0.$$

To do so, let (U, h) be a holomoprhic chart around the chosen point x. Then

$$h : \ U \xrightarrow{\sim} U' \subseteq \mathbb{C}^n$$

induces

$$\begin{aligned} h_{\mathbb{R}} : U &\xrightarrow{\sim} \mathbb{R}^{2n} \\ x &\mapsto (\Re(h(x_1)), \Im(h(x_1)), ..., \Re(h(x_n)), \Im(h(x_n))). \end{aligned}$$

The map $h_{\mathbb{R}}$ is real analytic, therefore rh is also smooth. It sufficies to prove the above claim in the origin, i.e. we shall prove that $T_0\mathbb{C}^n \simeq T_{\mathbb{R}}^{2n}$. Let

$$b_{\mathbb{C}}^0 = \{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_n}\}$$

be a basis for $T_0\mathbb{C}^n$, and

$$b_{\mathbb{R}}^0 = \{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\}$$

be a basis for $T_0\mathbb{R}^{2n}$.
To the choice of a basis correspond a choice of isomorphism, hence we have the following commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\sim"] \arrow[d, equal] & T_0\mathbb{C}^n \arrow[d, "\simeq"'] \\ \mathbb{R}^{2n} \arrow[r, "\sim"] & T_0\mathbb{R}^{2n} \end{tikzcd}$$

Because of previous example we see that $J_x : T_0\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the desired complex structure. We shall now prove that $J_x : T_0X_0 \to T_0X_0$ does not depend on the chosen holomorphic chart. To do so, let $f : N \xrightarrow{\sim} N$ be a biholomorphism, N is a open neighbourhood of the origin of $\mathbb{C}^n$ and assume that $f(0) = 0$. We can write f as follows

$$f(z) = \zeta = \Re(\zeta) + \Im(\zeta) = \xi + i\eta, \quad \xi = u(x, y),\ \eta = v(x, y).$$

Where ξ and η are diffeomorphisms associated to the change of coordinates prescribed from the map f. The Jacobian matrix of f represents that change of coordinates. Hence, to prove the claim, it sufficies to show that the Jacobian matrix of f commutes with the matrix associated to the linear map J_x. we have

$$M = \begin{pmatrix} \frac{\partial u^\alpha}{\partial x^\alpha} & \frac{\partial v^\alpha}{\partial x^\alpha} \\ \frac{\partial u^\alpha}{\partial y^\alpha} & \frac{\partial v^\alpha}{\partial y^\alpha} \end{pmatrix}$$

Using the Cauhy-Riemann equations the entrances of M become

$$\begin{pmatrix} \frac{\partial v^\alpha}{\partial x^\alpha} & \frac{\partial v^\alpha}{\partial x^\alpha} \\ -\frac{\partial v^\alpha}{\partial x^\alpha} & \frac{\partial v^\alpha}{\partial x^\alpha} \end{pmatrix}$$

So it is of type

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Now it is easy to see that $MJ = JM$ as wanted.

We have seen that $\mathbb{C}$ can be seen as a vector space with basis $\{1, i\}$. Then the tensor product

$$V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$$

for a real vector space V is called *complexification* of V. With the following definition

$$\alpha \cdot (v \otimes x) := v \otimes (\alpha z),\ \forall \alpha, z \in \mathbb{C},\ \forall v \in V$$

$V_\mathbb{C}$ becomes a complex vector space. The conjugation in that space is defined as

$$\overline{v \otimes \alpha} := v \otimes \bar{\alpha}.$$

Let (V, J) be a complex structure, consider the complexification of V, $V_\mathbb{C}$, by the properties of tensor product there exists a linear map induced from J in $V_\mathbb{C}$

$$J = J \otimes id_\mathbb{C} : V_\mathbb{C} \to V_\mathbb{C}.$$

It is easy to see that J is still a complex structure. Moreover J has eigenvalues $\{-i, i\}$. Call $V^{1,0}$ the eigenspace relative to the eigenvalue i and $V^{0,1}$ the eigenspace relative to the eigenvalue $-i$. Then, because of the spectral theorem, we find

$$V_{\mathbb{C}} \simeq V^{1,0} \oplus V^{0,1}.$$

Since complex conjugation is a linear isomorphism, then we deduce that $V^{1,0} \simeq V^{0,1}$. These considerations extend to every real vector space. We now want to consider the exterior algebra of these complex vector spaces. Namely, $\wedge V_{\mathbb{C}}$, $\wedge V^{1,0}$ and $\wedge V^{0,1}$. Then we have natural injections

$$\wedge V^{1,0} \hookrightarrow \wedge V_{\mathbb{C}}$$
$$\wedge V^{0,1} \hookrightarrow \wedge V_{\mathbb{C}}$$

and we let $\wedge^{p,q} V$ be the subspace of $\wedge V_{\mathbb{C}}$ generated by elements of the form $\{u \wedge w\}$, where $u \in \wedge^p V^{1,0}$ and $w \in \wedge^q V^{0,1}$. Thus we have the direct sum

$$\wedge V_{\mathbb{C}} = \bigoplus_{r=0}^{2n} \bigoplus_{p+q=r} \wedge^{p,q} V.$$

In the next subsection we carry out the above algebraic construction on the tangent bundle to a manifold.

♠

1.3.2 Almost complex structures

Definition 1.12. *Let X be a smooth manifold of dimension $2n$. Suppose that $J : TX \to TX$ is a vector bundle equivalence, to be called* almost complex structure*, such that $\forall x \in X$ $J_x : T_x X \to T_x X$ is a linear complex structure and $J^2 = -I$. If X is equipped of a almost complex structure then the tuple (X, J) will be christened* almost complex manifold

One thinks about almost complex structure as a family $J = \{J_x\}_{x \in X}$ of linear complex structures smoothly parametrized by points of X. Furthermore, having a complex structure on a manifold X it is equivalent to prescribe a complex vector bundle on the real tangent bundle to the manifold.

Proposition 1.3.1. *A complex manifold X induces a almost complex structure on the underlying smooth manifold X_0.*

For a proof of the above proposition the reader may see [6]

Remark 1.3.1. *Because of the above proposition we see that a complex manifold determines a almost complex structure, the converse is not true in general, for counterexamples see [11]*

♠

1.3.3 Complex differential forms

Let X be a smooth manifold of dimension m. Consider the complexifications of tangent and cotangent bundles of X, i.e. $TX_{\mathbb{C}}$ and $T^*X_{\mathbb{C}}$ respectively. We define the space of differential forms complex valued of total degree r by

$$\mathcal{E}^r(X)_{\mathbb{C}} = \mathcal{E}(X, \wedge^r T^*X_{\mathbb{C}}).$$

In particular $\omega \in \mathcal{E}^r(X)_{\mathbb{C}}$ if and only if

$$\omega(x) = \sum_I \omega_I(x) dx_I$$

in a local chart (U, h). $\omega_I(x)$ are complex valued smooth functions. The de Rham operator is defined as usual, therefore we have the correspondent de Rham complex

$$\mathcal{E}^0_{\mathbb{C}} \xrightarrow{d} \mathcal{E}^1_{\mathbb{C}} \xrightarrow{d} \dots \xrightarrow{d} \mathcal{E}^m_{\mathbb{C}} \to 0.$$

Let (X, J) be a almost complex manifold. Consider $TX_{\mathbb{C}}$,and we apply fiberwise the linear algebra acquired in Subsection 1.3.1. Denote by $TX^{1,0}$ the vector bundle relative to the $\{i\}$-eigenspace and by $TX^{0,1}$ the vector bundle relative to the $\{-i\}-$eigenspace. We define naturally, always fiberwise, a conjugation

$$Q : TX_{\mathbb{C}} \to TX_{\mathbb{C}}$$

then $TX^{1,0} \simeq TX^{0,1}$. Consider the exterior bundles $\wedge T^*X_{\mathbb{C}}$, $\wedge T^*X^{1,0}$ and $\wedge T^*X^{0,1}$. We have that

$$T^*X_{\mathbb{C}} = T^*X^{1,0} \oplus T^*X^{0,1}$$

Therefore we have two natural injections

$$\wedge T^*X^{1,0} \hookrightarrow \wedge T^*X_{\mathbb{C}}$$

$$\wedge T^*X^{0,1} \hookrightarrow \wedge T^*X_{\mathbb{C}}.$$

Consider the vector bundle $\wedge^{p,q}T^*X$, its sections are the differential forms of type (p,q) on X that will be denoted by

$$\mathcal{E}^{p,q}(X) = \mathcal{E}(X, \wedge^{p,q}T^*X),$$

moreover we have that

$$\mathcal{E}^r(X)_{\mathbb{C}} = \bigoplus_{p+q=r} \mathcal{E}(X, \wedge^{p,q}T^*X).$$

We now look for a local representation for these forms. To do so we will use particular sections:

Definition 1.13. *Let $E \to X$ be a vector bundle of rank r, a family of local sections $s_j \in S(U, E)$, $1 \leq j \leq r$, is called* local frame *if* $\forall x \in U$ $\{s_j(x)\}_{j=1}^r$ *is a basis of* E_x.

Remark 1.3.2. *In a vector bundle to give a local trivialization it is equivalent to prescrive a local frame. For details see [1], [9]*

Consider a almost complex manifold (X, J), let $\{w_1, ..., w_r\}$ be a local frame of $T^*X^{1,0}$ and $\{\bar{w}_1, ..., \bar{w}_r\}$ be a local frame of $T^*X^{0,1}$. Then $\{w^I \wedge \bar{w}^J\}$ with $|I| = p$ and $|J| = q$ is a local frame of $\wedge^{p,q}T^*X$. Therefore, every $s \in \mathcal{E}^{p,q}(X)_{\mathbb{C}}$ can be locally represented by

$$s = \sum_I \sum_J a_{IJ} w^I \wedge \bar{w}^J, \quad a_{IJ} \in \mathcal{E}^0(U)_{\mathbb{C}}.$$

Apply the de Rham operator

$$ds = \sum_{IJ} (da_{IJ} w^I \wedge \bar{w}^J + a_{IJ} d(w^I \wedge \bar{w}^J)),$$

we see that the second term of the sum in the right hand side can be non zero, since the local sections w^i may not be constant. This will lead to the next and last Subsection. ♠

1.3.4 The ∂ and $\bar{\partial}$ operators and integrability

Let (X, J) be a almost complex manifold, we have seen that

$$\mathcal{E}^r(X) = \bigoplus_{p+q=r} \mathcal{E}^{p,q}(X)$$

since the direct sum is a biproduct, then it makes sense to define the following natural projection:

$$\pi_{p,q} : \mathcal{E}^r(X) \to \mathcal{E}^{p,q}(X).$$

Consider the de Rham operator

$$d : \mathcal{E}^{p,q}(X) \to \mathcal{E}^{p+q+1}(X)$$

we can have a decomposition of the above operator as follows

$$\partial := \pi_{p+1,q} \circ d : \mathcal{E}^{p,q} \to \mathcal{E}^{p+1,q}(X) \tag{1.7}$$

$$\bar{\partial} := \pi_{p,q+1} \circ d : \mathcal{E}^{p,q} \to \mathcal{E}^{p,q+1}(X) \tag{1.8}$$

These definitions naturally extends to all the complex $\mathcal{E}^{\bullet}(X)$.

Definition 1.14. *Let* (X, J) *be a almost complex manifold. If*

$$d = \partial + \bar{\partial}$$

then we say that (X, J) *is* integrable.

Since every complex manifold determines a almost complex structure in the underlying smooth manifold, one can ask if the determined almost complex structure is also integrable as the following result shows

Theorem 1.3.1. *Let* X *be a complex manifold, then* (X_0, J) *is an integrable almost complex structure*

For a proof of this theorem see [6]. The converse of this theorem is a profound result [6], [11].

Theorem 1.3.2. *(*Newlander-Nirenberg*) Let* (X, J) *be an integrable almost complex manifold. Then there exists a unique complex structure* $\mathcal{O}_X$ *on* X *which induces the almost complex structure* J*.*

Chapter 2

Sheaf Theory

In this chapter, we will se some useful notion of sheaf theory, that will be used during this work. The major virtue of sheaves is that they unify and give a mechanism for dealing with many problems concerned with passage from local information to global information. For this treatment, the reader should be familiar with the basic concept of homological algebra and category theory that can be found in [10]. For a deeper treatment of Sheaf Theory the reader may see [8].

2.1 Presheaves and Sheaves

In this section we will give the definition of presheaf and sheaf, and we will give numerous examples. Henceforth we denote by $\mathrm{Ob}(-)$ the objects of some category.

Definition 2.1. *Let X be a topological space, i.e. $X \in \mathrm{Ob}(\mathcal{T}\mathrm{op})$, a* presheaf *$\mathcal{F}$ on X consist of*

a. an assignment
$$X \supset U \mapsto \mathcal{F}(U) \in \mathrm{Ob}(\mathrm{Set}).$$

b. $\forall V \subset U$ open we have a restriction homomorphism
$$r_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$$

such that

1. $r_U^U = 1_{\mathcal{F}(U)}$.
2. *If $W \subset V \subset U$*
$$r_W^U = r_W^V \circ r_V^U.$$

Remark 2.1.1. *The definition of presheaf naturally adapts to all algebraic structures. Indeed, it is possible to define presheaves of abelian groups, rings and modules, where the assignment would be a functor from the category of the open set of a topological space X, where morphisms are the inclusions, to the category of abelian groups, or rings, or modules. The restriction homomorphisms will then be a morphism between objects of the considered category.*

Examples of presheaves are in rich supply. For our targets, we are interested in a particular kind of presheaves that permit to transform local datas into global datas.

Definition 2.2. *A* sheaf *on $X \in \mathrm{Ob}(\mathcal{T}\mathrm{op})$, is a presheaf $\mathcal{F}$ on X, such that for every collection of open sets U of X where U is the union of other open sets, i.e. $U = \cap_i U_i$, the following axioms are satisfied:*

S1. $\forall s, t \in \mathcal{F}(U)$ *such that* $r^U_{U_i}(s) = r^U_{U_i}(t)$, *then* $s = t$.

S2. given $s_i \in \mathcal{F}(U_i)$ *such that* $r^{U_i}_{U_i \cap U_j}(s_i) = r^{U_j}_{U_i \cap U_j}(s_j)$, $\forall i, j$. *Then, there exists a* $s \in \mathcal{F}(U)$ *such that* $s_{|U_i} = s_i$.

Remark 2.1.2. *From the above definition, we can see that axiom S1 tells that elements defined on large open sets that locally coincide are globally indistinguishable. Axiom S2 tells that local elements can be glued together to form a global element.*

Definition 2.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be (pre)sheaves on $X \in \mathrm{Ob}(X)$. A morphism of (pre)sheaves*

$$h : \mathcal{F} \longrightarrow \mathcal{G}$$

is a collection of maps

$$h_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U),$$

where U is an open subset of X, such that for any other open subset $V \subset U$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{h_U} & \mathcal{G}(U) \\ {\scriptstyle r^U_V}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{r}^U_V} \\ \mathcal{F}(V) & \xrightarrow{h_V} & \mathcal{G}(V) \end{array}$$

Moreover, if h_U is an inclusion, then we say that $\mathcal{F}$ is a sub(pre)sheaf of $\mathcal{G}$. An isomorphism of (pre)sheaves is defined in obvious way.

Example 2.1.1. Let $X, Y \in \mathrm{Ob}(\mathcal{T}op)$, consider

$$C_{X,Y} := \{f : X \to Y : f \ \text{ is continuous}\} \tag{2.1}$$

Let $\{U_i\}$ be an open cover of X. The assignment

$$X \supset U \mapsto C_{X,Y}(U) = C_{U,Y},$$

is clearly a presheaf. Furthermore, if, for all index i we have that $f, g \in C_{X,Y}$ are such that $f_{|U_i} = g_{|U_i}$, then $f = g$. Therefore, axiom $S1$ is satisfied. Moreover, given $f_i \in C_{U_i,Y}$, if $f_{i_{|U_i \cap U_j}} = f_{j_{|U_i \cap U_j}}$, then by the gluing lemma for continuous functions [13] there exists $f \in C_{X,Y}$ such that $f_{|U_i} = f_i$. Thus, axiom S2 is satisfied. Hence (2,1) is a sheaf.

Example 2.1.2. In the previous example, if we let $Y = \mathbb{K}$ be a field that can be $\mathbb{R}$ or $\mathbb{C}$, then we have a sheaf of commutative algebras $C_{X,\mathbb{K}} := C_X$.

Example 2.1.3. Let X be a $\mathcal{S}$-manifold, then $\mathcal{S}_X$ is a subsheaf of C_X called the *structure sheaf* of X.

Example 2.1.4. Let $X \in \mathrm{Ob}(\mathcal{T}op)$. If G is an abelian group, then the assignment

$$X \supset U \mapsto G$$

defines a sheaf, called the *constant sheaf*. The same construction could be done with any other algebraic structure.

Definition 2.4. *Let $\mathcal{R}$ be a sheaf of commutative rings on $X \in \mathrm{Ob}(\mathcal{T}op)$, and let $\mathcal{M}$ be a sheaf of abelian groups on X. We define the sheaf of $\mathcal{R}$-modules by the assignment:*

$$X \supset U \mapsto \mathcal{R}(U)$$

and restriction homomorphism

$$r_V^U(\alpha f) = \rho_V^U(\alpha) r_V^U(f).$$

The above holds for every open subset $V \subset U$, $\forall \alpha \in \mathcal{M}(U)$, $\forall f \in \mathcal{R}(U)$. The maps ρ_V^U are the restriction homomorphisms of $\mathcal{M}$.

Example 2.1.5. Let $\pi : E \to X$ be a vector bundle of rank r. The module of $\mathcal{S}$-section on E, i.e. $\mathcal{S}(E)$ is a sheaf of $\mathcal{S}_X-$modules by the assignment

$$X \supset U \mapsto \mathcal{S}(U) := \mathcal{S}(U, E) = \{s : U \to E : \pi \circ s = 1_U\}.$$

Directly from the above definition, we see that $\mathcal{S}(E)$ is a subsheaf of $C_{X,E}$.

Example 2.1.6. A special case of previous example is $\mathcal{E}_X^\bullet$ of a smooth manifold. Or $\mathcal{E}^{p,q}(X)$ on a complex manifold X. These are sheaf of $\mathcal{E}_X-$modules.

Remark 2.1.3. *If $\mathcal{F}$ is a sheaf over $X \in \mathrm{Ob}(\mathcal{T}op)$, then there is a natural restriction of $\mathcal{F}$ on the open set $U \subset X$, and it will be denoted by $\mathcal{F}_{|U}$.*

Definition 2.5. *Let $\mathcal{R}$ be a sheaf of cummutative rings on $X \in \mathrm{Ob}(\mathcal{T}op)$. We define:*

(a) Direct sum sheaf *the sheaf of $\mathcal{R}-$modules*

$$X \supset U \mapsto \mathcal{R}^p(U) = \underbrace{\mathcal{R}(U) \oplus ... \oplus \mathcal{R}(U)}_{p-times}$$

where p is a positive integer.

(b) If $\mathcal{M}$ is a sheaf of $\mathcal{R}-$modules, we say that is free *if $\mathcal{M} \simeq \mathcal{R}^p$. We say that $\mathcal{M}$ is* locally free *if $\mathcal{M}_{|U} \simeq \mathcal{R}^p$, for $U \subset X$ open.*

Theorem 2.1.1. *Let X be a connected $\mathcal{S}-$manifold. Then, there is a one to one correspondence between isomorphism classes of $\mathcal{S}-$vector bundles on X and isomorphism classes of sheaf of locally free $\mathcal{S}-$modules on X.*

Proof. ($\Rightarrow$) Consider the sheaf of sections $\mathcal{S}(E)$ of a vector bundle $E \to X$. We show that $\mathcal{S}(E)$ is locally free. Since $E \to X$ is a vector bundle, then for every open trivializing set $U \subset X$ we have $\mathcal{S}(E)_{|U} \simeq \mathcal{U} \times \mathbb{K}^{\nabla}$, since $E_U \simeq U \times \mathbb{K}^r$. Then, it sufficies to prove that $\mathcal{S}(U \times \mathbb{K}^r)$ is free. Let $f \in S(U \times \mathbb{K}^r)(V)$, then $f(x) = (x, g(x))$, where g is a vector valued function, i.e. $g(x) = (g_1(x), ..., g_r(x)) \in \mathbb{K}^r$, and $g_j \in \mathcal{S}(V)$. Thus, from the definition of trivializing section we have the following bijection

$$\mathcal{S}(U \times \mathbb{K}^r) \ni f \mapsto (g_1, ..., g_r) \in \mathcal{S}_U(V) \oplus ... \oplus \mathcal{S}_U(V).$$

Therefore, $\mathcal{S}(E)$ is a locally free sheaf of $\mathcal{S}-$modules.
($\Leftarrow$) Let $\mathcal{L}$ be a locally free sheaf of $\mathcal{S}-$modules on X. Consider a open cover $\{U_\alpha\}$ of X. Being $\mathcal{L}$ locally free, then we have the following isomorphism

$$g_\alpha : \ \mathcal{L}_{|U_\alpha} \xrightarrow{\simeq} S^r_{|U_\alpha}.$$

Since X is connected the above ismorphism does not depend on α. Therefore, we can define

$$g_{\alpha\beta} : \ \mathcal{S}_{|U^r_\alpha \cap U_\beta} \to \mathcal{S}_{|U^r_\alpha \cap U_\beta},$$

that induces a liner map

$$g_{\alpha\beta} : \ U_\alpha \cap U_\beta \to GL(r, \mathbb{K})$$

that satisfies the Čech cocycle condition. Hence, because of the properties of vector bundles, there exists a vector bundle $E \to X$ that has $\{g_{\alpha\beta}\}$ as transition functions. $\square$

2.2 Resolutions

In this section we will learn how local datas of a certain topological space can give global information. To do so, we require some preparation.

2.2.1 Sheafifications

In the following, we will learn how to turn a presheaf into a sheaf on a given topological space X.

Definition 2.6. *Let $X \in \mathrm{Ob}(\mathcal{T}op)$, a* etale *space on X consist of*

(a) a topological space $Y \in \mathrm{Ob}(\mathcal{T}op)$,

(b) a surjective continuous map

$$\pi : Y \to X$$

that is also a local homeomorphism.

A *section* $s : U \to Y$ for a etale space $\pi : Y \to X$ is a continuous function such that $\pi \circ s = 1_U$. Denote by $\Gamma(U, Y)$ the space of sections of the etale space $\pi : Y \to X$. Cleary $\Gamma(U, Y)$ is a subsheaf of $C_{X,Y}$, therefore it is a sheaf. By using the notion of etale space we can associate a sheaf to a topological space X in which a presheaf is prescribed. This process, that we will describe in the following, is called *sheafification*. Let $X \in \mathrm{Ob}(\mathcal{T}op)$ and $\mathcal{F}$ be a presheaf of X. On every open set $U \subset X$ consider the direct limit:

$$\varinjlim_{x \in U} \mathcal{F}(U) := \mathcal{F}_x.$$

Where the direct limit is considered with respect the restriction homomorphisms of $\mathcal{F}$. The induced restriction homomorphism define the representatives of the classes of $\mathcal{F}_x$, namely

$$r_x^U(s) := s_x$$

where $s \in \mathcal{F}(U)$ and s_x is called *germ* of s at x. Consider

$$\tilde{\mathcal{F}} := \bigcup_{x \in X} \mathcal{F}_x.$$

Then, we have a natural projection

$$\pi : \tilde{\mathcal{F}} \to X$$

defined in the natural way, i.e. $\pi(s_x) = x$. Since π is naturally surjective, then we have some right inverses $\tilde{s} : U \to \tilde{\mathcal{F}}$, defined as $\tilde{s}(x) = s_x$. Consider the family of sets $\{\tilde{s}(U)\}_{U \subset X}$. Observe that such a family covers $\tilde{F}$, and given $\tilde{s}(U_1) \cap \tilde{s}(U_2)$, being U_1, U_2 open sets of X, they can be expressed by unions of other subsets $U_1 = \cup_j U_{1j}$ and $U_2 = \cup_j U_{2j}$. Thus $\tilde{s}(U_1) \cap \tilde{s}(U_2) = \cup_j \tilde{s}(U_{1j}) \cap \tilde{s}(U_{2j})$. Then, there exists a unique topology on $\tilde{\mathcal{F}}$ that has $\{\tilde{s}(U)\}_{U\subset X}$ as basis. With respect to this topology it is easy to see that the map π becomes a continuous map and it is a local homeomorphism. Therefore, $\pi : \tilde{\mathcal{F}} \to Y$ is a etale space. In the above construction we did not consider any algebraic structure on the prescribed presheaf $\mathcal{F}$. In case $\mathcal{F}$ has some algebraic structure, then the elements of $\mathcal{F}_x$ are called *stalks* and they inherit the algebraic structure. To give an insight, suppose that $\mathcal{F}$ is a presheaf of abelian groups, then the associated etale space satisfy the following properties:

- $\mathcal{F}_x$ is a abelian group $\forall x \in X$.
- If $\tilde{\mathcal{F}} \circ \tilde{\mathcal{F}} = \{(s,t) \in \tilde{\mathcal{F}} \times \tilde{\mathcal{F}} : \pi(s) = \pi(t)\}$ then we have an continuous operation

$$\begin{aligned} \mu :& \tilde{\mathcal{F}} \circ \tilde{\mathcal{F}} \to \tilde{\mathcal{F}} \\ & (s_x, t_x) \mapsto s_x - t_x \end{aligned}$$

- For every open subset $U \subset X$, $\Gamma(U, \tilde{\mathcal{F}})$ is an abelian group, by defining pointwise the operation on sections, i.e.

$$(s-t)(x) = s(x) - t(x), \ \forall x \in U.$$

In the above construction we have seen that in a topological space X where a presheaf $\mathcal{F}$ is prescribed, we can associate a etale space and therefore a sheaf of continuous function $\tilde{\mathcal{F}} = \Gamma(U, \tilde{\mathcal{F}})$. Then, is rather natural to define a map

$$\tau : \mathcal{F} \to \tilde{\mathcal{F}},$$

defined as follows, for every open subset U of X

$$\begin{gathered} \tau_U : \mathcal{F}(U) \to \tilde{\mathcal{F}}(U) \\ [\tau_U(s)](x) = [\tau_U(s(x))] = \tilde{s}(x) = r_x^U(s) = s_x. \end{gathered}$$

Being defined on the presheaves then we have the following commutative diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\tau_U"] \arrow[d, "r_V^U"'] & \tilde{\mathcal{F}}(U) \arrow[d, "\tilde{r}_V^U"] \\ \mathcal{F}(V) \arrow[r, "\tau_V"] & \tilde{\mathcal{G}}(V) \end{tikzcd}$$

Proposition 2.2.1. *In the above situation, in $\mathcal{F}$ is a sheaf then τ is a bijection.*

Proof. (a) We prove that τ_U is injective. Let $s, t \in \mathcal{F}(U)$ such that $\tau_U(s) = \tau_u(t)$. Then by definition $[\tau_U(s)](x) = [\tau_U(t)](x) \Leftrightarrow r_x^U(s) = r_x^U(t)$ that is true also in the neighbourhood of x, namely there is a neighbourhood of x, $V \subset U$, such that $r_V^U(s) = r_V^U(t)$. Thus, we can cover U by open sets U_i that behaves like V. Namely, for every index i we have $r_{U_i}^U(s) = r_{U_i}^U(t)$. Therefore, since $\mathcal{F}$ is a sheaf, then by axiom S1, we have that $s = t$.
(b) We prove that τ_U is surjective, let $\sigma \in \Gamma(U, \tilde{\mathcal{F}}), x \in U$. Then, there exists an open neighbourhood V of x such that $\sigma_{|V}(x) = [\tau_V(s)](x)$. If two sections coincide in a point, then they still coincide in a neighbourhood of that point, say V^*.Namely,

$$\sigma_{V^*} = \tau_V(s)_{|V^*} = \tau_{V^*}(r_{V^*}^V(s)). \tag{2.2}$$

That is true $\forall x \in U$. Therefore, we can cover U with open sets U_i that behaves like in (2.2). Thus, there exist $s_i \in \mathcal{F}(U_i)$ such that $\tau_{U_i}(s_i) = \sigma_{|U_i}$. Moreover, in the overlap $U_i \cap U_j$ we have $\tau_{U_i}(s_i) = \tau_{U_j}(s_j)$, then

$$r_{U_i \cap U_j}^{U_i}(s_i) = r_{U_i \cap U_j}^{U_j}(s_j).$$

Since $\mathcal{F}$ is a sheaf then by axiom S2 there exists $s \in \mathcal{F}(U)$ such that $r_{U_i}^U(s) = s_i$. Thus,

$$\tau_{U_i}(r_{U_i}^U(s)) = \tau_U(s)_{|U_i} = \tau_{U_i}(s_i) = \sigma_{|U_i} \;\Rightarrow \sigma = \tau_U(s).$$

□

Using the above construction, we can construct another sheaf. Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaf of abelian groups on a topological space X. Let $\mathcal{G}$ be a subsheaf of $\mathcal{F}$. Consider the sheaf of sections generated by

$$\{X \supset U \mapsto \mathcal{F}(U)/\mathcal{G}(U)\}.$$

The above generates the *quotient sheaf* $\mathcal{F}/\mathcal{G}$. We have a natural projection on the quotient $\forall U \subset X$

$$\mathcal{F}(U) \to \mathcal{F}(U)/\mathcal{G}(U),$$

by taking the direct limits, and using Proposition 2.2.1 we have a well defined map of sheaves

$$\mathcal{F} \to \mathcal{F}/\mathcal{G}.$$

♠

2.2.2 Short exact sequences of abelian sheaves

Definition 2.7. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be sheaves of abelian groups over a topological space X. A sequence of abelian groups and homomorphism h, g*

$$\mathcal{A} \xrightarrow{h} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

is called exact *in $\mathcal{B}$ if $\forall x \in X$ the relative sequence of stalks is exact, i.e. $\forall x \in X$*

$$\mathcal{A}_x \xrightarrow{h_x} \mathcal{B}_x \xrightarrow{g_x} \mathcal{C}_x$$

we have $\ker g_x = \mathrm{im} f_x$.
Similarly the sequence

$$0 \to \mathcal{A} \xrightarrow{h} \mathcal{B} \xrightarrow{g} \mathcal{C} \to 0$$

is called short exact *if $\forall x \in X$, the sequence*

$$0 \to \mathcal{A}_x \xrightarrow{h_x} \mathcal{B}_x \xrightarrow{g_x} \mathcal{C}_x \to 0$$

is short exact.

Example 2.2.1. (*The exponential sequence*) Let X be a complex connected manifold, consider the following sheaves of abelian groups

$$\begin{aligned} \mathcal{O} &:= \{f : X \to \mathbb{C} : f \text{ is holomorphic}\}, \\ \mathcal{O}^* &:= \{f : X \to \mathbb{C}_* : f \text{ is holomorphic}\}, \\ &\mathbb{Z} \ \text{ is the constant sheaf}. \end{aligned}$$

Consider the following sequence:

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0, \tag{2.3}$$

where i is the canonical inclusion and the map $\exp : \mathcal{O} \to \mathcal{O}^*$ is defined on every open set $U \subset X$ as

$$\exp_U(f)(x) = \exp(2\pi i f(z)).$$

In order to show exactness in $\mathcal{O}^*$, is sufficient to prove that $\exp_x$ is surjective $\forall x \in X$. To do so, we observe that if we invert the exponential mapping, the complex logarithm is a multivalued function. To avoid that, we should restrict the range of invertibility by using a fundamental result of covering space theory [14]

Theorem 2.2.1. *Let $p : \tilde{X} \to X$ be a covering space, consider $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0)$, and let Y be a connected and locally path connected topological space. Consider y_0 and $f : Y \to X$ be a continuous function such that $f(y_0) = x_0$. Then, there exists a unique lifting $\tilde{f}$ of f with respect to the map p if and only if $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$.*

Therefore, in our case the unique possibility to invert the $\exp_x$ is in a symply connected open subset $U \subset X$. In that case, we have $f_x = \left(\frac{1}{2\pi i} logg\right)_x$, thus $\exp_x f_x = g_x$.Hence, $\exp_x$ is surjective $\forall x \in X$. It is easy to see that $\ker(\exp_x) = \mathbb{Z} = \mathrm{im} i$, and that i is injective. This shows exactness in $\mathcal{O}$ and $\mathbb{Z}$ respectively. Thus, (2.3) is a short exact sequence and is called *exponential sequence.*

Example 2.2.2. Let $\mathcal{A}$ be a subsheaf of abelian groups of the sheaf of abelian groups $\mathcal{B}$, then it is easy to see that the following sequence is short exact:

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{B}/\mathcal{A}.$$

Example 2.2.3. From the short exact sequence of previous example, we can consider a particular case. Let $X = \mathbb{C}$, consider the sheaf $\mathcal{O}$ of holomorphic functions and let $\mathcal{I}$ be the subsheaf of $\mathcal{O}$ of holomorphic functions that vanish in zero. Then we have the following exact sequence

$$0 \to \mathcal{I} \to \mathcal{O} \to \mathcal{O}/\mathcal{I} \to 0,$$

where

$$(\mathcal{O}/\mathcal{I})_x = \begin{cases} \mathbb{C}, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$$

Example 2.2.4. Let X be a connected Hausdorff space, let $a, b \in X$ such that $a \neq b$. Let $\mathbb{Z}$ the constant sheaf con X, and let $\mathcal{I}$ be the constant subsheaf of $\mathbb{Z}$ that is zero in a and b. Then the following sequence is short exact:

$$0 \to \mathcal{I} \to \mathbb{Z} \to \mathcal{Z}/\mathcal{I} \to 0,$$

where

$$(\mathbb{Z}/\mathcal{I})_x = \begin{cases} \mathbb{Z}, & \text{if } x \neq a \text{ and } x \neq b \\ 0, & \text{if } x = a \text{ or } x = b \end{cases}$$

♠

2.2.3 Graded sheaves, differential sheaves and resolutions

Definition 2.8. *Let X be a topological space.*

- *A* graded *sheaf of abelian groups o modules is a indexed family of sheaves $\{\mathcal{F}^\alpha\}_{\alpha \in \mathbb{Z}}$ together with a sequence connected by sheaf morphism:*

$$\ldots \to \mathcal{F}^0 \xrightarrow{\alpha_0} \mathcal{F}^1 \xrightarrow{\alpha_1} \mathcal{F}^2 \xrightarrow{\alpha_2} \ldots \tag{2.4}$$

- *A* differential *sheaf is a graded sheaf whose sequence (2.4) has* $\alpha_i \circ \alpha_{i-1} = 0$ *at every level* i*.*

- *A* resolution *of a sheaf* $\mathcal{F}$ *is an exact sequence of the form*

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to \dots$$

that we will indicate as

$$0 \to \mathcal{F} \to \mathcal{F}^\bullet$$

Example 2.2.5. (*The resolution of the complex and real constant sheaves*). Let X be a smooth manifold of real dimension n. Let $\mathcal{E}_X^p$ be the sheaf of differential form of degree p. Then there is a resolution of the constant sheaf $\mathbb{R}$ given by:

$$0 \to \mathbb{R} \xrightarrow{i} \mathcal{E}_X^0 \xrightarrow{d} \mathcal{E}_X^1 \xrightarrow{d} \mathcal{E}_X^2 \xrightarrow{d} \dots \xrightarrow{d} \mathcal{E}_X^n \to 0. \tag{2.5}$$

In the above d is the exterior derivative, we know that $d^2 = 0$, therefore (2.5) is a differential sheaf. For the Poincaré's Lemma [5], [15], if $U \subseteq \mathbb{R}^n$ is a star shaped open set where for $f \in \mathcal{E}_X^2, df = 0$, then there exists a differential form $g \in \mathcal{E}_X^{p-1}$ such that $dg = f$. We can always restrict the above situation in a local chart where the Poincaré's Lemma is satisfied. The exactness in $\mathcal{E}_X^0$ follows from the basic theory of ordinary differential equations. Similarly follows for the constant sheaf $\mathbb{C}$. Hence, we have the two desired resolutions

$$0 \to \mathbb{R} \to \mathcal{E}_X^\bullet, \tag{2.6}$$

$$0 \to \mathbb{C} \to \mathcal{E}_X^\bullet. \tag{2.7}$$

Example 2.2.6. (*The reslution of the constant sheaf of abelian groups*). Let X be a topological manifold. We want to find a resolution for the constant sheaf of abelian groups G on X, where G is a abelian group. On every open set $U \subset X$, we consider the set of standard $p-$simplices $\{f : \Delta^p \to U\}$, we take the free abelian group of that linearization and we can construct the singular $p-$chain, i.e. the abelian group with coefficient in $\mathbb{Z}$, that will be denoted by $S_p(U, \mathbb{Z})$. With this in mind, we define the singular cochain by writing

$$S^p(U, G) := \mathrm{Hom}_{\mathbb{Z}}(S_p(U, \mathbb{Z}), G).$$

The assignment $U \mapsto S^p(U, G) = S(G)(U)$, determines the sheaf of singular conchains. Together with the coboundary operator

$$\delta^p : S^p(U, G) \to S^{p+1}(U, G),$$

that satisfies $\delta^{p+1} \circ \delta^p = 0$, we have a differential sheaf

$$... \to S^{p-1}(U,G) \xrightarrow{\delta^{p-1}} S^p(U,G) \xrightarrow{\delta^p} S^{p+1}(U,G) \xrightarrow{\delta^{p+1}} ... \tag{2.8}$$

The above sequence is used to define the cohomology groups of the singular cohains

$$H^p(S^p(U,G)) := \frac{\ker(\delta^p)}{\operatorname{im}(\delta^{p-1})}. \tag{2.9}$$

If U is an open ball in a euclidean space then the cohomology groups (2.9) are trivial $\forall p > 0$, and the sequence (2.8) is exact at every level [10]. Then for $p > 0$ we have the resolution of G

$$0 \to G \to S^0(G) \to S^1(G) \to S^2(G) \to ...$$

where $\ker(\delta : S^0(G) \to S^1(G)) \simeq G$. The same considerations hold also in case X is a smooth manifold. For notational brevity we denote the resolution of G, in case X is a smooth manifold, as

$$0 \to G \to S^\bullet_\infty(G) \tag{2.10}$$

Example 2.2.7. (*The resolution of the sheaf of differential forms*). Let X be a complex manifold of complex dimension m. Let $\mathcal{E}^{p,q}$ be the sheaf of $(p,q)-$differential forms on X. Consider the following sequence

$$0 \to \Omega^p \xrightarrow{i} \Omega^{p,0} \xrightarrow{\bar\partial} \Omega^{p,1} \xrightarrow{\bar\partial} \Omega^{p,2} \xrightarrow{\bar\partial} ... \xrightarrow{\bar\partial} \Omega^{p,m} \to 0 \tag{2.11}$$

Where $\omega^p = \ker(\bar\partial : \mathcal{E}^{p,0} \to \mathcal{E}^{p,1})$, and that is the sheaf of holomorphic forms of type $(p,0)$. The sequence (2.11) is also exact. This can be seen using the "complex version" of the Poincaré Lemma, that is the Dolbeault's Lemma. Namely, if ω is a (p,q)-differential form defined in a polydisk of $\mathbb{C}^n$, i.e. $\Delta = \{z \in \mathbb{C} : |z| < i\}$, such that $\bar\partial\omega = 0$. Then, there exist a (p,p-1)-form u defined in $\Delta' \subset \Delta$ such that $\bar\partial u = \omega$ in Δ'. Hence,

$$0 \to \Omega^p \to \mathcal{E}^{p,\bullet} \tag{2.12}$$

is a resolution for Ω^p. Similar considerations hold for

$$0 \to \Omega^p \to \mathcal{E}^{\bullet,p}$$

and for the constant sheaf

$$0 \to \mathbb{C} \to \Omega^\bullet$$

Consider two differential sheaves $\mathcal{L}^\bullet$ and $\mathcal{M}^\bullet$ on a topological space X. A morphism of differential sheaves $f_\bullet : \mathcal{L}^\bullet \to \mathcal{M}^\bullet$ is a sequence of sheaf morhphism $f_j : \mathcal{L}^j \to \mathcal{M}^j$ for which the following diagram is commutative at every level j

$$\begin{array}{ccc}
\mathcal{L}^j & \xrightarrow{f_j} & \mathcal{M}^j \\
{\scriptstyle d_{\mathcal{L}^j}}\downarrow & & \downarrow{\scriptstyle d_{\mathcal{M}^j}} \\
\mathcal{L}^{j+1} & \xrightarrow{f_{j+1}} & \mathcal{M}^{j+1}
\end{array}$$

Similarly, given two resolutions for sheaves $\mathcal{A}$, $\mathcal{B}$ on X

$$0 \to \mathcal{A} \to \mathcal{A}^\bullet$$
$$0 \to \mathcal{B} \to \mathcal{B}^\bullet.$$

We define a homomorphism of resolution as a morphism of sheaves f together with a homomorphism of differential sheaves $f_\bullet$

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathcal{A} & \longrightarrow & \mathcal{A}^\bullet \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle f_\bullet}\downarrow \\
0 & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{B}^\bullet
\end{array}$$

Example 2.2.8. Let X be a smooth manifold, consider the resolutions

$$0 \to \mathbb{R} \to \mathcal{E}_X^\bullet,$$

and

$$0 \to \mathbb{R} \to S_\infty^\bullet(\mathbb{R}).$$

We have a natural morphism of resolutions

$$\begin{array}{ccccc}
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \mathcal{E}_X^\bullet \\
 & & \Big\| & & {\scriptstyle I}\downarrow \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & S_\infty^\bullet
\end{array}$$

Where I is given by integration of differential forms on the singular cochains, that is

$$I_U(\varphi)(c) = \int_c \varphi.$$

We shall verify that the following diagram is commutative

$$\begin{array}{ccc}
\mathcal{E}_X^p & \xrightarrow{I_p} & S_\infty^p(\mathbb{R}) \\
{\scriptstyle d}\downarrow & & {\scriptstyle \delta}\downarrow \\
\mathcal{E}_X^{p+1} & \xrightarrow{I_{p+1}} & S_\infty^{p+1}(\mathbb{R})
\end{array}$$

By using Stoke's theorem [5], [2]

we have

$$
\begin{aligned}
(I_{p+1} \circ d)_U(\varphi)(c) &= \int_c d\varphi \\
&= \int_{\partial c} \varphi = (\delta \circ I_p)(\varphi)(c).
\end{aligned}
$$

A very usefull result in complex geometry and cohomology is given by the following

Lemma 2.2.1. *($\partial\bar{\partial}-$Lemma). Let $\varphi \in \mathcal{E}^{p,q}(U)$, $U \subseteq \mathbb{C}^n$ open such that $d\varphi = 0$. Then, for all points $p \in U$ there exists a neighbourhood near p N, and $\psi \in \mathcal{E}^{p-1,q-1}(U)$ such that*

$$\partial\bar{\partial}\psi = \varphi, \text{ in } N.$$

Proof. By assumption $d\varphi = 0$, by Dolbeault Lemma there exist $u \in \mathcal{E}_x^{r-1}(U), r = p + q)$, such that $du = \varphi$. Notice that u can be written as

$$u = u^{r-1,0} + ... + u^{0,r-1}.$$

Then, $du = \bar{\partial}u^{p,q-1} + \partial u^{p-1,q}$. We have $\partial u^{p,q-1} = \bar{\partial}u^{p-1,q} = 0$. By applying again the Dolbeault Lemma, there exists $\eta_1 \in \mathcal{E}_x^{p-1,q-1}$ such that

$$u^{p,q-1} = \partial\eta_1$$

. For the same argument, there exists $\eta_2 \in \mathcal{E}_x^{p-1,q-1}$ such that

$$u^{p-1,q} = \bar{\partial}\eta_2$$

. Therefore

$$
\begin{aligned}
du &= \bar{\partial}u^{p,q-1} + \partial u^{p-1,q} \\
&= \bar{\partial}\partial\eta_1 + \partial\bar{\partial}\eta_2 = \partial\bar{\partial}(\eta_2 - \eta_1).
\end{aligned}
$$

□

♠

2.3 Sheaf Cohomology

In this last section we will study the sheaf cohomology, in order to do that we will have to introduce some kind of sheaves in the next subsection. In the last subsection of this section we will give an outline of Sheaf Cohomology in algebraic topology, namely we will talk about Čhec cohomology.

2.3.1 Soft, Fine and Flabby

In this subsection we will introduce the fundamental motivation behind sheaf cohomology by defining some kind of sheaves and the so called *canonical resolution*. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian sheaves over a topological space X. Suppose we have a short exact sequence of these sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0.$$

Consider the induced sequence

$$0 \to \mathcal{A}(X) \to \mathcal{B}(X) \to \mathcal{C}(X) \to 0, \tag{2.13}$$

We can see that this sequence is not in general short exact anymore. We see that, in general, we have exactness at $\mathcal{A}(X)$ and in $\mathcal{B}(X)$ but in general we lose exactness at $\mathcal{C}(X)$. Indeed, consider the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

on $X = \mathbb{C}_*$, then the induced sequence

$$0 \to \mathbb{Z} \to \mathcal{O}(X) \xrightarrow{\exp_X} \mathcal{O}^*(X) \to 0$$

fails to be exact because $\exp_X$ can't be inverted. Sheaf cohomology permits to study the obstruction of (2.13) to be exact. Henceforth we assume that X is a paracompact Hausdorff topological space. We introduce a class of sheaves for which there will be no obstruction to exactness.
Let $\mathcal{F}$ be a sheaf over a topological space X, and let $S \subset X$ be a closed subset. Consider the direct limit

$$\mathcal{F}(S) := \varinjlim_{S \subset U \subset X} \mathcal{F}(U).$$

Definition 2.9. *A sheaf $\mathcal{F}$ on X, is said to be* soft *if for all closed subset $S \subset X$ the map*

$$\mathcal{F}(X) \to \mathcal{F}(S)$$

is surjective. Namely, every section of $\mathcal{F}(S)$ can be extended to a section of $\mathcal{F}(X)$.

Definition 2.10. *A sheaf $\mathcal{F}$ on X, is said to be* flabby*, if for all open subset $U \subset X$ the map*

$$\mathcal{F}(X) \to \mathcal{F}(U)$$

is surjective.

Remark 2.3.1. *It follows from the definition that a flabby sheaf is also soft, in general the converse is not true.*

Definition 2.11. *A sheaf of abelian groups $\mathcal{F}$ on X, is said to be* fine *if for every open locally finite cover $\{U_i\}$ of X there exists a family of morphism of sheaves*

$$\{\eta_i : \mathcal{F} \to \mathcal{F}\}$$

called partition of unity *that satisfies:*

a. $\sum \eta_i = 1$.

b. $\eta_i(\mathcal{F}_x) = 0, \ \forall x \in N \subset X \setminus U_i$.

We will explore now some properties of soft sheaves

Theorem 2.3.1. *If $\mathcal{A}$ is a soft sheaf and*

$$0 \to \mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{h} \mathcal{C} \to 0 \tag{2.14}$$

is a short exact sequence of abelian sheaves. Then the induced sequence

$$0 \to \mathcal{A}(X) \xrightarrow{g_X} \mathcal{B}(X) \xrightarrow{h_X} \mathcal{C}(X) \to 0 \tag{2.15}$$

is also exact.

Proof. For the above discussion, it sufficies to prove that for every $c \in \mathcal{C}(X)$ there exists an element $b \in \mathcal{B}(X)$, such that $h_X(b) = c$. Namely, h_X is surjective. If (2.14) is exact, then by definition of exactness, se induced sequence on the stalks has to be exact $\forall x \in X$. Therefore, $\forall x \in X$ there exists a open neighbourhood U of x such that $h_U(b) = c_{|U}$. Choose some open cover $\{U_i\}$ of X such that $h_{U_i}(b) = c_{|U_i}$ holds. Being X paracompact by assumption, then there exist a locally finite refinement $\{S_i\}$ of closed subsets of X. Consider the set of tuples $\{(b, S)\}$ where S is the union of S_i and $h(b) = c_{|S}$. On that set we can put a partial order: we say that $(b, S) \preceq (b', S')$ if $S \subseteq S'$ and $b'_{|S} = b$. For the axiom S2 of the definition of sheaf, every chain of $\{(b, S)\}$ admits a maximal element. Therefore, by the Zorn's Lemma there exists a maximal element S, such that $h(b) = C_{|S}$. We shall now prove that $X = S$. Suppose, by contradiction, that $X \neq S$, then there exists $S_j \in \{S_i\}$ such that $S_j \not\subset S$ and in $S \cap S_j$ we have $h(b - b_j) = c - c = 0$, then $b - b_j \in \ker h_{S \cap S_j} = \mathrm{im} g_{S \cap S_j}$. Therefore there exists a element $a \in S \cap S_j$ such that $g(a) = b - b_j$. Since $\mathcal{A}$ is soft then we can extend a to all X. Thus, we can define $\tilde{b} \in \mathcal{B}(S \cup S_j)$ such that

$$\tilde{b} = \begin{cases} b, & \text{in } S \\ b_j + g(a), & \text{in } S \cap S_j \end{cases}$$

But then $h(\tilde{b}) = C_{|S \cup S_j}$, that means S is not maximal, so the desired contradiction. $\square$

Corollary 2.3.1. *Let* $0 \to \mathcal{A} \xrightarrow{g} \mathcal{B} \xrightarrow{h} \mathcal{C} \to 0$ *be a short exact sequence of abelian sheaves. If* $\mathcal{A}$ *and* $\mathcal{B}$ *are soft, then also* $\mathcal{C}$ *is soft.*

Proof. Being $\mathcal{A}$ soft by assumption, then for Theorem 2.3.1 the following induced sequences are short exact

$$0 \to \mathcal{A}(X) \xrightarrow{g_X} \mathcal{B}(X) \xrightarrow{h_X} \mathcal{C}(X) \to 0$$

$$0 \to \mathcal{A}(S) \xrightarrow{g_S} \mathcal{B}(S) \xrightarrow{h_S} \mathcal{C}(S) \to 0$$

Then, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}(X) \arrow[r, "g_X"] \arrow[d, "\alpha"'] & \mathcal{B}(X) \arrow[r, "h_X"] \arrow[d, "\beta"'] & \mathcal{C}(X) \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & \mathcal{A}(S) \arrow[r, "g_S"] & \mathcal{B}(S) \arrow[r, "h_S"] & \mathcal{C}(S) \arrow[r] & 0
\end{tikzcd}$$

From the above diagram, we shall prove that γ is surjective. Let $c \in \mathcal{C}(S)$, since h_S is surjective, then there exist a $b \in \mathcal{B}(S)$ such that $h_S(b) = c$. By assumption $\mathcal{B}$ is soft, therefore β is surjective, then there exists a $b' \in \mathbb{B}(X)$ such that $\beta(b') = b$. Hence,

$$c = h_S(b) = h_S(\beta(b')) = \gamma(h_X(b)).$$

Because of the arbitrariness of the choices we conclude that γ is surjective and hence $\mathbb{C}$ is a soft sheaf. $\square$

Corollary 2.3.2. *If* $0 \to S_0 \to S_1 \to S_2 \to ...$ *is a long exact sequence of soft sheaves, then the induced sequence* $0 \to S_0(X) \to S_1(X) \to S_2(X) \to ...$ *is also exact.*

Proof. We prove the claim by an inductive argument. We let

$$K_i = \ker(S_i \to S_{i+1}).$$

With this choice we have an induced short exact sequence

$$0 \to K_i \to S_i \to K_{i+1} \to 0.$$

We shall see that the induced sequence

$$0 \to K_i(X) \to S_i(X) \to K_{i+1}(X) \to 0 \tag{2.16}$$

is exact. In $i = 0$ we have $K_1 = S_0$, by assumption S_0 is soft then by Theorem 2.3.1

$$0 \to K_1(X) \to S_1(X) \to K_2(X) \to 0$$

is short exact. Since K_1 and S_1 are soft, then by Corollary 2.3.1 K_2 is soft. By inductive hypothesis suppose that

$$0 \to K_{i-1}(X) \to S_{i-1}(X) \to K_i(X) \to 0$$

is short exact. Then, immediately from Corollary 2.3.1 it follows that (2.16) is exact. By splicing together all of these sequences, we obtain the long induced exact sequence of the claim. $\square$

Proposition 2.3.1. *If $\mathcal{F}$ is a fine sheaf on X, then $\mathcal{F}$ is soft on X.*

Proof. Let $S \subset X$ be a closed subset and let $s \in \mathcal{F}(S)$. Thus there exists a open cover $\{U_i\}$ of S such that $s_i \in \mathcal{F}(U_i)$ has

$$s_{i_{|S \cap U_i}} = s_{j_{|S \cap U_j}}, \ \forall i, j.$$

We can exted such covering to all X by letting $U_0 = X \setminus S$. X is paracompact, then we can choose $\{U_i\}$ to be locally finite. Let $\{\eta_i\}$ be a partition of unity subordinated to $\{U_i\}$, define

$$\tilde{s} = \sum_i \eta_i(s) = \begin{cases} s, & \text{in } N \subset U_i \\ b_j + g(a), & \text{in } X \setminus U_i \end{cases}$$

Then $\tilde{s} \in \mathcal{F}(X)$ is the desired extension. $\square$

Example 2.3.1. The following sheaves are all fine, since it is always possible to define a partition of unity

1. C_X,
2. $\mathcal{E}_X$
3. $\mathcal{E}_X^{p,q}$ and X is an almost complex manifold.
4. Locally free sheaves of $\mathcal{E}_X-$ modules.

Example 2.3.2. Let $X = \mathbb{C}$, consider the sheaf of holomorphic function $\mathcal{O}$ on X. Then X is not soft. Indeed, consider the holomorphic function

$$f(z) = \sum_n z^{n!}.$$

f is holomorphic in the open unit disc, hence it is holomorphic in the closed disc with radius 1/2, but it can't be extended outside the open unit disk. Indeed, take some root of unity w, then for sufficiently large n and all $m > n$ we have $w^{m!} = 1$, so the series is not convergent in w. Roots of unity are dense in the boundary of unit disc, so f cannot be extended to the unit disc as well as to the complex plane.

Let $\mathcal{S}$ be a sheaf on X. Consider the associated etale space $\pi : \tilde{\mathcal{S}} \to X$. We define the *sheaf of discountinuous function* by the assignment

$$X \supset U \mapsto \mathcal{C}^0(\mathcal{S})(U) := \{f : U \to \tilde{\mathcal{S}} : \pi \circ f = 1_U\}$$

By construction $\mathcal{C}^0(\mathcal{S})$ is flabby, hence is soft. Moreover there is a natural inclusion

$$\mathcal{S} \hookrightarrow \mathcal{C}^0(\mathcal{S}).$$

Therefore, it makes sense to take the quotient:

$$\mathcal{F}^1 = \mathcal{C}^0(\mathcal{S})/\mathcal{S},$$

from which we obtain the short exact sequence

$$0 \to \mathcal{S} \to \mathcal{C}^0(\mathcal{S}) \to \mathcal{F}^1(\mathcal{S}) \to 0.$$

We iterate this reasoning by defining $\mathcal{C}^0(\mathcal{F}^1(\mathcal{S})) := \mathcal{C}^1(\mathcal{S})$, from which we obtain the short exact sequence

$$0 \to \mathcal{F}^1(\mathcal{S}) \to \mathcal{C}^1(\mathcal{S}) \to \mathcal{F}^2(\mathcal{S}) \to 0.$$

Therefore, by an inductive reasoning we have $\mathcal{F}^{i+1}(\mathcal{S}) = \mathcal{C}^i(\mathcal{S})/\mathcal{F}^i$ and

$$0 \to \mathcal{F}^i(\mathcal{S}) \to \mathcal{C}^i(\mathcal{S}) \to \mathcal{F}^{i+1}(S) \to 0.$$

Splicing together, we obtain the long exact sequence

$$0 \to \mathcal{S} \to \mathcal{C}^0(\mathcal{S}) \to \mathcal{C}^1(\mathcal{S}) \to ...$$

The above is called *canonical soft resolution* of $\mathcal{S}$ that, henceforth, will be denoted by

$$0 \to \mathcal{S} \to \mathcal{C}^\bullet(\mathcal{S}). \tag{2.17}$$

By taking global sections we obtain a cochain complex

$$0 \to \Gamma(X, \mathcal{S}) \to \Gamma(X, \mathcal{C}^0(\mathcal{S})) \to \Gamma(X, \mathcal{C}^1(\mathcal{S})) \to ...$$

If $\mathcal{S}$ is soft then by Corollary 2.3.2 is a everywhere exact complex. We let $C^\bullet(X, \mathcal{S}) := \Gamma(X, \mathcal{C}^\bullet(\mathcal{S}))$ and we rewrite (2.17) as

$$0 \to \Gamma(X, \mathcal{S}) \to C^\bullet(X, \mathcal{S}). \tag{2.18}$$

Now we are ready to introduce sheaf cohomology. ♠

2.3.2 The definition of Sheaf Cohomology and properties

For the rest of the discussion, assume that X is a paracompact Hausdorff topological space. Let $\mathcal{S}$ be a sheaf on X and consider the canonical soft resolution of that sheaf like in (2.18). We define the $q-th$ *derived cohomology group* with coefficient in $\mathcal{S}$ as

$$H^q(C^\bullet(X, \mathcal{S})) := H^q(X, \mathcal{S}) = \frac{\ker(C^q \to C^{q+1})}{\mathrm{im}(C^{q-1} \to C^q)}$$

and $C^{-1} = 0$.
We will now see the basic properties of sheaf cohomology

- $H^0(X, \mathcal{S}) = \Gamma(X, \mathcal{S})$. If $\mathcal{S}$ is soft, then $\forall q \geq 0, H^q(X, \mathcal{S}) = 0$. Indeed, being $C^{-1} = 0$ and (2.18) is exact in $\Gamma(X, \mathcal{S})$, therefore

$$\ker(C^0 \to C^1) = \Gamma(X, \mathcal{S}).$$

  If $\mathcal{S}$ is soft, then for Corollary 2.3.2 it followis that (2.18) is everywhere exact. Hence, $H^q(X, \mathcal{S}) = 0, \forall q \geq 0$.

- Sheaf Cohomology is functorial. Indeed, let $\mathcal{A}$ and $\mathcal{B}$ two abelian sheaves on X and $h : \mathcal{A} \to \mathcal{B}$ a morphism of sheaves. Then $\forall q \geq 0$ there is a induced morphism in cohomology

$$h_q : H^q(X, \mathcal{A}) \to H^q(X, \mathcal{B})$$

  such that

  1. $h_0 = h_X : \mathcal{A}(X) \to \mathcal{B}(X)$,
  2. h_q is the identity if h is the identity
  3. for every other morphism of sheaves $g : \mathcal{B} \to \mathcal{C}$ we have

$$(g \circ h)_q = g_q \circ h_q.$$

  To see the functoriality property, let $C^\bullet(X, \mathcal{A}) = C^\bullet(\mathcal{A})$. Define

$$\begin{aligned} h^0 : \ & C^0(\mathcal{A}) \to C^0(\mathcal{B}) \\ & s_x \mapsto (h \circ s)_x \end{aligned}$$

  Observe that, we have the following commutative diagram

$$\begin{array}{ccc} C^0(\mathcal{A}) & \xrightarrow{h^0} & C^0(\mathcal{B}) \\ \downarrow & & \downarrow \\ \mathcal{F}^1(\mathcal{A}) & \xrightarrow{\tilde{h}^0} & \mathcal{F}^1(\mathcal{B}) \end{array}$$

Then the map $\tilde{h}^0$ induces

$$h^1 : C^1(\mathcal{A}) \to C^1(\mathcal{B}).$$

By iterating the same reasoning we induce

$$h^\bullet : C^\bullet(\mathcal{A}) \to C^\bullet(\mathcal{B}).$$

With this in mind we immediately understand the functorial character of the construction.

- Given a short exact sequence of abelian sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0,$$

according to the previous point, we induce another short exact sequence,

$$0 \to C^\bullet(\mathcal{A}) \to C^\bullet(\mathcal{B}) \to C^\bullet(\mathcal{C}) \to 0.$$

Using the Snake Lemma we induce the so called *long exact sequence in cohomology*

$$0 \to \Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{B}) \to \Gamma(X, \mathcal{C}) \xrightarrow{\delta} H^1(X, \mathcal{A}) \to H^1(X, \mathcal{B}) \to H^1(X, \mathcal{C}) \to ...$$

Where δ is called *Bockstein operator*, that is the usual connecting homomorphism defined in any cohomology theory. With this in mind, given a commutative diagram of sheaves

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B} \arrow[r] \arrow[d] & \mathcal{C} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{A}' \arrow[r] & \mathcal{B}' \arrow[r] & \mathcal{C}' \arrow[r] & 0 \end{tikzcd}$$

we have an induced commutative diagram in cohomology

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X, \mathcal{A}) \arrow[r] \arrow[d] & \Gamma(X, \mathcal{B}) \arrow[r] \arrow[d] & \Gamma(X, \mathcal{C}) \arrow[r, "\delta"] \arrow[d] & H^1(X, \mathcal{A}) \arrow[r] \arrow[d] & ... \\ 0 \arrow[r] & \Gamma(X, \mathcal{A}') \arrow[r] & \Gamma(X, \mathcal{B}') \arrow[r] & \Gamma(X, \mathcal{C}') \arrow[r, "\delta"] & H^1(X, \mathcal{A}') \arrow[r] & ... \end{tikzcd}$$

Definition 2.12. *A resolution of a sheaf $\mathcal{S}$ on X*

$$0 \to \mathcal{S} \to \mathcal{A}^\bullet$$

is called acyclic *if* $H^q(X, \mathcal{A}^p) = 0$, $\forall q > 0$, *and* $\forall p \geq 0$.

Theorem 2.3.2. *Let $\mathcal{S}$ be a sheaf on x and let $0 \to \mathcal{S} \to \mathcal{A}^\bullet$ be a resolution. Then there is a natural isomorphism*

$$\gamma^p : H^p(\Gamma(X, \mathcal{A}^\bullet)) \to H^p(X, \mathcal{S}),$$

moreover if $0 \to \mathcal{S} \to \mathcal{A}^\bullet$ is acyclic, then γ^p is an isomorphism.

Proof. Let $K^p = \ker(\mathcal{A}^p \to \mathcal{A}^{p+1}) = \mathrm{im}(\mathcal{A}^{p-1} \to \mathcal{A}^p)$. Then $K^0 = \mathcal{S}$. With this definition we have the following short exact sequence of sheaves:

$$0 \to K^{p-1} \to \mathcal{A}^{p-1} \to K^p \to 0.$$

Using the long exact sequence in cohomology we see that

$$\ker(\Gamma(X, \mathcal{A}^p) \to \Gamma(X, \mathcal{A}^{p+1}) \simeq \Gamma(X, K^p)$$

.
Thus,

$$H^p(\Gamma(X, \mathcal{A}^\bullet) = \Gamma(X, K^p)/\mathrm{im}(\Gamma(X, \mathcal{A}^{p-1} \to \Gamma(X, \mathcal{A}^p)).$$

The long exact sequence in cohomology induces the following map

$$\gamma_1^p : H^p(\Gamma(X, \mathcal{A}^\bullet) \to H^1(X, K^{p-1})$$

Therefore, if the resolution is acyclic γ_1^p becomes an isomorphism. For $2 \leq r \leq p$ consider the short exact sequence

$$0 \to K^{p-r} \to \mathcal{A}^{p-r} \to K^{p-r+1} \to 0.$$

From the long exact sequence in cohomology there is an induced morphism

$$\gamma_r^p : H^{r-1}(X, K^{p-r+1}) \to H^r(X, K^{p-r})$$

such that, when the resolution is acyclic, then γ_r^p is an isomorphism.
Define,

$$\gamma^p = \gamma_p^p \circ \gamma_{p-1}^p \circ ... \circ \gamma_2^p \circ \gamma_1^p$$

and the claim follows. $\square$

Remark 2.3.2. *The word "natural" from Theorem 2.3.2 means that, if we have a morphism of resolutions*

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{A}^\bullet \\ & & {\scriptstyle f}\Big\downarrow & & {\scriptstyle g}\Big\downarrow \\ 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{B}^\bullet \end{array}$$

then, we have the following commutative diagram in cohomology

$$\begin{array}{ccc} H^p(\Gamma(X,\mathcal{A}^\bullet)) & \xrightarrow{\gamma^p} & H^p(X,\mathcal{S}) \\ {\scriptstyle g_p}\Big\downarrow & & {\scriptstyle f_p}\Big\downarrow \\ H^p(\Gamma(X,\mathcal{B}^\bullet)) & \xrightarrow{\gamma^p} & H^p(X,\mathcal{I}) \end{array} \tag{2.19}$$

Thus, if f is a isomorphism of sheaves and the resolutions are acyclic, then the map g^p in the above diagram is an isomorphism $\forall p$.

Theorem 2.3.3. *(*de Rham*) Let X be a smooth manifold. Then the natural map induced from the integration of singular cohains is an isomorphism, i.e.*

$$I : H^p(\mathcal{E}^p) \xrightarrow{\simeq} H^p(S^\bullet_\infty(X,\mathbb{R})).$$

In order to prove the above theorem we need the following

Lemma 2.3.1. *In $\mathcal{M}$ is a sheaf of modules over a soft sheaf of rings $\mathcal{R}$, then $\mathcal{M}$ is soft.*

Proof. Let $K \subset X$ be a closed subset. If $s \in \Gamma(K,\mathcal{M})$, we define the function $\rho \in \Gamma(K \cup (X \setminus U), \mathcal{R})$ as

$$\rho = \begin{cases} 1, & \text{in } K \\ 0, & \text{in } X \setminus U \end{cases}$$

Then, $\rho \cdot s$ is the desired extension of s. Because of the arbitrariness of the choices, it follows that $\mathcal{M}$ is soft. □

Proof. (of Theorem 2.3.3.) Consider the morphism of resolution of sheaves of example 2.2.8. By using naturality (Remark 2.3.2) we find the map

$$I_p = g_P : \; H^p(\mathcal{E}^\bullet(X)) \to H^p(S^\bullet_\infty(X,\mathbb{R})).$$

It remains to show that $\mathcal{E}^\bullet(X)$ and $S^\bullet_\infty(X,\mathbb{R})$ are soft. But, we know that $\mathcal{E}^\bullet(X)$ is fine, hence by Proposition 2.3.1, is soft. $S^p_\infty(X,\mathbb{R})$ is a $S^0_\infty-$module, where $S^0_\infty = C^0(X,\mathbb{R})$ is a flabby sheaf of rings (hence is soft), then by

Lemma 2.3.1 we deduce that $S^p_\infty(X,\mathbb{R})$ is soft. Hence, I_p is an isomorphism. $\square$

The *de Rham* Theorem generalizes in the following

Theorem 2.3.4. *(*Dolbeault*) Let X be a complex manifold. Then*

$$H^q(X,\Omega^p) \simeq \frac{\ker(\mathcal{E}^{p,q}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q+1}(X))}{\operatorname{im}(\mathcal{E}^{p,q-1}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q})}.$$

We want generalize the *Dolbeault* cohomology. To do that, we need the following

Definition 2.13. *Let $\mathcal{M}$ and $\mathcal{N}$ be sheaves of modules over a sheaf of commutative rings $\mathcal{R}$. Then $\mathcal{M}\otimes_{\mathcal{R}}\mathcal{N}$ would be the sheaf generated by*

$$U \mapsto \mathcal{M}(U)\otimes_{\mathcal{R}}\mathcal{N}(U).$$

Lemma 2.3.2. *If $\mathcal{I}$ is a locally free sheaf of $\mathcal{R}-$modules and*

$$0 \to \mathcal{A} \to \mathcal{A}' \to \mathcal{A}'' \to 0$$

is a short exact sequence of $\mathcal{R}-$modules, then the following sequence

$$0 \to \mathcal{A}\otimes_{\mathcal{R}}\mathcal{I} \to \mathcal{A}'\otimes_{\mathcal{R}}\mathcal{I} \to \mathcal{A}''\otimes_{\mathcal{R}}\mathcal{I} \to 0$$

is short exact.

Using this latter result, we want to generalize the *Dolbeault* cohomology. Consider a complex manifold X of complex dimension n, and the resolution

$$0 \to \Omega^p \to \mathcal{E}^{p,\bullet}.$$

Let $E \to X$ be a holomorphic vector bunlde, then the sheaf of holomorphic sections $\mathcal{O}(E)$ is a locally free sheaf of $\mathcal{O}-$modules. Therefore, using Lemma 2.3.2 we have the following resolution

$$0 \to \Omega^p\otimes_{\mathcal{O}}\mathcal{O}(E) \to \mathcal{E}^{p,0}\otimes_{\mathcal{O}}\mathcal{O}(E) \xrightarrow{\bar\partial\otimes 1} ... \xrightarrow{\bar\partial\otimes 1} \mathcal{E}^{p,n}\otimes_{\mathcal{O}}\mathcal{O}(E) \to 0.$$

Directly from the properties of tensor product, we deduce that

$$\Omega^p\otimes_{\mathcal{O}}\mathcal{O}(E) \simeq \mathcal{O}(\wedge^p T^*X\otimes_{\mathcal{C}} E)$$
$$\mathcal{E}^{p,q}\otimes_{\mathcal{O}}\mathcal{O}(E) \simeq \mathcal{O}(\wedge^{p,q} T^*X\otimes_{\mathcal{C}} E)$$

Moreover, $\Omega^p \otimes_{\mathcal{O}} \mathcal{O}(E) = \mathcal{E}(E)$ is a $\mathcal{E}-$module and is the sheaf of differentiable section on E. Sections of $\mathcal{O}(\wedge^p T^* X \otimes_{\mathcal{C}} E)$ are holomorphic forms with coefficients in E. We denote this latter by

$$\Omega^p(X, E) := \mathcal{O}(\wedge^p T^* X \otimes_{\mathcal{C}} E)$$

and by $\bar{\partial}_E = \bar{\partial} \otimes 1$, then we have the following resolution

$$0 \to \Omega^p(X, E) \to \mathcal{E}^{p,\bullet}(E).$$

The above resolution is fine, hence soft. Therefore we have the following generalization of Dolbeault cohomology

$$H^q(X, \Omega^p(E)) = \frac{\ker(\mathcal{E}^{p,q}(X, E) \xrightarrow{\bar{\partial}_E} \mathcal{E}^{p,q+1}(X, E))}{\operatorname{im}(\mathcal{E}^{p,q-1}(X, E) \xrightarrow{\bar{\partial}_E} \mathcal{E}^{p,q}(X, E))}.$$

♠

2.4 Čech Cohomology

In this section we will give an outline of Čheck cohomology, for a deeper treatment the reader may see [8]. Consider a topological space X and suppose that $\mathcal{S}$ is a sheaf on X. Let $\mathcal{U} = \{U_\alpha\}$ be a covering of X. A *q-simplex* σ, is a ordered collection of $q+1$ elements of the covering $\mathcal{U}$ with non empty intersection, i.e.

$$\sigma = (U_0, ..., U_q), \text{ and } \bigcap_{i=0}^{q} U_i \neq \emptyset.$$

The set $\bigcap_{i=0}^{q} U_i$ is called *support* of the simplex σ and it will be denoted by $|\sigma|$. A q-*cochain* of $\mathcal{U}$ with coefficients in $\mathcal{S}$ is a map f that assigned to every $q-$simplex σ,i.e.

$$f(\sigma) \in \mathcal{S}(|\sigma|).$$

The set of all $q-$cochains will be denoted by $C^q(\mathcal{U}, \mathcal{S})$ and is an abelian group. We define the coboundary operator by:

$$\delta : \ C^q(\mathcal{U}, \mathcal{S}) \to C^{q+1}(\mathcal{U}, \mathcal{S})$$

such that, if $f \in C^q(\mathcal{U}, \mathcal{S})$, $\sigma = (U_0, ..., U_{q+1})$ then

$$\delta f(\sigma) = \sum_{i=0}^{q+1} (-1)^i r_{|\sigma|}^{|\sigma_i|} f(\sigma_i).$$

Where $\sigma_i = (U_0, ..., U_{i-1}, U_{i+1}, ..., U_{q+1})$. Moreover, δ is a group homomorphism and $\delta^2 = 0$. Therefore, we have a cochain complex

$$C^\bullet(\mathcal{U},\mathcal{S}) = C^0(\mathcal{U},\mathcal{S}) \xrightarrow{\delta} C^1(\mathcal{U}.\mathcal{S}) \xrightarrow{\delta} C^2(\mathcal{U},\mathcal{S}) \xrightarrow{\delta} ...$$

The cohomology of that cochain complex defines the *Čech cohomology* with coefficients in $\mathcal{S}$, i.e.

$$\check{H}^q(\mathcal{U},\mathcal{S}) := H^q(C^\bullet(\mathcal{U},\mathcal{S})) = \frac{\ker(\delta : \; C^q(\mathcal{U},\mathcal{S}) \to C^{q+1}(\mathcal{U},\mathcal{S}))}{\operatorname{im}(\delta : \; C^{q-1}(\mathcal{U},\mathcal{S}) \to C^q(\mathcal{U},\mathcal{S}))}$$

Here we present some general properties of Čech cohomology:

- if $\mathfrak{M}$ is a refinement of $\mathcal{U}$, then there is a natural homomorphism

  $$\mu^{\mathcal{U}}_{\mathfrak{M}} \check{H}^q(\mathcal{U},\mathcal{S}) \to \check{H}(\mathfrak{M},\mathcal{S}),$$

  and

  $$\varinjlim_{\mathcal{U}} \check{H}^q(\mathcal{U},\mathcal{S}) = \check{H}(X,\mathcal{S}).$$

- If $\mathcal{U}$ is a covering such that $H^q(|\sigma|,\mathcal{S}) = 0$ for $q \geq 1$, then

  $$\check{H}^q(X,\mathcal{S}) \simeq \check{H}^q(\mathcal{U},\mathcal{S}), \forall q \geq 0.$$

  In this case $\mathcal{U}$ is called *Leray covering.*

- If X is paracompact and $\mathcal{U}$ is locally finite cover for X, then

  $$\check{H}^q(\mathcal{U},\mathcal{S}) = 0$$

  for $q > 0$ and $\mathcal{S}$ is a fine sheaf on X.

Chapter 3

Hermitian Differential Geometry

In this Chapter, we will see the basic notions of hermitian differential geometry, such as connections, curvature and Chern classes. Henceforth, for the rest of this discussion, we consider vector bundles with complex fibers over a smooth manifold X.

3.1 Local representations, Hermitian metrics

Let $E \to$ X be a vector bundle of rank r and let $f = (e_1, ..., e_r)$ be a local frame over the open set $U \subset X$. Consider the smooth map

$$g : U \to GL(r, \mathbb{C}),$$

that acts on the set of local frames above U, by

$$f \longmapsto f \cdot g.$$

Where that action is defined $\forall x \in U$ by

$$(f \cdot g)(x) := f(x)g(x) = \left(\sum_{\rho=1}^{r} g_{\rho 1}(x)e_\rho(x), ..., \sum_{\rho=1}^{r} g_\rho(x)e_\rho(x) \right).$$

We can immediately see that, with the above, we obtain another frame above U. The map g will be called henceforth *change of frame*. Local frames are important because give the possibility to locally represent sections of vector bundles. Namely, let $E \to X$ a vector bundle of rank r, consider a section $\xi \in \mathcal{E}(U, E)$ and a local frame $f = (e_1, ..., e_r)$ above U. Then, the *local representation* of ξ in the frame f is given by

$$\xi(f) = \sum_{\rho=1}^{r} \xi^{\rho}(f) e_{\rho}, \tag{3.1}$$

where $\xi(f)$ denotes the section ξ with respect to the frame f, and $\xi^{\rho}(f)$ are smooth functions on the open set U. By fixing a frame f above U, we can think about the local representation of ξ as a map

$$\begin{aligned} l_f :\ \mathcal{E}(U,E) &\longrightarrow \mathcal{E}(U)^r = \mathcal{E}(U, U \times \mathbb{C}^r) \\ \xi &\longmapsto l_f(\xi) := \xi(f) = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_r \end{pmatrix}. \end{aligned}$$

Consider a change of frame g, then $f \cdot g$ is another frame for E_U. Therefore, the transformation law for (3.1) is given by

$$\xi^{\rho}(f \cdot g) = \sum_{\sigma=1}^{r} g^{-1}_{\rho\sigma} \xi^{\sigma}(f),$$

hence,

$$\xi(f \cdot g) = g^{-1}\xi(f) \quad \Leftrightarrow \quad \xi(f) = g\xi(f \cdot g).$$

Similar consideration can be made also for holomorphic vector bundle. Namely, given a local frame f over $U \subset X$, where X is a complex manifold, then the holomorphic representation of a section $\xi \in \mathcal{O}(U, E)$ with respect to the local frame f is like in (3.1) where $\xi^{\rho}(f)$ are holomorphic functions for all $\rho = 1, ..., r$.

Definition 3.1. *Let $E \to X$ be a vector bundle of rank r. A hermitian metric cosist of*

1. *an entailment of a hermitian inner product*

$$X \ni x \mapsto < \cdot, \cdot >_x : E_x \times E_x \to \mathbb{C}, \quad \forall x \in X.$$

2. *For every open neighbourhood $U \subset X$ and $\forall \xi, \eta \in \mathcal{E}(U, E)$ the function*

$$\begin{aligned} < \xi, \eta >:\ U &\longrightarrow \mathbb{C} \\ x &\longmapsto < \xi(x), \eta(x) >_x \end{aligned}$$

is smooth $\forall x \in U$.

Moreover, a vector bundle $E \to X$ endowed with a hermitian metric is called hermitian vector bundle *and it is denoted by* (E, h).

One can think about a hermitian metric as a family of hermitian inner products smoothly parametrized by points of X. To a given a local frame $f = (e_1, ..., e_r)$ above an open neighbourhood $U \subset X$ for a hermitian vector bundle (E, h), we can associate a $r \times r$ matrix with smooth components with respect the local frame f:

$$h(f) = h(f)_{\rho\sigma} =< e_\rho, e_\sigma > .$$

The above matrix is positive definite, hermitian and symmetric. In particular $\forall \xi, \eta \in \mathcal{E}(U, E)$ we can write

$$\begin{aligned}
< \xi, \eta > &=< \sum_\rho \xi^\rho(f) e_\rho, \sum_\sigma \xi^\rho(f) e_\sigma > \\
&= \sum_{\rho\sigma} \xi^\rho(f) \bar{\eta}^\sigma(f) < e_\rho, e_\sigma >= \sum_{\rho\sigma} \xi^\rho(f) \bar{\eta}^\sigma(f) h_{\rho\sigma}(f) \\
&= \sum_{\rho\sigma} \bar{\eta}^\sigma(f) h_{\rho\sigma}(f) \xi^\rho(f) = \bar{\eta}^\sigma(f) h_{\rho\sigma}(f) \xi^\rho(f) \\
&=^t \bar{\eta}(f) h(f) \xi(f).
\end{aligned}$$

Moreover, if g is a change of frame, then

$$h(f \cdot g) =^t \bar{g} h(f) g. \tag{3.2}$$

Theorem 3.1.1. *Every vector bundle admits a hermitian metric*

Proof. Let $E \to X$ be a vector bundle of rank r. Choose a open cover $\{U_\alpha\}$ for X. Being X paracompact, then we can assume that the chosen open cover is locally finite. Let $\{f_\alpha\}$ be a family of local frames defined above U_α,$\forall \alpha$, and define the hermitian inner product $h_{\alpha|U_\alpha}$ as follows: given $\xi, \eta \in E_x$, $x \in U_\alpha$

$$< \xi, \eta >_x^\alpha =^t \bar{\eta}(f_\alpha)(x) \xi(f_\alpha)(x).$$

Choose some partition of unity $\{\rho\}$ subordinated to the open cover $\{U_\alpha\}$, define

$$< \xi, \eta >_x = \sum_\alpha \rho_\alpha(x) < \xi, \eta >_x^\alpha = \sum_\alpha \rho_\alpha(x) \bar{\eta}(f_\alpha)(x) \xi(f_\alpha)(x).$$

It is clear that the entailment $\forall \xi, \eta \in \mathcal{E}(U, E)$

$$x \mapsto < \xi(x), \eta(x) >_x = \sum_\alpha \rho_\alpha(x) < \xi(x), \eta(x) >_x^\alpha$$

is smooth, since it depents only on smooth quantities. Therefore, h is a hermitian metric on $E \to X$ $\square$

Consider a p-differential form with coefficients in a vector bundle. Consider a verctor bundle $E \to X$ and let

$$\mathcal{E}^p(X, E) = \mathcal{E}(X, \wedge^p T^* X \otimes_{\mathbb{C}} E).$$

We want to find a local representation for a p-form on E.

Lemma 3.1.1. *Let $E, E^{'}$ be vector bundles above X. Then, the following isomorphism holds*

$$\tau : \mathcal{E}(E) \otimes_{\mathcal{E}} \mathcal{E}(E^{'}) \xrightarrow{\sim} \mathcal{E}(E \otimes E^{'})$$

Proof. We define τ_U on presheaves:

$$\tau_U : \mathcal{E}(U, E) \otimes_{\mathcal{E}(U)} \mathcal{E}(U, E^{'}) \to \mathcal{E}(U, E \otimes E^{'})$$

by letting

$$\tau_U(\varphi \otimes \xi)(x) := \varphi(x) \otimes \xi(x) \in E_x \otimes E_x.$$

Notice that if $f = (e_1, ..., e_r)$ and $f^{'} = (e^{'}_1, ..., e^{'}_r)$ are two local frames of E and $E^{'}$ over U respectively, then $\forall \gamma \in \mathcal{E}(U, E \otimes E^{'})$ we can write

$$\gamma(x) = \sum \alpha\beta\gamma_{\alpha\beta}(x) e_\alpha(x) \otimes e^{'}_\beta(x),$$

with $\gamma_{\alpha\beta} \in \mathcal{E}(U)$. Therefore, $\gamma \in \mathcal{E}(U, E) \otimes_{\mathcal{E}(U)} \mathcal{E}(U, E^{'})$. Hence, using the S2 axiom of the definition of sheaf we see that τ is an isomorphism. $\square$

A immediate consequence of the above Lemma is

Corollary 3.1.1. $\mathcal{E}^p \otimes_{\mathcal{E}} \mathcal{E}(E) \simeq \mathcal{E}^p(E)$.

By this corollary we can understand how locally p-forms on E look like. Henceforth we adopt the following notation $\tau(\varphi \otimes \xi) = \varphi cdot \xi$, where $\varphi \in \mathcal{E}^p(X)$ and ξ is a smooth section of E. Consider an element $\gamma \in \mathcal{E}^p(U, E)$, that means by definition an element of $\mathcal{E}(U, \wedge^p T^* X \otimes_{\mathbb{C}} E)$. Then, for representing γ locally, we need a coframe of $T^* X \otimes_{\mathbb{C}} E$, $(\omega_1, ..., \omega_s)$. Therefore $\forall x \in U$ we have

$$\gamma(x) = \sum \alpha\beta\gamma_{\alpha\beta}(x) \omega_\alpha(x) \otimes e_\beta(x),$$

where $\gamma_{\alpha\beta} \in \mathcal{E}(U)$. Clearly we have that $\gamma \in \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathcal{E}(U, E)$. Therefore, using the notation we have fixed before we have

$$\gamma^{\beta} = \sum_{\alpha} \gamma_{\alpha\beta}\omega_{\alpha}$$

then,

$$\gamma = \sum_{\beta} \gamma^{\beta}(f) \cdot e_{\beta},$$

so for a fixed frame f we can define a map like before

$$\begin{aligned} l_f :\ \mathcal{E}^p(U,E) &\longrightarrow \mathcal{E}^p(U)^r \\ \gamma &\longmapsto l_f(\gamma) := \gamma(f) = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_r \end{pmatrix}. \end{aligned}$$

It can be easily observed that the coefficients γ^{β} does not depend on the choice of the coframe.

3.2 Connections and Curvature

We begin with the following

Definition 3.2. *Let $E \to X$ be a vector bundle, then a* connection *D on E is a $\mathbb{C}-$linear map*

$$D : \mathcal{E}(X, E) \longrightarrow \mathcal{E}^1(X, E),$$

that satisfies the Leibniz rule: $\forall \varphi \in \mathcal{E}(X), \forall \xi \in \mathcal{E}(X, E)$

$$D(\varphi\xi) = d\varphi \cdot \xi + \varphi D\xi.$$

Remark 3.2.1. *In the above definition, if E is the trivial bundle, e.g. $E = X \times \mathbb{C}$ then a connection D becomes the exterior derivative. Indeed*

$$\begin{array}{ccc} \mathcal{E}(X, X \times \mathbb{C}) & \longrightarrow & \mathcal{E}(X, T^*X \otimes_{\mathbb{C}} (X \times \mathbb{C}) \\ \simeq\big\uparrow & & \simeq\big\uparrow \\ \mathcal{E}(X) & \xrightarrow{d} & \mathcal{E}(X, T^*X \otimes \mathbb{C}) \simeq \mathcal{E}^1(X) \end{array}$$

Hence, we can think about connections as the generalization of the exterior derivative.

We give now a local representation of a connection D. Let $E \to X$ a vector bundle and D a connection on E. Consider a local frame $f = (e_1, ..., e_r)$ above an open neighbourhood $U \subset X$. We define the *connection matrix* with respect to the chosen frame

$$\theta(D, f) = (\theta_{\rho\sigma}(D, f)), \quad \theta_{\rho\sigma} \in \mathcal{E}^1(U).$$

This matrix can be defined by the action of the connection D on an element of the frame. Namely,

$$De_\sigma = \sum_{\rho=1}^{r} \theta_{\rho\sigma}(D, f) e_\rho.$$

For notational brevity, in order to indicate the connection matrix with respect to the frame f we simply write

$$\theta(D, f) = \theta(f).$$

Let $\xi \in \mathcal{E}(U, E)$, then with respect to a local frame f above U we get

$$\xi(f) = \sum_{\rho=1}^{r} \xi^\rho(f) e_\rho,$$

we can now apply the connection D and have

$$\begin{aligned}
D\xi &= D(\sum_\rho \xi^\rho(f) e_\rho) = \sum_\rho D\xi^\rho(f) e_\rho \\
&= \sum_\rho (d\xi^\rho(f) e_\rho + \xi^\rho(f) De_\rho) \\
&= \sum_\rho \left(d\xi^\rho(f) e_\rho + \xi^\rho(f) \sum_\sigma \theta_{\rho\sigma}(f) e_\sigma \right) \\
&= \sum_\sigma \left(d\xi^\sigma(f) + \sum_\rho \xi^\rho(f) \theta_{\rho\sigma}(f) e_\sigma \right) \cdot e_\sigma \\
&= \sum_\sigma \left(d\xi(f) + \xi(f)\theta(f) \right) \cdot e_\sigma.
\end{aligned}$$

We see that if we perform the sum in σ we have

$$D\xi = d\xi(f) + \theta(f)\xi(f) = (d + \theta(f))\xi(f).$$

Therefore, we can think about $(d + \theta(f))$ as an operator that acts on the smooth vector valued functions.

3.2.1 Intermezzo: the local representation of a p-form with values in $\mathrm{End}(E)$.

Let $E \to X$ be a vector bundle of rank r, consider the End-bundle and the sheaf of differential p-forms on that bundle $\mathcal{E}^p(X, \mathrm{End}(E))$. Let $\chi \in \mathcal{E}(X, \mathrm{End}(E))$. We want to find a local representation of χ. To do so, let $f = (e_1, ..., e_r)$ be a local frame above a open neighbourhood $U \subset X$, then we have a basis for the free $\mathcal{E}(U)$-module:

$$\mathcal{E}^p(U, \mathrm{End}(E)) \simeq \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathcal{E}(U, \mathrm{End}(E).$$

We know that given a local trivialization, is equivalent to give a local frame. Then, we have the following isomorphism

$$\mathcal{E}(U, \mathrm{E}) \simeq \mathcal{M}_r(U) = \mathcal{M}_r \otimes_{\mathbb{C}} \mathcal{E}(U),$$

where $\mathcal{M}_r(U)$ are $r \times r$ matrices with coefficients in $\mathcal{E}(U)$. Therefore,

$$\mathcal{E}^p(U, \mathrm{E}(E)) \simeq \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathcal{M}_r(U).$$

That means, given a trivialization, to the chosen element χ we have a matrix with respect to the frame f, i.e.

$$\chi(f) := (\chi(f)_{\rho\sigma}, \quad \text{where } \chi(f)_{\rho\sigma} \in \mathcal{E}^p(U).$$

Hence, we have a globally defined homomorphism of vector bundles:

$$\chi : \mathcal{E}(X, E) \to \mathcal{E}^p(X, E),$$

and locally we have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{E}(U, E) & \xrightarrow{\chi} & \mathcal{E}^p(U, E) \\
\simeq\big\downarrow & & \simeq\big\downarrow \\
\mathcal{E}(U, E)^r & \xrightarrow{\chi(f)} & \mathcal{E}^p(U, E)^r
\end{array}$$

Where, the map $\chi(f)$ is defined as

$$\mathcal{E}(U, E)^r \ni \xi(f) \mapsto \chi(f)\xi(f) = \eta(f)$$

with coefficients

$$\eta^\rho(f) = \sum_\sigma \chi(f)_{\rho\sigma} \xi^\sigma(f).$$

If we perform a change of frame g, then $\chi(f)$ follows the following transformation rule

$$
\begin{aligned}
\eta(f \cdot g) &= g^{-1}\eta(f) = g^{-1}\chi(f)\xi(f) \\
&= \chi(gf)\xi(gf) = \chi(gf)g^{-1}\xi(f) \\
&\Rightarrow g^{-1}\chi(f)\xi(f))\chi(gf)g^{-1}\xi(f) \\
&\Rightarrow \chi(gf) = g^{-1}\chi(f)g.
\end{aligned}
$$

Conversely, at every matrix of a p-form $\chi(f)$ is defined for all frame f, this determines completely the element $\chi \in \mathcal{E}^p(X, \mathrm{End}(E))$. ♠

The aim of the previous subsection is to consistently introduce the *curvature* operator of a connection D defined on a vector bundle. Consider a vector bundle $E \to X$ of rank r and let D be a connection on E. We want to show that D induces naturally an element

$$\Theta_E(D) \in \mathcal{E}^2(X, \mathrm{End}(E))$$

to be called *curvature tensor*. Let f be a local frame above U, like before. Define

$$\Theta(f) = \Theta(D, f) := d\theta(f) + \theta(f) \wedge \theta(f)$$

This is the *curvature matrix* associated to the connection D with respect to the local frame f. This is a 2-form matrix $r \times r$:

$$\Theta_{\rho\sigma} = d\theta_{\rho\sigma} + \sum_k \theta_{\rho k} \wedge \theta_{k\sigma}$$

Lemma 3.2.1. *Let g be a change of frame and $\Theta(f)$ like above. Then the following facts hold*

(a.) $dg + \theta(f)g = g\theta(fg)$

(b.) $\Theta(fg) = g^{-1}\Theta(f)g$

Proof. (a.) By a change of frame we get another frame, i.e.

$$fg = \left(\sum g_{\rho 1} e_\rho, ..., \sum g_{\rho r} e_r\right) = (e_1^{'}, ..., e_r^{r}).$$

Then, we get

$$De_r^{'} = \sum_\nu \theta_{\nu\sigma}(fg)e_\nu^{'} = \sum_{\nu\rho} \theta_{\nu\sigma}(fg)g_{\rho\sigma}e_\rho = g\theta(fg).$$

On the other hand we have that

$$
\begin{aligned}
De_r' &= D(\sum_\rho g_{\rho\sigma} e_\rho) = \sum_\rho (dg_{\rho\sigma} e_\rho + g_{\rho\sigma} De_\rho) \\
&= \sum_\rho \left(dg_{\rho\sigma} e_\rho + \sum_\tau g_{\rho\sigma} \theta_{\rho\tau} e_\tau \right) = \sum_\tau \left(dg_{\rho\tau} + \sum_\rho g_{\rho\sigma} \theta_{\rho\tau} \right) e_\tau \\
&= dg + \theta(f) g.
\end{aligned}
$$

Therefore, putting all together we get

$$dg + \theta(f)g = g\theta(fg) = g\theta(fg),$$

so (a.) holds. Now we prove (b.) by applying the exterior differential to (a.)

$$dg \wedge \theta(fg) + gd\theta(fg) = d\theta(f)g - \theta(f) \wedge dg.$$

Introduce in the above

$$\theta(fg) = g^{-1}dg + g^{-1}\theta(f)g.$$

We then get an algebraic expression for $gd\theta(fg)$, in terms of the quantities $d\theta(f)$, $\theta(f)$, dg, g and g^{-1}. Then we write

$$g(d\theta(fg) + \theta(fg) \wedge \theta(fg).$$

With the same previous quantities and simplifying we have

$$(d\theta(fg) + \theta(fg) \wedge \theta(fg))g,$$

so by combining these two results we obtain (b.). □

Lemma 3.2.2. $(d + \theta(f))(d + \theta(f))\xi(f) = \Theta(f)\xi(f)$.

Proof.

$$
\begin{aligned}
(d + \theta) \cdot (d + \theta)\xi &= (d^2 + \theta d + d \cdot \theta + \theta \wedge \theta)\xi \\
&= d^2\xi + \theta \wedge d\xi + d(\theta \cdot \xi) + \theta \wedge \theta \cdot \xi \\
&= \theta \wedge d\xi + d\theta \cdot \xi - \theta \wedge d\xi + \theta \wedge \theta \cdot \xi \\
&= (d\theta + \theta \wedge \theta)\xi \\
&= \Theta \cdot \xi.
\end{aligned}
$$

□

Definition 3.3. *Let D be a connection on a vector bundle $E \to X$. Then the* curvature *$\Theta_E(D)$ is defined as the element $\Theta \in \mathcal{E}^2(X, \mathrm{End}(E))$ such that the $\mathbb{C}-$linear map*

$$\Theta : \mathcal{E}(X, E) \longrightarrow \mathcal{E}(X, E)$$

has the following local representation with respect to a frame f of E

$$\Theta(f) = \Theta(D, f) = d\theta(f) + \theta(f) \wedge \theta(f).$$

Remark 3.2.2. *Because of assertion (b.) of Lemma 3.2.1 we see that $\Theta_E(D)$ is well defined, and because of the previous intermezzo we immediately see that it is an element of $\mathcal{E}^2(X, \mathrm{End}(E))$*

We can extend the action of a connection D, defined on a vector bundle $E \to X$, to differential form of arbitrary degree by letting

$$D\xi(f) = d\xi(f) + \theta(f) \wedge \xi(f), \text{ where } \xi \in \mathcal{E}^p(X, E).$$

Then, the map

$$D : \mathcal{E}^p(X, E) \longrightarrow \mathcal{E}^{p+1}(X, E)$$

is well defined. We should check that by a change of frame we still get a $(p+1)-$form E-valued.

$$\begin{aligned} g(d\xi(fg) + \theta(fg)\xi(fg)) &= d(g\xi(fg)) - dg \cdot \xi(fg) + g^{-1}(dg + \theta(f)g) \wedge \xi(f) \\ &= d(gg^{-1}\xi(f)) - dg \cdot \xi(fg) + g^{-1}dg\xi(f) + \theta(f) \wedge \xi(f) \\ &= d\xi(f) - g^{-1}dg\xi(f) + g^{-1}dg\xi(f) + \theta(f) \wedge \xi(f) \\ &= d\xi(f) + \theta(f) \wedge \xi(f). \end{aligned}$$

The operator $D : \mathcal{E}^p(X, E) \longrightarrow \mathcal{E}^{p+1}(X, E)$ such that $D\xi(f) = d\xi(f) + \theta(f) \wedge \xi(f)$, is called *covariant derivative.*

Proposition 3.2.1. *In the above situation $D^2 = \Theta$ as an operator that maps $\mathcal{E}^p(X, E)$ to $\mathcal{E}^{p+2}(X, E)$.*

Proof. This is a direct computation, indeed

$$\begin{aligned} D^2\xi(f) &= D(D\xi(f)) = D(d\xi(f) + \theta(f) \wedge \xi(f)) \\ &= d(d\xi(f) + \theta(f) \wedge \xi(f)) + \theta(f) \wedge (d\xi(f) + \theta(f) \wedge \xi(f)) \\ &= d^2\xi(f) + d\theta(f) \wedge \xi(f) - \theta(f) \wedge d\xi(f) + \theta(f) \wedge d\xi(f) + \theta(f) \wedge \theta(f) \wedge \xi(f) \\ &= (d\theta(f) + \theta(f) \wedge \theta(f)) \wedge \theta(f) = \Theta(f)\xi(f). \\ &\Rightarrow D^2 = \Theta. \end{aligned}$$

□

By previous proposition we see that the curvature represent the obstruction to the sequence

$$\mathcal{E}(X,E) \xrightarrow{D} \mathcal{E}^1(X,E) \xrightarrow{D} \mathcal{E}^2(X,E) \xrightarrow{D} ...$$

to be a chain complex.
The differential forms $\mathcal{E}^p(X, \mathrm{End}(E))$ are locally matrices of p-forms, we want to use this fact to define a Lie bracket. To do so, we proceed as follows: consider $\chi \in \mathcal{E}^p(X, \mathrm{End}(E))$, and a frame f above the open neighbourhood $U \subset X$. As we have already seen, $\chi(f)$ is an element of $\mathcal{M}_r \otimes_{\mathbb{C}} \mathcal{E}^p(U)$, and then if $\psi \in \mathcal{E}^q(X, \mathrm{End}(E))$ we define

$$[\chi(f), \psi(f)] = \chi(f) \wedge \psi(f) - (-1)^{p \cdot q} \psi(f) \wedge \chi(f).$$

If g is a change of frame then it is easy to verify that

$$[\chi(fg), \psi(fg)] = g^{-1}[\chi(f), \psi(f)]g$$

That brackets satisfy the Lie algebra axioms, see [insert biblio], and then $\mathcal{E}^\bullet(X, \mathrm{End}(E)$ becomes a differential graded Lie algebra.

Proposition 3.2.2. *(Bianchi's identity)* $d\theta(f) = [\Theta(f), \theta(f)]$

Proof. We know that $\Theta = d\theta + \theta \wedge \theta$, we can then compute

$$d\Theta = d^2\theta + d\theta \wedge \theta - \theta \wedge d\theta,$$

and

$$\begin{aligned}[\Theta, \theta] &= [d\theta + \theta \wedge \theta, \theta] = [d\theta, \theta] + [\theta \wedge \theta, \theta] \\ &= d\theta \wedge \theta - \theta \wedge d\theta + \theta \wedge \theta \wedge \theta - \theta \wedge \theta \wedge \theta. \\ &\Rightarrow d\Theta = [\Theta, \theta].\end{aligned}$$

□

Definition 3.4. *Let $E \to X$ be a hermitian vector bundle with hermitian metric h, endowed with a connection D. We say that the connection D is $h-$*compatible *if $\forall \xi, \eta \in \mathcal{E}(X,E)$ we have*

$$d < \xi, \eta > = < D\xi, \eta > + < \xi, D\eta > .$$

Consider a hermitian vector bundle $E \to X$ with hermitian metric h and connection D. Let $f = (e_1, ..., e_r)$ be a frame above U, denote by $h(f) = h$ and $\theta(f) = \theta$ the metric and the connection matrices respectively. Then we have the following

Proposition 3.2.3. *D is $h-$compatible $\Leftrightarrow$ $dh = h\theta +^t \bar{\theta}h$*

Proof. $(\Rightarrow)$

$$\begin{aligned} dh_{\rho\sigma} &= d < e_\sigma, e_\rho > = < De_\sigma, e_\rho > + < e_\sigma, De_\rho > \\ &= < e_\sigma, \sum_\mu \theta_{\mu\rho} e_\mu > = \sum_\tau \theta_{\tau\sigma} < e_\tau, e_\rho > + \sum_\mu \bar{\theta}_{\tau\sigma} < e_\tau, e_\rho > \\ &= \sum_\tau \theta_{\tau\sigma} h_{\rho\tau} + \sum_\mu \bar{\theta}_{\tau\sigma} h_{\tau\rho} = (h\theta)_{\rho\sigma} + ({}^t\bar{\theta}h)_{\rho\sigma} \\ &\Rightarrow dh = h\theta +^t \bar{\theta}h. \end{aligned}$$

$(\Leftarrow)$ Assume that $dh = h\theta +^t \bar{\theta}h$, then the claim holds by the following computation

$$\begin{aligned} d < \xi, \eta > = d({}^t\bar{\eta}h\xi) &=^t (d\bar{\eta})h\xi +^t \bar{\eta}dh\xi +^t \bar{\xi}hd\xi \\ &=^t (d\bar{\eta})h\xi +^t \bar{\eta}(h\theta +^t \bar{\theta}h)\xi +^t \bar{\eta}hd\xi \\ &=^t (d\bar{\eta})h\xi +^t \bar{\eta}h\theta\xi +^t \bar{\eta}^t\bar{\theta}h\xi +^t \bar{\eta}hd\xi \\ &=^t (d\bar{\eta})h\xi +^t \bar{\eta}h\theta\xi +^t (\bar{\theta\eta})h\xi +^t \bar{\eta}hd\xi \\ &=^t (d\bar{\eta} + \bar{\theta\eta})h\xi +^t \bar{\eta}h(\theta\xi + d\xi) \\ &= < \xi, D\eta > + < D\xi, \eta > \end{aligned}$$

That means, D is h-compatible. $\square$

Theorem 3.2.1. *Let $E \to X$ be a hermitian vector bundle. Then there exist a connection D which is h-compatible.*

Proof. Having a hermitian metric, is always possible to construct a unitary local frame f, by using the Gram-Schmidt process. Let f be a local unitary frame above U, let $\{U_\alpha\}$ be a locally finite open cover of X and $\{f_\alpha\}$ a family of unitary frames defined on every U_α. If we could locally define a h-compatible connection, then by previous proposition then we would have $h = I$, $dh = 0$ and $\theta = -{}^t\bar{\theta}$, i.e. θ is skew-hermitian. Then, above every U_α we can take $\theta_\alpha = 0$, i.e. $\theta(f_\alpha) = 0$. If g is a change of frame above f_α, then by Lemma 3.2.1 we have

$$g^{-1}dg = \theta(f_\alpha g).$$

And, notice that

$$h(f_\alpha g) =^t \bar{g} h(f_\alpha) g =^t \bar{g} g.$$

So we have

$$\begin{aligned} dh(f_\alpha g) &= d({}^t\bar{g}g) = d({}^t\bar{g})g +^t \bar{g}dg \\ &=^t \bar{\theta}(f_\alpha g)h(f_\alpha g) + h(f_\alpha g)\theta(f_\alpha g). \end{aligned}$$

Then, by previous proposition, we have a natural h-compatible connection on every U_α

$$(D_\alpha \xi)(f_\alpha) = d\xi(f_\alpha).$$

Let $\{\varphi\}$ be a partition of unity subordinated to $\{U_\alpha\}$. Define

$$D = \sum_\alpha \varphi_\alpha D_\alpha,$$

this is a connection on $E \to X$. Is left to show that D is h-compatible:

$$\begin{aligned} < D\xi, \eta > + < \xi, D\eta > &= \sum_\alpha \varphi_\alpha (< D_\alpha \xi, \eta > + < \xi, D_\alpha \eta >) \\ &= \sum_\alpha \varphi_\alpha d < \xi, \eta >= d < \xi, \eta > . \end{aligned}$$

□

3.3 Canonical Connection and Curvature for Holomorphic Verctor Bundles

Let $E \to X$ be a holomorphic vector bundle on the complex manifold X. If E as a smooth vector bundle has a hermitian metric, then this latter together with the hermitian metric is called hermitian holomorphic vector bundle. Recall that

$$\mathcal{E}^\bullet(E) = \bigoplus_r \mathcal{E}^r(E) = \bigoplus_{p,q} \mathcal{E}^{p,q}(E),$$

where $\mathcal{E}^{p,q}(E) = \mathcal{E}(X, \wedge^{p,q}T^*X \otimes_{\mathbb{C}} E)$. Because of Corollary 3.1.1 we have that $\mathcal{E}^{p,q}(E) \simeq \mathcal{E}_X^{p,q} \otimes_{\mathcal{E}_X} \mathcal{E}(E)$. Then, a connection on the given holomorphic hermitian vector bundle will be

$$D: \ \mathcal{E}(X,E) \to \mathcal{E}^1(X,E) = \mathcal{E}^{1,0}(X,E) \oplus \mathcal{E}^{0,1}(X,E).$$

That means $D = D^{'} + D^{''}$, where $D^{'} : \mathcal{E}(X,E) \to \mathcal{E}^{1,0}(X,E)$, and $D^{''} : \mathcal{E}(X,E) \to \mathcal{E}^{0,1}(X,E)$.

Theorem 3.3.1. *Let $E \to X$ be a holomorphic hermitian vector bundle, with hermitian metric h. Then h induces a connection $D(h)$ on E, that satisfies, in all open neighbourhood $W \subset X$, the following conditions*

(a) $\xi, \eta \in \mathcal{E}(W,E)$, $d < \xi, \eta > = < D\xi, \eta > + < \xi, D\eta >$.

(b) $\xi \in \mathcal{O}(W,E) \ \Rightarrow \ D^{''}\xi = 0.$

Proof. The condition (b) can be equivalently formulated by saying that the connection matrix, with respect to a holomorphic frame f, is of type $(1,0)$. Indeed, for a section $\xi \in \mathcal{O}(W,E)$ we have:

$$D\xi(f) = (d + \theta(f))\xi(f) = (\partial + \theta^{1,0}(f))\xi(f) + (\bar{\partial} + \theta^{0,1}(f))\xi(f).$$

By assumption, the section ξ is holomorphic. Therefore, $\bar{\partial}\xi(f) = 0$. Hence,

$$D^{''}\xi(f) = \theta^{0,1}(f)\xi(f).$$

Suppose to have a connection that satisfies (a) and (b). Let f be a holomorphic frame above the open neighbourhood $U \subset X$. Because of condition (a), D is $h-$compatible, therefore by Proposition 3.2.3 $dh = h\theta +^t \bar{\theta}h$. For condition (b), we have $\partial h = h\theta$, and $\bar{\partial}h =^t \bar{\theta}h$. Therefore,

$$\theta = h^{-1}\partial h. \tag{3.3}$$

We define θ like in (3.3) and we show that by a change of frame g the condition ($a.$) of Lemma 3.2.1 holds, it will follow that θ is the desired connection. Firstly, we observe that

$$h(fg) =^t \bar{g}h(f)g \ \Rightarrow \ h^{-1}(fg) = g^{-1}h(f)^{-1}({}^t\bar{g})^{-1}.$$

Then, by writing $g^{\dagger} =^t \bar{g}$

$$\begin{aligned} \theta(fg) &= h^{-1}(fg)\partial h(fg) = g^{-1}h(f)^{-1}(g^{\dagger})^{-1}\partial(g^{\dagger}h(f)g) \\ &= g^{-1}h(f)^{-1}(g^{\dagger})^{-1}(\partial g^{\dagger}h(f)g + g^{\dagger}\partial h(f)g + g^{\dagger}h(f)\partial g) \\ &= g^{-1}h(f)^{-1}\partial h(f)g + dg = \theta(f)g + dg. \end{aligned}$$

Hence, θ has the desired properties. □

In the proof of previous theorem, we see that the local expression for the canonical connection, in a holomorphic hermitian vector bundle, is the following

$$\theta(f) = h(f)^{-1}\partial h(f)$$

with respect to a holomorphic frame f. Moreover

$$D^{'} = \partial + \theta(f) \text{ and } D^{''} = \bar{\partial}.$$

Furthermore, it is clear from the construction that the canonical connection is uniquely determined.

Proposition 3.3.1. *Let D be a canonical connection in a holomorphic vector bundle $E \to X$. Let $\theta(f)$ and $\Theta(f)$ be the connection and curvature matrices respectively, with respect to a holomorphic frame f. Then, it follows that*

1. *$\theta(f)$ is of type (1,0) and $\partial\theta(f) = -\theta(f) \wedge \theta(f)$.*
2. *$\Theta(f) = \bar{\partial}\theta(f)$ and $\Theta(f)$ is of type (1,1).*
3. *$\bar{\partial}\Theta(f) = 0$ and $\partial\Theta(f) = [\Theta(f), \theta(f)]$.*

Proof. For notational brevity let $h(f) = f$ and $\theta(f) = \theta$. By previous discussions we see that $\theta = h^{-1}\partial h$ is of type (1,0). Then we have

$$\begin{aligned}\partial\theta &= \partial h^{-1}\partial h = \partial h^{-1} \wedge \partial h + h^{-1}\partial^2 h \\ &= (-h^{-1}\partial h h^{-1}) \wedge \partial h = -h^{-1}\partial h \wedge h^{-1}\partial h \\ &= -\theta \wedge \theta.\end{aligned}$$

That proves the first assertion. For the second assertion, we have directly from 1.

$$\begin{aligned}\Theta(f) = \Theta &= d\theta + \theta \wedge \theta \\ &= d\theta - \partial\theta = (\partial + \bar{\partial})\theta - \partial\theta = \bar{\partial}\theta.\end{aligned}$$

It is now clear that Θ is of type (1,1). The third assertion follows easily from 2. and from the Bianchi's identity. $\square$

Consider a holomorphic vector bundle $E \to X$ of rank r. Let $p \in X$ and $f = (e_1, ..., e_r)$ a holomorphic local frame near p. Choose some coordinates of point p such that $(z_1, ..., z_n) = z = 0$ at p. In order to indicate the dependence on z we write

$$f(z) = (e_1(z), ..., e_r(z)).$$

So, if h is a hermitian metric for the considered vector bundle we have, in the above notation, that

$$h(z) = h(f(z))$$

near $z = 0$.

Lemma 3.3.1. *In the above situation, there exists a holomorphic frame such that*

a. $h(z) = I + O(|z|^2)$,

b. $\Theta(0) = \bar{\partial}\partial h(0)$.

Proof. Notice that if a. holds then b. follows. Indeed, observe that $h^{-1}(z) = I + O(|z|^2)$. By Proposition 3.3.1 whe have

$$\begin{aligned}
\Theta(z) &= \bar{\partial}\theta(z) = \bar{\partial}h^{-1}(z)\partial h(z)\\
&= \bar{\partial}(I + O(|z|^2))\partial h(z) = \bar{\partial}\partial h(z)\\
&\Rightarrow \Theta(0) = \bar{\partial}\partial h(0).
\end{aligned}$$

Now, we prove the first assertion. Observe that $h(0)$ is a hermitian positive matrix, then there exists a $g \in GL_r(\mathbb{C})$ such that ${}^t\bar{g}h(o)g = I$. Furthermore, g induces a change of frame, i.e. $f \mapsto f \cdot g := \tilde{f}$. Therefore, $\hat{h}(z) = h(\tilde{f}(z)) = h(fg) =^t \bar{g}hg$. implies

$$\hat{h}(z) = I + O(|z|^2) \tag{3.4}$$

Suppose that $h(z)$ satisfies (3.4). Consider a change of frame of the following type

$$g = I + A(z),$$

where $A(z) = \left(\sum_j A^j_{\rho\sigma} z_j,\right)$ and $A(0) = 0$. Then, this change of frame preserves (3.4) if we choose $A(z)$ such that

$$\tilde{h}(z) =^t \bar{g}(z)h(z)g(z) = I + O(|z|^2). \tag{3.5}$$

Because of Taylor expansion, $d\tilde{h}(0) = 0$. Then we have

$$\begin{aligned}
d\tilde{h}(z) &= dh(z) + d^t\bar{A}(z) \cdot h(z) + h(z)dA(z) + O(|z|^2)\\
d\tilde{h}(0) &= \partial h(0) + d^t\bar{A}(0) + dA(0) + \bar{\partial}h(0).
\end{aligned}$$

Then, we choose $A(z)$ as follows

$$dA(0) = -\partial h(0), \ d^t\bar{A}(0) = -\bar{\partial} h(0).$$

Hence, (3.5) holds. □

The previous Lemma asserts that we can easily calculate the curvature Θ in a particular point by choosing the *right* frame. In the following example we show how we can calculate connection and curvature for a holomorphic vector bundle.

Example 3.3.1. Consider the Universal Bundle $U_{r,n} \to G_{r,n}$. A frame for this bundle consist of a open neighbourhood $U \subset G_{r,n}$ and smooth functions

$$e_j : U_j \to \mathbb{C}^n, \text{ such that } e_1 \wedge ... \wedge e_r \neq 0.$$

Then, a frame $f = (e_1, ..., e_r)$ could be interpreted as a $n \times r$ that has smooth functions as coefficients, namely $e_j, j = 1, ..., r$ are its column. f has maximal rank $\forall z \in U$. A holomorphic frame would simply have holomorphic function as coefficients. We defina a metric $h(f) =^t \bar{f} f$, for all frames f of $U_{r,n}$. That metric comes from the fact that $U_{r,n}$ sits in the trivial bundle $G_{r,n} \times \mathbb{C}^n$, therefore we obtain a hermitian metric on $U_{r,n}$ simply by restricting the canonical hermitian metric on $\mathbb{C}^n$ to the fibers of $U_{r,n} \to G_{r,n}$. Since f has maximal rank, then $h(f)$ has maximal rank. Indeed,

$${}^t\bar{z} h(f) z =^t \bar{z}^t \bar{f} f z =^t (\bar{f} z)(f z) = |fz|^2 > 0, \text{ for } z \neq 0.$$

Moreover, if g is a change of frame, we have

$$h(fg) =^t (\bar{f}\bar{g})(fg) =^t \bar{g}^t \bar{f} f g =^t \bar{g} h(f) g.$$

Hence, h is a well defined hermitian metric on the considered vector bundle. Now we can calculate the connection and the curvature with respect to the metric h. Computations shows that the canonical connection is given by

$$\theta(f) = \theta = h^{-1}(f) \partial h(f).$$

Therefore, by Proposition 3.3.1 we have

$$\Theta = \Theta(f) = \bar{\partial}\theta = h^{-1} \cdot^t d\bar{f} \wedge df - h^{-1} \cdot^t d\bar{f} \cdot f \cdot h^{-1} \wedge^t \bar{f} \cdot df.$$

For $r = 1$ we can obtain a more explicit formula for the curvature. Indeed, if $\varphi \in [\mathcal{E}^p(W)]^n$, $\psi \in [\mathcal{E}^q(W)]^n$, for $W \subset \mathbb{C}^n$, we let

$$< \varphi, \psi >= (-1)^{pq} \cdot^t \bar{\psi} \wedge \varphi.$$

Then, the curvature form becomes

$$\Theta(f) = -\frac{< f, f >< df, df > - < df, f > \wedge < f, df >}{< f, f >^2} \tag{3.6}$$

where f is a holomorphic frame for $U_{1,n}$. If $f = (\xi_1, ..., \xi_n)$, where $\xi_j \in \mathcal{O}(U)$, ${}^t\bar{df} = (d\bar{\xi_1}, ..., d\bar{\xi_n})$, we obtain

$$\Theta(f) = -\frac{|f|^2 \sum_{i=1}^n d\xi_i \wedge d\bar{\xi_i} - \sum_{ij=1}^n \bar{\xi_i}\xi_j d\xi_i \wedge d\bar{\xi_j}}{|f|^4} \tag{3.7}$$

$\xi_1, ..., \xi_n$ are the functions with respect the local coordinates of $G_{1,n} = \mathcal{P}_{n-1}$.

3.4 Chern Classes

The Chern Classes of a smooth vector bundle $E \to X$, over a complex manifold X are a very important invariant that in some peculiar cases permits to classify the class of vector bundles above X. In order to introduce sistematically this argument, we need a brief digression.

3.4.1 Linear Intermezzo

Consider the space of matrices $\mathcal{M}_r$. A k-linear form is a mapping

$$\tilde{\varphi} : \underbrace{\mathcal{M}_r \times ... \times \mathcal{M}_r}_{\text{k -times}} \longrightarrow \mathbb{C},$$

linear in each argument. We say that $\tilde{\varphi}$ is *invariant* if $\forall g \in GL_r(\mathbb{C})$ we have that

$$\tilde{\varphi}(gA_1g^{-1}, ..., gA_kg^{-1}) = \tilde{\varphi}(A_1, ..., A_k).$$

Denote by $\tilde{I}_k(\mathcal{M}_r)$ the space of k-invariant forms of $\mathcal{M}_r$.

Example 3.4.1. Let $k = 1$, consider the determinant function

$$\det : \ \mathcal{M}_r \to \mathbb{C}.$$

It is clear that $\det \in \tilde{I}_1(\mathcal{M}_r)$. Indeed, for the Binet's rule, for every $g \in GL_r(\mathbb{C})$, and forall $A \in \mathcal{M}_r$, we have

$$\det(g^{-1}Ag) = \det(gg^{-1})\det(A) = \det(A).$$

Given any $\tilde{\varphi} \in \tilde{I}_r(\mathcal{M}_r)$, we can induce a map $\varphi : \mathcal{M}_r \to \mathbb{C}$, by taking the diagonal, i.e.

$$\tilde{\varphi}(A, ..., A) := \varphi(A).$$

We see that $\varphi(A)$ is a complex number and φ is a homogeneous polynomial of degree k. In particular, φ is also invariant. Indeed, $\forall g \in GL_r(\mathbb{C})$

$$\varphi(gAg^{-1}) = \tilde{\varphi}(gAg^{-1}, ..., gAg^{-1}) = \tilde{\varphi}(A, ..., A) = \varphi(A).$$

Denote by $I_k(\mathcal{M}_r)$ the space of invariant homogeneous polynomial of degree k.

Example 3.4.2. The determinant function on previous example is clearly an element of $I_r(\mathcal{M}_r)$. Now, let $A \in \mathcal{M}_2$ and observe that

$$\det(I + A) = \det \begin{pmatrix} x+1 & y \\ z & t+1 \end{pmatrix} = 1 + t + x + tx - zy.$$

Denote by $\Phi_0(A) = 1$, $\Phi_1(A) = t + x$, $\Phi_2(A) = tx - zy$. Then

$$\det(I + A) = \sum_{k=0}^{2} \Phi_k(A),$$

where $\Phi_k(A)$ are homogeneous polynomial of degree k. Moreover, arguing by induction, it can be shown that for every $A \in \mathcal{M}_r$ we have the so called *determinant formula*:

$$\det(I + A) = \sum_{k=0}^{r} \Phi_k(A).$$

♠

3.4.2 Invariant Polynomial on $\mathcal{E}^\bullet(\mathrm{End}(E))$

Let $E \to X$ be a vector bundle of rank r. We now want to extend an element $\varphi \in I_k(\mathcal{M}_r)$ to an element of $\mathcal{E}^\bullet(\mathrm{End}(E))$. Recall that, we have the following isomorphism:

$$\mathcal{E}^p(\mathrm{End}(E)) \simeq \mathcal{M}_r \otimes_{\mathcal{E}} \mathcal{E}^p.$$

Let $U \subset X$ be an open set, we define the extension of φ on $\mathcal{M}_r(U) \otimes_{\mathcal{E}(U)} \mathcal{E}^p(U)$ by

$$\varphi_U(A_1 w_1, ..., A_k w_k) = w_1 \wedge ... \wedge w_k \varphi_U(A_1, ..., A_k),$$

where $A_i w_i \in \mathcal{M}_r(U) \otimes_{\mathcal{E}(U)} \mathcal{E}^p(U), i = 1, ..., k$. This is a k-form on $\mathcal{M}_r \otimes \mathcal{E}^p$. Let $\xi_j \in \mathcal{E}^p(\mathrm{End}(E)), j = 1, ..., k$ and write

$$\varphi_U(\xi_1, ..., \xi_k) := \varphi_U(\xi_1(f), ..., \xi_k(f)),$$

where f is a frame above U. We can easily see that the above does not depend on the choice of the frame f. Indeed, for any change of frame $g \in GL(r, \mathbb{C})$ we find

$$\varphi_U(\xi_1(fg), ..., \xi_k(fg)) = \varphi_U(g^{-1}\xi_1(f)g, ..., g^{-1}\xi_k(f)g)$$
$$= \varphi_U(\xi_1(f), ..., \xi_k(f)).$$

Therefore, because of this invariance, we obtain the invariance of φ on all X:

$$\varphi_x : \mathcal{E}^p(\mathrm{End}(E)) \times ... \times \mathcal{E}^p(\mathrm{End}(E)) \to \mathcal{E}^{pk}(X)$$

so, the invariant polynomial is defined: $\varphi_x(\xi, ..., \xi) = \varphi_X(\xi)$, as a map we have

$$\varphi_X : \; \mathcal{E}^p(X, \mathrm{End}(E)) \to \mathcal{E}^{pk}(X).$$

Consider a connection D on the given vector bundle. As we have previously seen, to a connection D we can associate a curvature, i.e. a element $\Theta_E(D) \in \mathcal{E}^2(\mathrm{End}(E))$. Therefore, because of previous discussion we see that $\varphi_X(\Theta_E(D))$ is a global $2k$-form on X. ♠

In order to introduce the Chern classes for a vector bundle $E \to X$ of rank r, we need the following result

Theorem 3.4.1. *(*Weil*) Let $E \to X$ be a vector bundle and D be a connection on E. Consider a invariant polynomial $\varphi \in I_k(\mathcal{M}_r)$ like before. Then*

(a) $\varphi_X(\Theta_E(D))$ is closed

(b) The image of $\varphi_X(\Theta_E(D))$ in $H^{2k}(X, \mathbb{C})$ does not depend on D.

In order to prove the above theorem we require two Lemmas

Lemma 3.4.1. *Every $\varphi \in I_k(\mathcal{M}_r)$ satisfies the following equality*

$$\sum_j \varphi(A_1, ..., [A_j, B], ..., A_k) = 0, \; \forall A_j, B \in \mathcal{M}_r.$$

Proof. We restrict the proof when $k = 2$, since the generalization will be rather immediate. Recall that from the Baker-Campbell-Hausdorff formula [isert biblio] we have that

$$e^{-tB} A e^{tB} - A = t[A, B] + O(|t|^2).$$

Here A, B are $r \times r$ matrices. Consider $\varphi \in I_2(\mathcal{M}_r)$. Directly from the invariance of φ we have that

$$\varphi(e^{-tB} A_1 e^{tB}, e^{-tB} A_2 e^{tB}) - \varphi(A_1, A_2) = 0.$$

We now add and subtract the term $\varphi(e^{-tB} A_1 e^{tB}, A_2)$ to the previous relation and we get

$$\varphi(e^{-tB}A_1e^{tB}, e^{-tB}A_2e^{tB} - A_2) + \varphi(e^{-tB}A_1e^{tB} - A_1, A_2) = 0.$$

That means,

$$\varphi(e^{-tB}A_1e^{tB}, t[A_2, B] + O(|t|^2)) + \varphi(t[A, B] + O(|t|^2), A_2) = 0.$$

Using the invariance of φ we have

$$\varphi(A_1, t[A_2, B]) + \varphi(t[A_1, B], A_2) + O(|t|^2) = 0,$$

$$t(\varphi(A_1, [A_2, B]) + \varphi([A_1, B], A_2)) + O(|t|^2) = 0, \quad t \neq 0$$

Hence,

$$\varphi(A_1, [A_2, B]) + \varphi([A_1, B], A_2) = \sum_{j=1}^{2} \varphi(A_1, [A_j, B]) = 0.$$

The general case can be obtained like the case when $k = 2$ adding and subtracting some $k - 1$ terms from the difference

$$\varphi(e^{-tB}A_1e^{tB}, ..., e^{-tB}A_ke^{tB}) - \varphi(A_1, ..., A_k).$$

□

Lemma 3.4.2. *Let D_t be a one parameter family of connections on E. $\forall t \in \mathbb{R}$ consider the one parameter family of induced curvatures Θ_t. Then $\forall \varphi \in I_k(\mathcal{M}_r)$ we have that*

$$\varphi_X(\Theta_b) - \varphi_X(\Theta_a) = d(\int_a^b \varphi'(\Theta_t, \dot{D}_t)dt),$$

where $\varphi'(\xi, \eta) = \sum_\alpha \varphi(\xi, \xi, ..., \xi, \eta, \xi, ..., \xi)$, for $\xi, \eta \in \mathcal{E}^\bullet(X, \mathrm{End}(E))$.

Proof. Sufficies to prove that for a frame f above an open set $U \subset X$ we have

$$\dot{\varphi}_U(\Theta) = d\varphi'(\Theta, \dot{\theta}),$$

where $\Theta = \Theta_t(D_t, f)$, $\theta = \theta(D_t, f)$.

$$\begin{aligned} d\varphi'_U(\Theta, \dot{\theta}) &= d\left(\sum_\alpha \varphi_U(\Theta, ..., \dot{\theta}, ..., \Theta)\right) \\ &= \sum_\alpha \left(\sum_{i<\alpha} \varphi_U(\Theta, ..., d\Theta, ..., \dot{\theta}, ..., \Theta) + \varphi_U(\Theta, ..., d\dot{\theta}, ..., \Theta) - \sum_{i>\alpha} \varphi_U(\Theta, ..., \dot{\theta}, ..., d\Theta, ..., \Theta)\right) \end{aligned}$$

By adding and subtracting $\sum_\alpha \varphi_U(\Theta, ..., [\dot{\theta}, \theta], ..., \theta)$, to the previous relation, and by observing that $\dot{\Theta} = d\dot{\theta} + [\dot{\theta}, \theta]$, and $d\Theta = [\Theta, \theta]$, we easily obtain the claim. □

Proof. (of Theorem 3.4.1.) Using Lemma 3.4.1 we can easily obtain (a). Indeed,

$$\begin{aligned} d\varphi_U(\Theta) = d\varphi_U(\Theta, ..., \Theta) &= \sum \varphi_U(\Theta, ..., d\Theta, ..., \Theta) \\ &= \sum \varphi_U(\Theta, ..., [\Theta, \theta], ..., \Theta) = 0. \end{aligned}$$

For part (b) we use Lemma 3.4.2. Indeed, let D_1 and D_2 be connections for $E \to X$. Then set

$$D_t = tD_1 + (1-t)D_2.$$

That is clearly a one parameter family of connections, then from Lemma 3.4.2 the claim holds immediately. □

3.4.3 The definition of Chern Classes and their properties

Consider invariant polynomials defined by the equation

$$\det(I + A) = \sum_{k=0}^{r} \Phi_k(A).$$

Definition 3.5. *Let $E \to X$ be a vector bundle of rank r endowed with a connection D. The k-th* Chern form *of E with respect to the connection D is defined as*

$$c_k(E, D) = (\Phi_k)_X \left(\frac{i}{2\pi} \Theta_E(D) \right) \in \mathcal{E}^{2k}(X). \tag{3.8}$$

The total Chern form *of E with respect to the connection D is defined as*

$$c(E, D) = \sum_{k=0}^{r} c_k(E, D). \tag{3.9}$$

The k-th Chern Class *of E is denoted by $c_k(E)$, is the cohomology class of (3.8) in $H^{2k}(X, \mathbb{C}$. The* total Chern Class *is denoted by $c(E)$ and is the cohomology class of (3.9) in $H^\bullet(X, \mathbb{C}$, i.e. $c(E) = \sum_{k=0}^{r} c_k(E)$.*

Remark 3.4.1. *For the Weil's theorem (Theorem 3.4.1) we see that Chern classes are well defined and are independent on the choice of the connection D. Therefore, the Chern classes are cohomology classes associated to the base manifold of a vector bundle. Roughly speaking, Chern classes tells how a vector bundle fails to be the trivial bundle.*

In the following we will explore the basic properties of Chern classes.

Proposition 3.4.1. *Let D be a $h-$compatible connection for the hermitian vector bundle $E \to X$, with hermitian metric h. Then the Chern form is a differential real form, and it follows that $c(E) \in H^\bullet(X, \mathbb{R})$ under the canonical inclusion of cohomology rings*

$$H^\bullet(X, \mathbb{C}) \hookrightarrow H^\bullet(X, \mathbb{C}).$$

Proof. It sufficies to prove the assertion on a local frame f using the matrix representations. Denote by $h(f) = h$, $\theta(f) = \theta$ and $\Theta(f) = \Theta$ the hermitian metric matrix, the connection matrix and the curvature matrix respectively. Being D $h-$compatible by assumprion, then $dh = h\theta +^t \bar{\theta}h$. Applying the exterior differential to this latther we find that

$$\begin{aligned}
0 &= dh \wedge \theta + hd\theta + d^t\bar{\theta}h -^t \bar{\theta} \wedge dh \\
&= (h\theta +^t \bar{\theta}h) \wedge \theta + hd\theta + d^t\bar{\theta}h -^t \bar{\theta} \wedge (h\theta +^t \bar{\theta}h) \\
&= h(d\theta + \theta \wedge \theta) + (d\bar{\theta} +^t \bar{\theta} \wedge^t \bar{\theta})h \\
&= h\Theta +^t \bar{\Theta}h.
\end{aligned}$$

Therefore, if f is a unitary frame then, by the previous computations, we see that Θ is skew-hermitian. Let $c = c(E, f, D) = \det\left(I + \frac{i}{2\pi}\Theta\right)$, we have that

$$\begin{aligned}
\det\left(h + \frac{i}{2\pi}\Theta h\right) &= \det\left(I + \frac{i}{2\pi}\Theta\right)\det h = c \cdot \det h \\
\det\left(h - \frac{i}{2\pi}\Theta h\right) &= \det\left(I - \frac{i}{2\pi}\Theta\right)\det h = \bar{c} \cdot \det h.
\end{aligned}$$

But, $\det\left(h + \frac{i}{2\pi}\Theta h\right) = \det\left(h - \frac{i}{2\pi}\Theta h\right)$, then $c = \bar{c}$, that is c is real. $\square$

In the following, we explore the functorial properties of Chern classes.

Theorem 3.4.2. *Let E, E' be vector bundles above a smooth manifold X.*

1. *if $\varphi : Y \to X$ is a smooth map, then*

$$c(\varphi^* E) = \varphi^* c(E).$$

2. $c(E \oplus E') = c(E) \cdot c(E')$.

3. *$c(E)$ depends only on the isomorphism classes of E.*

4. *If E^* is the dual bundle, then*

$$c_j(E^*) = (-1)^j c_j(E).$$

Proof. (1) Let D be any connection on $E \to X$. To prove the first assertion it will be sufficient to prove that $\varphi^*\Theta(D) = \Theta(D^*)$, where D^* denotes the connection on φ^*E. Firstly, we define the connection D^*. To do so, let $f = (e_1, ..., e_r)$ be a frame above U, then $f^* = (e_1^*, ..., e_r^*$, with $e_i^* = e_i \circ \varphi$ is a frame above $\varphi^{-1}(U)$. If g is a change of frame on U, then $g^* = g \circ \varphi$ is a change of frame on $\varphi^{-1}(U)$. Define

$$\theta^*(f^*) = \varphi^*\theta(f).$$

This is the desired connection since it verifies the condition (a) of Lemma 3.2.1. Therefore the curvature is given by

$$\begin{aligned}\Theta(D^*, f^*) &= d\theta^*(f^*) + \theta(f^*) \wedge \theta(f^*) = d\varphi^*\theta(f) + \varphi^*\theta(f) \wedge \varphi^*\theta(f) \\ &= \varphi^*(d\theta(f) + \theta(f) \wedge \theta(f)) = \varphi^*\Theta(D, f).\end{aligned}$$

(2) Let D and D' connections on E and E' respectively. Define on a frame f above U the matrix connection for $E \oplus E'$

$$\theta^{\oplus}(f) = \begin{pmatrix} \theta(f) & 0 \\ 0 & \theta'(f) \end{pmatrix}.$$

We now prove that the above is a connection. Let g be a change of frame, then we have

$$\begin{aligned}g\theta^{\oplus}(fg) &= g\begin{pmatrix} \theta(fg) & 0 \\ 0 & \theta'(fg) \end{pmatrix} = \begin{pmatrix} g\theta(f) & 0 \\ 0 & g\theta'(f) \end{pmatrix} = \begin{pmatrix} dg + \theta(f)g & 0 \\ 0 & dg + \theta'(f)g \end{pmatrix} \\ &= \begin{pmatrix} dg & 0 \\ 0 & dg \end{pmatrix} + \begin{pmatrix} \theta(f) & 0 \\ 0 & \theta'(f) \end{pmatrix} g = dgI_{2r} + \theta^{\oplus}(f)g.\end{aligned}$$

We see that condition (a) of Lemma 3.2.1 is satisfied so the above defines globally a connection. Therefore, the connection will be defined as

$$\Theta^{\oplus} = \begin{pmatrix} \Theta(f) & 0 \\ 0 & \Theta'(f) \end{pmatrix}.$$

$$
\begin{aligned}
C(E \oplus E', D^{\oplus})_U &= \det\left(I + \frac{i}{2\pi}\Theta^{\oplus}\right) \\
&= \det\begin{pmatrix} I + \frac{i}{2\pi}\Theta & 0 \\ 0 & I + \frac{i}{2\pi}\Theta'(f) \end{pmatrix} \\
&= \det\left(I + \frac{i}{2\pi}\Theta\right) \cdot \det\left(I + \frac{i}{2\pi}\Theta'\right) = c(E, D)_U \cdot c(E', D')_U.
\end{aligned}
$$

For Weil's theorem Chern classes do not depend on D, hence the second assertion follows.
The third assertion is rather obvious, so we prove directly the last assertion.
Let f be a frame and f^* be its dual frame. We define the dual connection as

$$\theta^* = \theta^*(D^*, f^*) = -{}^t\theta(D, f).$$

We shall verify that for $g^* ={}^t (g^{-1})$ change of frame, the condition (a) of Lemma 3.2.1 holds.

$$\theta^*(f^*g^*) = (g^*)^{-1}dg^* + (g^*)^{-1}\theta^*(f^*)g^*$$

is true if and only if

$$-{}^t\theta(fg) ={}^t g d{}^t g^{-1} -{}^t g \cdot{}^t \theta(f){}^t g^{-1}$$

By taking the transpose of the obove we have

$$-\theta(fg) = gdg^{-1} - g\theta(f)g^{-1}.$$

Observe that $dg^{-1} = -g^{-1}dgg^{-1}$. Therefore,

$$-\theta(fg) = -gg^{-1}dgg^{-1} - g\theta(f)g^{-1}$$

Hence,

$$\theta(fg) = g^{-1}dg + g\theta(f)g^{-1}$$

which is true. We calculate the curvature

$$
\begin{aligned}
\Theta^* &= d\theta^* + \theta^* \wedge \theta^* = -d{}^t\theta +{}^t \theta \wedge{}^t \theta \\
&= -d{}^t\theta -{}^t (\theta \wedge \theta) = -{}^t(d\theta + \theta \wedge \theta) ={}^t \Theta.
\end{aligned}
$$

Hence, the Chern class s given by

$$c_k(E^*, D^*)_{|U} = \Phi_k\left(-\frac{i}{2\pi}{}^t\Theta\right) = \Phi_k\left(-\frac{i}{2\pi}\Theta\right)$$
$$= (-1)^k\Phi_k\left(\frac{i}{2\pi}{}^t\Theta\right) = (-1)^k c_k(E, D)_{|U}.$$

□

The next result will tell about the geometrical intuition behind Chern classes.

Theorem 3.4.3. *Let $E \to X$ be a vector bundle of rank r. Then,*

a. $c_0(E) = 1$.

b. If $E \simeq X \times \mathbb{C}^r$, then $c_j(E) = 0$ for $j = 1, ..., r$.

c. If $E \simeq E' \oplus T_s$, where T_s is the trivial bundle of rank s, then $c_j(E) = 0$ for $j = r - s + 1, ..., r$.

Proof. The first assertion follows immediately from the definition of Chern class. For assertion b if E is the trivial bundle, then $\mathcal{E}(E) \simeq \mathcal{E}(X, X \times \mathbb{C}^r) \simeq \mathcal{E}(X)^r$. The connection is given by the exterior derivative, therefore $\theta = \Theta = 0$. Hence,
$c(E, D) = \det(I + 0) = 1 \quad \Rightarrow \quad c_j(E, D) = 0, \text{ for } j > 0.$
For assertion c. we use the functoriality of Chern classes explained in the previous theorem, and the just proven point b.

$$c(E' \oplus T_s) = c(E') \cdot c(T_s) = c(E') \cdot 1$$

E' ha rank $r - s$, hence $c_j(E) = 0$ for $j = r - s + 1, ..., r$. □

Example 3.4.3. Consider the tangent bundle to the complex projective line, i.e. $T(\mathcal{P}_1(\mathbb{C}))$. This is a holomorphic vector bundle of rank 1. We want to show that it is not isomorphic to the trivial bundle. A natural metric on $T(\mathcal{P}_1(\mathbb{C}))$ is the so called *chordal metric*, that is

$$h(z) = h\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = \frac{1}{(1 + |z|^2)^2}.$$

With respect to this metric the canonical conncection is given by

$$\theta(z) = h^{-1}(z)\partial h(z) = \frac{-2\bar{z}dz}{(1 + |z|^2)^2}.$$

Hence, the curvature is given by

$$\Theta(z) = \bar{\partial}\theta(z) = \frac{2dz \wedge d\bar{z}}{(1+|z|^2)^2}.$$

The first Chern form is therefore

$$c_1(E,h) = \frac{i}{2\pi}\Theta = \frac{dz \wedge d\bar{z}}{\pi(1+|z|^2)^2}.$$

In order to see that $T(\mathcal{P}_1(\mathbb{C}))$ is non trivial, sufficies to prove that $c_1(E,h)$ is not exact. This latter could be done by showing that the integral along any closed chain is non zero. Indeed, computations show that

$$\int_{\mathcal{P}_1} c_1(E,h) = 2$$

Hence $T(\mathcal{P}_1(\mathbb{C}))$ is not trivial.

3.5 Complex Line Bundles

In this section we will explain some basic fact of complex line bundles, that will be very important for the last Chapter of this work. We begin this section by giving two important results for the topology of vector bundles of finite rank.

Lemma 3.5.1. *Let $E \to X$ be a smooth vector bundle of rank r. Then, there exists a open finite covering $\{U_\alpha\}$ of open sets of X, such that E_{U_α} is trivial.*

Proof. If X is compact, then the claim follows immediately. Suppose X is not compact, let $\{V_\beta\}$ be a open covering that trivializes E, that is $\forall\beta, E_{V_\beta} \simeq V_\beta \times \mathbb{C}^r$. Since X is paracompact, then there exists a locally finite refinement $\{U_\alpha\}$ with the property that the intersection of $(n+2)$ elements of $\{U_\alpha\}$ is the empty set. Let $\{\varphi_\alpha\}$ be a partition of unity subordinated to $\{U_\alpha\}$. Let A_i be the non ordered set of indices of different elements of $\{\varphi_\alpha\}$. An element $a \in A_i$ means $a = \{\alpha_0, ..., \alpha_i\}$. Define

$$W_{ia} = \{x \in X : \ \varphi_\alpha(x) < \min(\varphi_{\alpha_0}(x), ..., \varphi_{\alpha_i}(x)), \alpha \neq \alpha_0, ..., \alpha_i\}.$$

W_{ia} is open. By construction for $a \neq b$, $W_{ia} \cap W_{ib} = \emptyset$, and

$$W_{ia} \subset \mathrm{supp}\varphi_\alpha \subset \mathrm{supp}\varphi_{\alpha 0} \cap ... \cap \mathrm{supp}\varphi_{\alpha i} \subset U_\alpha$$

We let

$$X_i = \bigcup_i W_{ia}, \quad i = 0, ..., n$$

We can immediately say that E_{X_i} is trivial by construction. We claim that $X = \bigcup_i X_i$. Let $x \in X$, then x is contained at most in $n+1$ subset of

$\{U_\alpha\}$, then $\{\varphi_\alpha\}$ has $n+1$ positive functions in x. Let $a = \{\alpha_0, ..., \alpha_i\}$ where $\varphi_{\alpha_0}, ..., \varphi_{\alpha_i}$ are the only positive function of $\{\varphi_\alpha\}$ in x, $0 \le i \le n$. Then, it follows that

$$0 = \varphi_\alpha(x) < \min(\varphi_{\alpha_0}(x), ..., \varphi_{\alpha_i}(x))$$

for every choice of $\alpha \neq \alpha_0, ..., \alpha_i$. So, $x \in W_{ia} \subset X_i$. Hence $\{X_i\}$ is a open finite cover of X. $\square$

Lemma 3.5.2. *Let $E \to X$ be a complex vector bundle of rank r. Then there exists an integer $N > 0$ and a smooth map $\Phi: X \to G_{r,N}$ such that $\Phi^*(U_{r,N}) \simeq E$.*

Proof. Consider the dual vector bundle $E^* \to X$ of $E \to X$. Because of Lemma 3.5.1 we can choose a finite open cover $\{U_\alpha\}$ of X and correspondingly a finite family of local frames $\{f_\alpha\}$ for E^*. Arguing by a partition of unity, we can find a finite number of global sections $\xi_1, ..., \xi_N \in \mathcal{E}(X, E^*)$, such that at every $x \in X$ there are r-sections $\{\xi_{\alpha_1}, ..., \xi_{\alpha_r}\}$ that are linearly independent in x and thus also in their neighbourhood. We use the global sections $\xi_1, ..., \xi_N$ to define the map

$$\Phi : X \longrightarrow G_{r,N}.$$

If f^* is a frame for E^* near x_0. Then

$$M(f^*) = (\xi_1(f^*)(x), ..., \xi_N(f^*)(x))$$

is a frame for $G_{r,N}$, that is, the rows generate a subspace of $\mathbb{C}^N$, that we denote by $\Phi(x)$. If g is a change of frame, then

$$M(f^*g) = (\xi_1(f^*g), .., \xi_N(f^*g)) = (g^{-1}\xi_1(f^*), ..., g^{-1}\xi_N(f^*)) = g^{-1}M(f^*).$$

Thus, $M(f^*g)$ and $M(f^*)$ generate the same $\Phi(x)$. Therefore, Φ is well defined and is smooth by construction. It is left to show that there exists a morphism of vector bundles

$$\tilde{\Phi}: \ E \longrightarrow U_{r,N}$$

that makes the following diagram commute

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\Phi}} & U_{r,N} \\ \big\downarrow & & \big\downarrow \pi \\ X & \xrightarrow{\Phi} & G_{r,N} \end{array}$$

It will follows, from the universality of the pullback bundle that $\Phi^*(U_{r,N}) \simeq E$. Firstly we see that $\tilde{\Phi}_{|E_x}$ is injective. Indeed, let $x \in X$ and f be a frame above x and f^* the dual frame. Thus, $\tilde{\Phi}$ can be represented by

$$\tilde{\Phi}(x, v) =^t v(f) \cdot M(f^*). \tag{3.10}$$

Since $M(f^*)$ has maximal rank then the injectivity follows. Moreover $\tilde{\Phi}(E_X) = \pi^{-1}(\Phi(x))$, indeed, because of (3.10) $\tilde{\Phi}(E_X)$ is generated by the rows of $M(f^*)$. □

Remark 3.5.1. *Directly from Lemma 3.5.2 and the functiorial properties of Chern classes, explained in Theorem 3.4.2, we see that the Chern class of a vector bundle $E \to X$ can be topologically defined as*

$$c(E) = \Phi^*(c(U_{r,N}))$$

Proposition 3.5.1. *Let $E \to$ be a complex line bundle. Then $c_1(E) \in \tilde{H}^2(X, \mathbb{Z})$, i.e. $c_1(E)$ is integral. (Here $\tilde{H}^2(X, \mathbb{Z})$ denotes the image of $H^2(X, \mathbb{Z})$ in $H^2(X, \mathbb{R})$, under the natural isomorphism induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$).*

Proof. For Lemma 3.5.2, we know that $c_1(E) = \Phi^*(c_1(U_{1,N}))$. Therefore, it sufficies to prove that $c_1(U_{1,N}) \in H^2(\mathcal{P}_{N-1}, \mathbb{Z})$. We know that

$$c_1(U_{1,N}, D(h)) = \frac{1}{2\pi i} \cdot \frac{|f|^2 \sum d\xi_j \wedge d\bar{\xi_j} - \sum \bar{\xi_j} \xi_k d\xi_j \wedge d\xi_k}{|f|^4} \tag{3.11}$$

The cohomology of the complex projective space is

$$H^q(\mathcal{P}_N, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for q even} \\ 0, & \text{for q odd} \end{cases}$$

Moreover, $\mathcal{P}_N$ is a CW-complex [10], thus $\mathcal{P}_j \subset \mathcal{P}_{N-1}$ is a generator for $H^{2q}(\mathcal{P}_N, \mathbb{Z})$. A differential form φ of degree $2j$ will a representative class of $H^{2q}(\mathcal{P}_{N-1}, \mathbb{Z})$ if and only if $\int_{P_j} \varphi \in \mathbb{Z}$.
Thus, in order to prove the claim we will show that

$$\int_{\mathcal{P}_1} c_1(U_{1,N}, D(h)) \in \mathbb{Z}.$$

We can think about $\mathcal{P}_1 \subset \mathcal{P}_{N-1}$ can be thought as the subspace generated by

$$\{(z_1, ..., z_N) : z_j = 0, j = 3, ..., N\}.$$

Let f be a frame for $U_{r,N} \to \mathcal{P}_{N-1}$ on $\{z : z_1 \neq 0\} = W$. Then

$$f([1, \xi_2, 0, ..., 0]) = (1, \xi_2, 0, ..., 0)$$

by letting $z = \xi_2$ we have

$$\alpha = c_1(U_{1,N}, D(h))_{|W \cap \mathcal{P}_1} = \frac{1}{2\pi i} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$

Thus, computations shows that

$$\int_{\mathcal{P}_1} \alpha = \frac{1}{2\pi i} \int_{W \cap \mathcal{P}_1} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} = -1.$$

□

3.5.1 Chern Classes for complex line bundles

Lemma 3.5.3. *There is a one to one correspondence between isomorphism classes of holomorphic complex line bundles above X, that will be christened together with the tensor product* Picard's group *[18], and elements of the group $H^1(X, \mathcal{O}^*)$.*

Proof. Let $E \to X$ be a complex holomorphic line bundle, then there exist a covering $\{U_\alpha\}$ and transition functions $\{g_{\alpha\beta}\}$ that satisfies the Čech cocycle conditions. By using the Bockstein operator of Čech cohomology we have
$\delta g(U_\alpha, U_\beta, U_\gamma) = g(U_\beta, U_\gamma) - g(U_\alpha, U_\gamma) + g(U_\alpha, U_\beta) = g_{\beta\gamma} g_{\gamma\alpha}^{-1} g_{\alpha\beta} = 1.$
Therefore, the transition functions $\{g_{\alpha\beta}\}$ define a cocycle, thus an element of $H^1(X, \mathcal{O}^*)$. Moreover, if $E' \to X$ is another holomorphic complex line bundle isomorphic to $E \to X$, then we can combine both through the equivalence $L : E \underset{\simeq}{\overset{E'}{\to}}$ and the transition functions for the two bundle defined in two open covering of X in a suitable refinement of the coverings. We find that $E' \to X$ defines the same class in $H^1(X, \mathcal{O}^*)$. Thus the mapping is well defined. Conversely, let $\xi \in H^1(X, \mathcal{O}^*)$ any cohomology class, represented by a cocycle $\{g_{\alpha\beta}$ in some open cover $\{U_\alpha\}$. Through the chosen cocycle we can build a holomorphic complex line bundle that has $\{g_{\alpha\beta}\}$ as transition functions by letting

$$\tilde{E} = \sqcup_\alpha U_\alpha \times \mathbb{C}$$

and considering, as always, the equivalence relation

$$(x, z) \in U_\alpha \times \mathbb{C}, (y, z) \in U_\beta \times \mathbb{C}$$

are equivalent, i.e.

$$(x,z) \sim (y,w) \Leftrightarrow y = x \text{ and } w = g_{\alpha\beta}(x)z \in U_\alpha \cap U_\beta.$$

Thus, $E = \tilde{E}/\sim$ is a holomorphic complex line bundle with the desired property. $\square$

As we have already previously observed, Chern classes depend only on the equivalence classes in cohomology. Thus, consider the exponential sequence for a connected manifold X

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

And consider the induced sequence in cohomology.

$$H^1(X,\mathcal{O}) \longrightarrow H^1(X,\mathcal{O}^*) \xrightarrow{\delta} H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathcal{O}) \longrightarrow H^2(X,\mathbb{R})$$

The first Chern class for a complex line bundle is defined as the group homomorphism

$$c_1 : H^1(X,\mathcal{O}^*) \longrightarrow H^2(X,\mathbb{R}).$$

Lemma 3.5.4. *The following diagram is commutative*

$$\begin{tikzcd} H^1(X,\mathcal{O}^*) \arrow[r, "\delta"] \arrow[dr, "c_1"'] & H^2(X,\mathbb{Z}) \arrow[d, "j"] \\ & H^2(X,\mathbb{R}) \end{tikzcd}$$

Where j is the canonical inclusion of cohomology groups, and δ is the Bockstein operator.

Proof. We represent the de Rham cohomology through the Čech cohomology and then we will calculate explicitly the Bockstein operator in this context. Suppose that $\mathcal{U}$,is a locally finite cover of X, and consider $\xi = \{\xi_{\alpha\beta\gamma} \in \mathcal{Z}^2(\mathcal{U},\mathbb{R})$. We want to associate to ξ a closed 2-form φ on X. Note that $\xi \in \mathcal{Z}^2(\mathcal{U},\mathcal{E})$, and $\mathcal{E}$ is a fine sheaf, therefore we can choose $\tau \in C^1(\mathcal{U},\mathcal{E})$ such that $\delta\tau = \xi$. This could be done by choosing some partition of unity $\{\varphi_\alpha\}$ subordinated to $\mathcal{U}$ and by letting

$$\tau_{\beta\gamma} = \sum_\alpha \xi_{\alpha\beta\gamma}.$$

The de Rham operator is well defined on the cochain groups $C^q(\mathcal{U},\mathcal{E})$ and commutes with δ. Thus,

$$\delta d\tau = d\delta\tau = d\xi = 0 \;\Rightarrow d\tau \in \mathcal{Z}^1(\mathcal{U}, \mathcal{E}^1).$$

But, also $\mathcal{E}^1$ is a fine sheaf, therefore we can repeat the same argument by choosing $\mu \in C^0(\mathcal{U}, \mathcal{E}^1)$ such that $\delta\mu = d\tau$, by letting $\mu_\beta = \sum_\alpha \varphi_\alpha d\tau_{\alpha\beta}$. Therefore,

$$\delta d\mu = d\delta\mu = d^2\tau = 0 \;\Rightarrow \varphi = -d\mu \in \mathcal{Z}^0(\mathcal{U}, \mathcal{E}^2) = \mathcal{E}^2(X).$$

That is a global closed 2-form on X. Hence, to a element $\xi \in \mathcal{Z}^2(\mathcal{U}, \mathbb{R})$ we associated a differential form $\varphi(\xi)$. This procedure induces a cohomology map

$$\check{H}^2(X, \mathbb{R}) \longrightarrow H^2(X, \mathbb{R})$$

which is an isomorphism. Suppose that $\mathcal{U}$ is a open cover of X with the propriety that every intersection of the covering is simply connected, such cover is called *Leray covering*. We use $\mathcal{U}$ to describe the Bockstein operator

$$\delta : \; H^1(X, \mathcal{O}^*) \longrightarrow H^2(X, \mathbb{Z})$$

Let $g = \{g_{\alpha\beta}\} \in \mathcal{Z}^1(\mathcal{U}, \mathcal{O}^*)$ represented by $\sigma = \sigma_{\alpha\beta}$ defined by

$$\sigma_{\alpha\beta} = \frac{1}{2\pi i} \log g_{\alpha\beta} = \exp^{-1}(g_{\alpha\beta}).$$

The above defines a element of $C^1(\mathcal{U}, \mathcal{O})$, therefore $\delta\sigma \in C^2(\mathcal{U}, \mathcal{O})$, and since $\delta^2 = 0$ we have that $\delta\sigma \in \mathcal{Z}^2(\mathcal{U}, \mathcal{O})$, but on the other hand

$$(\delta\sigma)_{\alpha\beta\gamma} = \frac{1}{2\pi i}(\log g_{\alpha\beta} - \log g_{\alpha\gamma} + \log g_{\alpha\beta})$$

is integer valued since $g_{\alpha\beta}$ is a cocycle. Therefore, $\delta\sigma \in \mathcal{Z}^2(\mathcal{U}, \mathbb{Z})$ is a representative for δg in $H^2(X, \mathbb{Z})$.

Let $g = \{g_{\alpha\beta}\}$ be transition functions for the holomorphic complex line bundle $E \to X$ endowed with a hermitian metric h. Being $\{U_\alpha\}$ a trivializing cover for E, we have, for every index α, frames f_α of E above every U_α. We let $h(f) = h_\alpha$ and h_α is positive definite in every U_α. Thus,

$$c_1(E, h) = \frac{i}{2\pi}\partial(h_\alpha \partial h_\alpha) = \frac{1}{2\pi i}\partial\bar{\partial} \log h_\alpha.$$

The function h_α satisfies $h_\alpha = |g_{\alpha\beta}|^2 h_\beta \in U_\alpha \cap U_\beta$. Consider $\delta\sigma \in \mathcal{Z}^2(\mathcal{U}, \mathbb{Z})$, like before, i.e.

$$\sigma_{\alpha\beta} = \frac{1}{2\pi i} \log g_{\alpha\beta}.$$

We want to associate to $\delta\sigma$ a closed 2-form through the isomorphism

$$\check{H}^2(X, \mathbb{R}) \longrightarrow H^2(X, \mathbb{R}),$$

which will be the Chern form on E. This will conclude the proof. We use τ and μ from the construction of the isomorphism letting $\tau = \mu$, and $\mu = \{\mu_\alpha\}$ where

$$\mu_\alpha = \frac{1}{2\pi i}\partial \log h_\alpha.$$

Note that

$$\begin{aligned}(\delta\mu)_{\alpha\beta} = \mu_\beta - \mu_\alpha &= \frac{1}{2\pi i}\partial \log g_{\alpha\beta}\bar{g}_{\alpha\beta} \\ &= \frac{1}{2\pi i}(\partial \log g_{\alpha\beta} + \partial \log \bar{g}_{\alpha\beta}) = \frac{1}{2\pi i}\partial \log g_{\alpha\beta} \\ &= \frac{1}{2\pi i} d \log g_{\alpha\beta} = d\left(\frac{1}{2\pi i}\log g_{\alpha\beta}\right) = d\sigma_{\alpha\beta} = d\tau_{\alpha\beta}.\end{aligned}$$

Thus, the closed 2-form associated to $\delta\sigma$ is given by

$$\varphi = -d\mu = d\left(\frac{i}{2\pi i}\log g_{\alpha\beta}\right) = \frac{1}{2\pi i}\partial\bar{\partial}\log h_\alpha = c_1(E, h).$$

□

Remark 3.5.2. *The above sequence, and the above Lemma holds also in the C^∞ category. Indeed, from the exponential sequence on a smooth manifold X, namely*

$$0 \to \mathbb{Z} \to \mathcal{E} \to \mathcal{E}^* \to 0$$

we can consider the induced sequence in cohomology

$$... \to H^1(X, \mathcal{E}) \to H^1(X, \mathcal{E}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{E}) \to ...$$

Since $\mathcal{E}$ is a fine sheaf, hence is also soft, we have that the above sequence restricts to the isomorphism

$$H^1(X, \mathcal{E}^*) \xrightarrow{\simeq} H^2(X, \mathbb{Z}),$$

that means, every complex smooth line bundle is completely determined by the first Chern class.

The above remark does not apply in general to complex holomorphic line bundles. Indeed, consider a complex manifold X and the sequence

$$H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}). \tag{3.12}$$

It may happen that $H^1(X, \mathcal{O})$ and $H^2(X, \mathcal{O})$ are non trivial, consequently the holomorphi complex line bundles are not completely determined by their Chern class. Consider the commutative diagram

$$\begin{tikzcd}
H^1(X,\mathcal{O}^*) \arrow[r, "\delta"] \arrow[dr, "c_1"'] & H^2(X,\mathbb{Z}) \arrow[d, "j"] \\
 & \tilde{H}^2(X,\mathbb{Z})
\end{tikzcd}$$

Denote by $\tilde{H}^{1,1}(X,\mathbb{Z})$ the cohomology classes in $\tilde{H}^2(X,\mathbb{Z})$ that admit d-closed forms of type $(1,1)$.

Proposition 3.5.2. *In the above situation*

$$c_1(H^1(X,\mathcal{O}^*)) = \tilde{H}^{1,1}(X,\mathbb{Z})$$

Proof. It is sufficient to prove that $\delta(H^1(X,\mathcal{O}^*)) = H^{1,1}(X,\mathbb{Z})$. To see that, we shall show that the image of $H^{1,1}(X,\mathbb{Z})$ in $H^2(X,\mathcal{O})$ is zero. Consider the commutative diagram of sheaves

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[dr] & \mathbb{C} \arrow[d] \\
 & \mathcal{O}
\end{tikzcd}$$

Take the induced cohomology diagram

$$\begin{tikzcd}
H^2(X,\mathbb{Z}) \arrow[r] \arrow[dr] & H^2(X,\mathbb{C}) \arrow[d] \\
 & H^2(X,\mathcal{O})
\end{tikzcd}$$

Now we see that $\tilde{H}^{1,1}(X,\mathbb{Z}) \subset H^2(X,\mathbb{C})$ and it is the image of $H^{1,1}(X,\mathbb{Z})$ in the above diagram. Therefore, we shall show that the image of $H^{1,1}(X,\mathbb{Z})$ in $H^2(X,\mathcal{O})$ is zero. Consider the homomorphism of resolution of sheaves

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "i"'] & \mathcal{E}^0 \arrow[r, "d"] \arrow[d, "i"'] & \mathcal{E}^1 \arrow[r] \arrow[d, "\pi_{0,1}"'] & \mathcal{E}^2 \arrow[d, "\pi_{0,2}"'] \\
0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{E}^{0,0} \arrow[r, "\bar{\partial}"] & \mathcal{E}^{0,1} \arrow[r, "\bar{\partial}"] & \mathcal{E}^{0,2}
\end{tikzcd}$$

The map $\pi_{0,q} : \mathcal{E}^q \to \mathcal{E}^{0,q}$ is a projection. Therefore, the cohomology mapping $H^2(X,\mathbb{C}) \to H^2(X,\mathcal{O})$ maps a $d-$closed differential form φ onto a $\bar{\partial}-$closed form $\pi_{0,2}\varphi$. Thus, it is clear that the image of $H^{1,1}(X,\mathbb{C})$ in $H^2(X,\mathcal{O})$ is zero, since a class in $H^{1,1}(X,\mathbb{C})$ is represented by a $d-$ closed form of type (1,1) φ, we must have that $\pi_{0,2}\varphi = 0$. $\square$

Chapter 4

The Kodaira Embedding Problem

4.1 Divisors

Let X be a complex manifold and consider the short exact sequence of multiplicative abelian sheaves

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^* \big/ \mathcal{O}^* \to 0$$

Where we denoted by $\mathcal{O}^*$ the sheaf of non vanishing holomorphic function on X and by $\mathcal{M}^*$ the sheaf of non vanishing meromorphic function on X. The quotient sheaf $\mathcal{M}^* \big/ \mathcal{O}^*$ defines the so called *sheaf of divisors* of X and we denote this latter by $\mathcal{D}$, a section $D \in \mathcal{D}$ is called a *divisor*. In particular if $D \in H^0(X, \mathcal{D})$ then there exists an open cover $\{U_\alpha\}$ of X and meromorphic non vanishing functions $f_\alpha \in \mathcal{M}^*(U_\alpha)$ such that

$$g_{\alpha\beta} = \frac{f_\beta}{f_\alpha} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$

It is rather immediate to verify that the above functions $g_{\alpha\beta}$ satisfy the Čech cocycle condition, consequently there exists a class of holomorphic complex line bundles above X. That means: a divisor $D \in H^0(X, \mathcal{D})$ determines $[D] \in H^1(X, \mathcal{O}^*)$, where we have previously seen that $H^1(X, \mathcal{O}^*)$ is isomorphic to the Picard's group of complex line bundles above X. Therefore, we say that two holomorphic line bundles determine the same class, i.e. are equivalent, if and only if they differ by a global non vanishing holomorphic function. We shall see that often divisors occur in the following way: given a complex manifold X, let $V \subset X$ be an algebraic submanifold of X, i.e. given an open covering $\{U_\alpha\}$ of X and holomorphic functions $f_\alpha \in \mathcal{O}^*(U_\alpha)$ such that $\frac{f_\beta}{f_\alpha} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, then V is determined by the zero locus of those functions. In this latter case V is called a divisor of X.

Example 4.1.1. Consider the complex projective space $\mathcal{P}_n$. Let $V \subset \mathcal{P}_n$ be an algebraic submanifold of P_n, e.g. V can be seen as a projective hyperplane determined by the equation $V = [z_0 = 0]$. Notice that in the open chart $U_\alpha \subset P_n$ the equations of V are determined by $[z_0/z_\alpha = 0]$. We see that the coordinate functions are clearly holomorphic functions on every coordinate chart $U_\alpha \subset \mathcal{P}_n$, and V is determined by the zero locus of these functions, that means V is a divisor of $\mathcal{P}_n$. Moreover, in the overlap $U_\alpha \cap U_\beta$ we have

$$g_{\alpha\beta} = \frac{z_0}{f_\alpha} \cdot \left(\frac{z_0}{z_\beta}\right)^{-1} = \frac{z_\beta}{z_\alpha},$$

and clearly we see that $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$. Therefore, there exists an holomorphic complex line bundle $L(V)$ above V.

4.2 Kähler Manifolds

A very important class of complex manifolds is surely determined by the Kähler manifolds. We give a very short and brief introduction to this topic, tailored directly for the purpouse of our discussion. The literature about Kähler manifolds is in rich supply, the reader may look at [6], [7]. In order to introduce what a Kähler manifold is, consider a complex manifold X and let h be an hermitian metric on X. Notice that we can see h as a smoothly parametrized family of 2-forms, $h = \{h_x\}_{x \in X}$, with

$$h_x : T_xX \times T_xX \to \mathbb{C}.$$

Therefore, that determines a form Ω that will be christened *fundamental form*, that is the holomorphic form of type $(1,1)$ with respect to the hermitian metric h.

Definition 4.1. *Let X be a complex manifold. An hermitian metric h on X whose fundamental form Ω is d-closed, i.e. $d\Omega = 0$, is called a* Kähler metric *and in this case Ω is called* Kähler form *. If there exists on X at least a Kähler metric, then X is said to be of* Kähler type. *A complex manifold endowed with a Kähler metric is called a* Kähler manifold.

The following result ensures that the class of Kähler manifolds is closed under the inclusion.

Proposition 4.2.1. *Let X be a Kähler manifold and $C \subset X$ be a complex submanifold of X. Then C is also a Kähler manifold.*

Proof. Consider the canonical inclusion of the submanifold C in X:

$$j : C \hookrightarrow X$$

Then if h is the Kähler metric on X, we see that the pullback of h through j determines a metric on C, i.e. $h_C = j^*h$, therefore since the pullback preserves types we see that the fundamental Kähler form Ω with respect to the metric h induces a fundamental form $\Omega_C = j^*\Omega$ on C with respect to the metric h_C. We can easily see that this form is also a Kähler form, indeed

$$d\Omega_C = dj^*\Omega = j^*d\Omega = 0.$$

□

On a complex manifold X a hermitian metric can be expressed in local coordinates by a hermitian symmetric tensor

$$h = \sum_{\mu\nu} h_{\mu\nu}(z) dz_\mu \otimes d\bar{z}_\nu,$$

where $h = (h_{\mu\nu})$ is a positively definite hermitian symmetric matrix, that is $h =^t \bar{h}$ and $\forall u \in \mathbb{C}^n$, ${}^t\bar{u}hu > 0$. The fundamental form Ω with respect to the hermitian metric h in local coordinates is given by

$$\Omega = \frac{i}{2} \sum_{\mu\nu} h_{\mu\nu}(z) dz \wedge d\bar{z},$$

where in this notation we see that the coefficients $h_{\mu\nu}$ as a function of the variable z are given by

$$h_{\mu\nu}(z) = h(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})(z).$$

Thus, if X is a Kähler manifold then

$$dh_{\mu\nu} = 0 \Leftrightarrow \partial h_{\mu\nu} = -\bar{\partial} h_{\mu\nu}.$$

Example 4.2.1. Let $X = \mathbb{C}^n$ and let $h = \sum_{\mu=1}^n dz_\mu \otimes d\bar{z}_\mu$. Then the fundamental form is given by

$$\Omega = \frac{i}{2} \sum_{\mu=1}^{n} dz_\mu \wedge d\bar{z}_\mu.$$

Since $h_{\mu\nu}$ are in this case constant we see that Ω is a d-closed differential form, so it is a Kähler form, hence h is a Kähler metric.

Example 4.2.2. One of the most important manifold of Kähler type is the complex projective space $\mathcal{P}_n$. We have already observed in Chapter 3 that $\mathcal{P}_n$ is a hermitian complex manifold. Denote by $(\xi_0, ..., \xi_n)$ the homogeneous coordinates for $\mathcal{P}_n$. Consider the following differential form

$$\tilde{\Omega} = \frac{i}{2}\frac{|\xi|^2 \sum_{\mu=0}^{n} d\xi_\mu \wedge d\bar{\xi}_\mu - \sum_{\mu\nu=0}^{n} \bar{\xi}_\mu \xi_\nu d\xi_\mu \wedge d\xi_\nu}{|\xi|^4}.$$

We have seen in Chapter 3 that the curvature form Θ associated to the universal bundle $U_{1,n+1} \to \mathcal{P}_n$ is a d-closed $(1,1)$-form, then $\tilde{\Omega}$ up to negative sign and to the factor $i/2$ coincide with the mentioned curvature form, therefore this latter is a Kähler form on $\mathcal{P}_n$. We can rewrite $\tilde{\Omega}$ in a particular set of coordinates by letting $w_j = \xi_j/\xi_0, j = 1, ..., n$.

$$\Omega(w) = \frac{i}{2}\frac{(1+|w|^2)^2 \sum_{\mu=0}^{n} dw_\mu \wedge d\bar{w}_\mu - \sum_{\mu\nu=0}^{n} \bar{w}_\mu w_\nu dw_\mu \wedge dw_\nu}{(1+|w|^2)^2}.$$

According to this notation, the metric tensor with respect the above form is given by

$$h = \left(\sum_{\mu\nu} h_{\mu\nu}(z) dw_\mu \otimes d\bar{w}_\nu \right) (1+|w|^2)^{-2},$$

where the coefficients of the matrix associated to the metric tensor, forgetting the positive denominator above, is given by

$$h_{\mu\nu}(w) = (1+|w|^2)\delta_{\mu\nu} - \bar{w}_\mu w_\nu.$$

We can see that $\tilde{h} = (h_{\mu\nu})$ is hermitian symmetric and positively definited. Indeed, $\forall u \in \mathbb{C}^n$ we can find that

$$\begin{aligned} {}^t\bar{u}\tilde{h}u = \sum_{\mu\nu} h_{\mu\nu} u_\mu \bar{u}_\nu &= \sum_{\mu\nu} (1+|w|^2)\delta_{\mu\nu} u_\mu \bar{u}_\nu - \left(\sum_\mu \bar{w}_\mu u_\mu \right) \left(\sum_\nu w_\mu \bar{u}_\nu \right) \\ &= |u|^2 + |u|^2 |w|^2 - < \bar{w}, \bar{u} >< w, u > . \end{aligned}$$

Then, by the Cauchy-Schwarz inequality we have

$${}^t\bar{u}\tilde{h}u \geq |u|^2 > 0$$

Then $\tilde{h}$ is positively defined. Therefore, $\tilde{h}$ defines a hermitian metric called the *Fubini-Study* metric and by previous discussions we see that it is a Kähler metric. Hence, $\mathcal{P}_n$ together with the Fubini-Study metric is a Kähler manifold.

4.3 Hodge Manifolds

A particular class of Kähler manifolds is called *Hodge* manifolds. In order to introduce the topic we shall firstly introduce the following class of differential forms.

Definition 4.2. *Let X be a complex manifold. A differential form φ of X is called* integral *if is d-closed and its cohomology class is in the image of the canonical inclusion of cohomology rings*

$$H^\bullet(X, \mathbb{Z}) \longrightarrow H^\bullet(X, \mathbb{C})$$

Remark 4.3.1. *Given a complex manifold X, if we want to verify that a certain differential form φ is integral then we could use the two following criteria, [5], [15]:*

1. *for every closed submanifold $S \subset X$ the integral of φ is integer valued, i.e.*

$$\int_S \varphi \in \mathbb{Z}.$$

2. *For every open cover $\mathcal{U} = \{U_i\}$ write $\varphi = d\theta_i$ in $U_i \cap U_j$, then we have that $\theta_i - \theta_j = dd\varphi = 0$, then $\theta_i - \theta_j = df_{ij}$. In $U_i \cap U_j \cap U_k$ we then must have $d(f_{ij} + f_{jk} + f_{ki}) = \theta_i - \theta_j + \theta_j - \theta_k + \theta_k - \theta_i = 0$, therefore, $(f_{ij} + f_{jk} + f_{ki}) = c_{ijk}$ then φ is integral if and only if $c_{ijk} \in \mathbb{Z}$*

Definition 4.3. *Let X be a complex manifold with hermitian metric h and let Ω be the associated fundamental form. If Ω is integral then it is called a* Hodge form *and h is called* Hodge metric. *If X is a manifold of Kähler type then it is called* Hodge manifold *if it admits an Hodge metric.*

We shall now give few basic examples of Hodge manifolds that are useful for our discussion.

Example 4.3.1. In the last section we proved that $\mathcal{P}_n$ is a manifold of Kähler type, the fundamental form associated to the Fubini-Study metric is indeed the Chern form associated to the universal bundle $U_{1,n+1}$ up to a sign, and in the previous Chapter we have seen that its integral is integer valued in any cell $\mathcal{P}_i$, therefore by point 1 of the above Remark we see that the fundamental form Ω is an Hodge form, hence $\mathcal{P}_n$ is a Hodge manifold. Moreover, let X be a compact complex projective algebraic manifold, then there exists a sufficiently large integer $N > 0$ and an embedding $j : X \hookrightarrow \mathcal{P}_{\mathcal{N}}$. If Ω is the fundamental form with respect to the Fubini-Study metric, then $j^*\Omega$ is a Hodge form on X. Indeed, it is d-closed

$$dj^*\Omega = j^* d\Omega = 0$$

and it is integral

$$\int_X j^*\Omega = \int_{j(X) \subseteq \mathcal{P}_N} \Omega \in \mathbb{Z},$$

where the last equality holds by the diffeomorphism invariance of the integral sign [insert biblio]. In general, by the same principle, a complex submanifold of a Hodge manifold is again a Hodge manifold.

Example 4.3.2. Let X be a complex compact manifold and let Y be an Hodge manifold. Then

- If X is an unramified covering for Y, that is ah holomorphic surjection $\pi : X \to Y$ such that $\pi^{-1}(y)$ is discrete $\forall y \in Y$ and π is a local biholomorphism, then if Ω is the Hodge form on Y we see that $\pi^*\Omega$ is an Hodge form on X.

- If $f : X \to Y$ is an immersion then $f^*\Omega$ is an Hodge form on X. This can be seen in two steps. Firstly suppose that $\dim(X) = \dim(Y)$ then the tangential map is an isomorphism, therefore there exist an open neighbourhood $U \subset X$ of any $x \in X$ and a local diffeomorphism $g : U \to f(U)$, therefore by the diffeomorphism invariance of integrals the claim follows. Now suppose that $\dim(X) < \dim(Y)$, then consider the following commutative diagram

$$\begin{array}{ccc} H^2(Y,\mathbb{C}) & \xrightarrow{f^*} & H^2(X,\mathbb{C}) \\ {\scriptstyle j_1}\big\uparrow & & \big\uparrow{\scriptstyle j_2} \\ H^2(Y,\mathbb{Z}) & \xrightarrow{f^*_{\mathbb{Z}}} & H^2(X,\mathbb{Z}) \end{array}$$

  Choose some closed integral representative ω of the cohomology class $[\omega] \in H^2(Y,\mathbb{C})$, then it sufficies to prove that $f^*\omega$ is integral, i.e. there exists an $\alpha \in H^2(X,\mathbb{C})$ such that $f^*\omega = j_2\alpha$. Since ω is integral, then there exist some $\beta \in H^2(Y,\mathbb{Z})$ such that $\omega = j_1\beta$, now apply f^* both sides and use the commutativity of the above diagram to find

$$j_2(f^*_{\mathbb{Z}}\beta) = f^*(j_1\beta) = f^*\omega,$$

  write $\alpha = f^*_{\mathbb{Z}}\beta$ and the claim follows as wanted.

Example 4.3.3. Let X be a Riemann surface compact and connected. Then X is a Hodge manifold. Indeed, since $dim_{\mathbb{R}}X = 2$ and by the fact that X is connected we know that $H^0(X,\mathbb{C}) \simeq \mathbb{C}$. Using Poincaré duality [5], [15] and the fact that X is also compact we have the following chain of isomorphisms

$$\mathbb{C} \simeq H^0(X,\mathbb{C}) \simeq H^2_c(X,\mathbb{C}) \simeq H^2(X,\mathbb{C}).$$

Let $\tilde{\Omega}$ be a $(1,1)$-form that is a basis element for the cohomology space $H^2(X,\mathbb{C})$, write $c = \int_X \tilde{\Omega}$, $c \neq 0$ (necessarily). Then $\Omega = c^{-1}\tilde{\Omega}$ is an Hodge form. Indeed

$$\int_X \Omega = \int_X c^{-1}\tilde{\Omega} = c^{-1}\int_X \tilde{\Omega} = c^{-1}c = 1 \in \mathbb{Z}.$$

This example, can be generalized as follows, every Kähler manifold such that $\dim H^{1,1}(X, \mathbb{C}) = 1$ is necessarily a Hodge manifold. Indeed, if Ω is the fundamental Kähler form, by letting $k = \int_X \Omega$ we get an Hodge form simply by writing $\tilde{\Omega} = k^{-1}\Omega$.

4.4 The Kodaira-Nakano Vanishing Theorem

In this section we want to introduce the Kodaira-Nakano Vanishing Theorem, which will be crucial for the proof of our main task, that is to prove the Kodaira embedding theorem. We shall firstly enfatize a particular class of holomorphic complex line bundles which are called *positive line bundles.* To do so, we shall firstly introduce the following notion

Definition 4.4. *Let X be a complex manifold. A holomorphic $(1,1)$ form φ is called* positive *if $\forall p \in X$ and for every coordinate system $z = (z_1, ..., z_n)$ near p we find that*

$$\varphi(z) = i\sum_{\mu\nu} \varphi_{\mu\nu}(z)dz_\mu \wedge d\bar{z}_\nu,$$

where $(\varphi_{\mu\nu}(z))$ must be hermitian symmetric and positively defined. For brevity, we shall indicate a positive differential form φ by $\varphi > 0$.

Definition 4.5. *Let $E \to X$ a holomorphic complex line bundle over a complex manifold X. Let $c_1(E) \in H^2(X, \mathbb{R})$ its first Chern class. We say that $E \to X$ is a* positive *complex line bundle if there exists a positive $(1,1)$-form ψ, with $\psi > 0$ which is a representative of the first Chern class, i.e. $\psi \in c_1(E)$. We say that $E \to X$ is negative if its dual bundle $E^* \to X$ is positive.*

In order to establish wether a holomorphic complex line bundle is positive we have the following criterium, for a proof of the following see [6]:

Proposition 4.4.1. *Let $E \to X$ be a holomorphic complex line bundle. Then $E \to X$ is positive if and only if there exists an hermitian metric h such that the curvature Θ_E induced by the hermitian metric h has $i\,\Theta_E > 0$.*

Example 4.4.1. Let $X = \mathcal{P}_n$ and consider the following three complex line bundles over $\mathcal{P}_n$:

1. The *Hyperplane section bundle* $H \longrightarrow \mathcal{P}_n$.
2. The *Universal bundle* $U = U_{1,n+1} \longrightarrow \mathcal{P}_n$.

3. The *Canonical bundle* $K = \wedge^n T^* \mathcal{P}_n \longrightarrow \mathcal{P}_n$.

We observe that the first complex line bundle it is defined by a divisor of $\mathcal{P}_n$, e.g. $[t_0 = 0]$ in homogeneous coordinates for $\mathcal{P}_n$. Then such a divisor will be defined by equations $[t_0/t_\alpha]$ in the coordinate neighbourhood

$$U_\alpha = \{[t_0, ..., t_\alpha, ..., t_n] \in \mathcal{P}_n : t_\alpha \neq 0\}.$$

Then, as we have seen on the first section of this Chapter, we see that the complex line bundle determined by this divisor has transition function

$$h_{\alpha\beta} = \left(\frac{t_0}{t_\alpha}\right) \cdot \left(\frac{t_0}{t_\beta}\right)^{-1} = \frac{t_\beta}{t_\alpha},$$

in the overlapping $U_\alpha \cap U_\beta$. As we know the transition functions in the same overlapping for the second complex line bundle are given by

$$u_{\alpha\beta} = \frac{t_\alpha}{t_\beta}.$$

Thus, we deduce that $H^* = U$.
We will explicitly calculate the transition functions for the third complex line bundle. Consider the coordinates $\zeta_j^\beta = t_j/t_\beta, j \neq \beta$. Then a basis for the restriction $K_{|U_\beta}$ is given by the n-form:

$$\Phi_\beta = (-1)^\beta d\zeta_0^\beta \wedge ... \wedge d\zeta_{\beta-1}^\beta \wedge d\zeta_{\beta+1}^\beta \wedge ... \wedge d\zeta_n^\beta.$$

Since

$$\zeta_j^\beta = \frac{t_j}{t_\beta} = \frac{t_j}{t_\alpha} \cdot \frac{t_\alpha}{t_\beta},$$

we have that

$$\zeta_j^\beta = \zeta_j^\alpha \cdot (\zeta_\alpha^\beta)^{-1}$$

in the overlapping $U_\alpha \cap U_\beta$. We can plug this latter in the Φ_β and we find the formula for the switch of coordinates

$$\Phi_\alpha = (\zeta_\beta^\alpha)^{n+1}(-1)^\beta d\zeta_0^\beta \wedge ... \wedge d\zeta_{\beta-1}^\beta \wedge d\zeta_{\beta+1}^\beta \wedge ... \wedge d\zeta_n^\beta = (\zeta_\beta^\alpha)^{n+1}\Phi_\beta.$$

Because of the arbitrariness of the above choices we see that this latter change of coordinates induces the desired transition function for the third line bundle:

$$k_{\alpha\beta} = (\zeta_\beta^\alpha)^{n+1}.$$

Therefore, because of the definition of the ζ_β^α, we deduce that the first and the second and the third bundle are $(n+1)$-tensor powers, that is

$$K = U^{n+1} = (H^*)^{n+1}.$$

We know that the universal bundle $U \to \mathcal{P}_n$ has as a curvature form the Kähler form, up to a sign, induced by the Fubini-Study metric. Therefore, by Proposition 4.4.1 we see that $i\Theta < 0$ therefore K, U, H^* are negative line bundles and then, by Definition 4.4, we deduce that H is positive.

In order to make an important remark about the previous example we shall give for granted the following result (for a proof see [6])

Theorem 4.4.1. *(*The Hodge decomposition theorem*) Let X be a compact complex manifold of Kähler type. Then, there is the following decomposition*

$$H^r(X, \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X)$$

and moreover,

$$\bar{H}^{p,q}(X, \mathbb{C}) = H^{q,p}(X, \mathbb{C}).$$

Immediately from the Hodge decomposition theorem follows that

$$H^1(\mathcal{P}_n, \mathcal{O}) = H^2(\mathcal{P}_n, \mathcal{O}) = 0.$$

Indeed, $H^1(\mathcal{P}_n, \mathbb{C}) = H^{1,0}(\mathcal{P}_n) \oplus H^{0,1}(\mathcal{P}_n)$.
By the de Rham cohomology of $\mathcal{P}_n$, we get $H^1(\mathcal{P}_n, \mathbb{C}) = 0$. Moreover,

$$\mathbb{C} \simeq H^2(\mathcal{P}_n, \mathbb{C}) = H^{2,0}(\mathcal{P}_n) \oplus H^{1,1}(\mathcal{P}_n) \oplus H^{0,2}(\mathcal{P}_n)$$

and since $\mathbb{C}[\Omega] = H^{1,1}(\mathcal{P}_n)$, where Ω is the fundamental form of $\mathcal{P}_n$ it follows that

$$H^2(\mathcal{P}_n, \mathcal{O} \simeq H^{0,2}(\mathcal{P}_n)) = 0.$$

Now consider the exponential sequence on $\mathcal{P}_n$

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

And consider the induced cohomology sequence

$$H^1(\mathcal{P}_n, \mathcal{O}) \to H^1(\mathcal{P}_n, \mathcal{O}^*) \xrightarrow{c_1} H^2(\mathcal{P}_n, \mathbb{Z}) \to H^2(\mathcal{P}_n, \mathcal{O})$$

by previous discussion we see that the above sequence restricts to the isomorphism

$$0 \to H^1(\mathcal{P}_n, \mathcal{O}) \underset{\sim}{\xrightarrow{c_1}} H^2(\mathcal{P}_n, \mathbb{Z}) \to 0.$$

Let $\mathcal{P}_1 \subset \mathcal{P}_n$ be a generator for $H^2(\mathcal{P}_n, \mathbb{Z})$. If we consider exterior powers of the Hyperplane section bundle H, e.g. H^m then we obtain

$$c_1(H^m)(\mathcal{P}_n) = m.$$

Indeed, for the properties of the Chern classes we have

$$c_1(H) = c_1(U^*) = -c_1(U), \text{ and } c_1(U)(\mathcal{P}_1) = \int_{\mathcal{P}_1} c_1(U) = -1,$$

then, $c_1(H)(\mathcal{P}_1) = 1$, therefore

$$c_1(H^m)(\mathcal{P}_n) = c_1(H \otimes ... \otimes H)(\mathcal{P}_n) = (c_1(H) + ... + c_1(H))(\mathcal{P}_n) = m.$$

Since, in this case, c_1 is an isomorphism of abelian groups it follows that every complex line bundle $L \to \mathcal{P}_n$ and in particular U and K of the previous example are tensor powers of H, that is $L = H^m$ and $c_1(L)(\mathcal{P}_n) = m$. Hence,

$$c_1(K)(\mathcal{P}_n) = c_1((H^*)^{n+1})(\mathcal{P}_n) = -c_1(H^{n+1})(\mathcal{P}_n) = -(n+1).$$

This means that we have a classification of all complex holomorphic line bundles of $\mathcal{P}_n$, namely we classified all holomorphic complex line bundles on $\mathcal{P}_n$.

We shall stress the fact that all these consideration could be done in the behalf of the vanishing of the cohomology spaces $H^1(\mathbb{P}_n, \mathcal{O})$ and $H^2(\mathbb{P}_n, \mathcal{O})$. Now, given a compact complex manifold X we wonder if we could find a sheaf such that the cohomology spaces are trivial. The following result was stated by Kodaira and proven by Nakano, for a proof the reader may have a look at [6].

Theorem 4.4.2. *(*The Kodaira-Nakano vanishing theorem*) Suppose that X is a compact complex manidold.*

1. *Let $E \to X$ be a holomorphic line bundle with the property that $E \otimes K^*$ is a positive line bundle. Then*

$$H^q(X, \mathcal{O}(E)) = 0, \quad q > 0$$

2. *Let $E \to X$ be a negative line bundle. Then*

$$H^q(X, \Omega^p(E)) = 0, \quad p + q < n$$

4.5 Quadratic Transformations

Let X be a complex manifold and let $p \in X$ be any point. Let $U \subset X$ be an open coordinate neighbourhood of the choosen point p with coordinates $z = (z_1, ..., z_n)$ such that $z(p) = z = 0$. Consider the product $U \times \mathcal{P}_{n-1}$ endowed with the product topology, denote by $[t_0, ..., t_n]$ the homegeneous coordinate for $\mathcal{P}_{n-1}$. Define the closed submanifold W of the product $U \times \mathcal{P}_{n-1}$ as

$$W := \{(z,t) \in U \times \mathcal{P}_{n-1} : z_\alpha t_\beta - z_\beta t_\alpha = 0\}. \tag{4.1}$$

From (1) there is a natural projection onto the first factor $\pi : W \to U$ such that $(z,t) \mapsto z$. This map has the following properties:

- $\pi^{-1}(0) = S = \{0\} \times \mathcal{P}_{n-1} \simeq \mathcal{P}_{n-1}$.
- $\pi_{|W/S} : W/S \xrightarrow{\sim} U/\{0\}$ is biholomorphic.

We define the *quadratic transformation* or the *Hopf blow up* $\tilde{X} = Q_p(X)$ of the complex manifold X at the chosen point $p \in X$ as

$$\tilde{X} := \begin{cases} W & \text{if } x \in U \\ X \setminus W & \text{if } x \in X \setminus U \end{cases}$$

Clearly, a quadratic transformation depends on the chosen point p. In order to enfatize that dependence we will sometimes denote the projection π_p. The submanifold W is called *local representation* of the quadratic transformation. Consider a quadratic transformation at the point p of the manifold X, and let $\pi : \tilde{X} \to X$ the projection onto the second factor like above. Notice that $\pi^{-1}(p) = S$ is a closed submanifold of $\tilde{X}$, so S is a divisor of $\tilde{X}$. Therefore, since S is a divisor, there exists a class of holomorphic complex line bundles, we denote by $L = L(S) \to S$ a representative of that class. Since $S \simeq \mathcal{P}_{n-1}$, then there is a canonical line bundle, the *hyperplane section bundle* $H \to S$ which is the line bundle determined by the divisor corresponding to a fixed linear hyperplane, e.g. $[t_1 = 0]$. Let $\sigma : W \to \mathcal{P}_{n-1}$ denote the projection onto the second factor and let $L_{|W}$ denote the restriction of the line bundle $L \to \tilde{X}$ to the local representation $W \subset \tilde{X}$. Then we have the following result:

Proposition 4.5.1. $L_{|W} = \sigma^* H^*$

Proof. Let U be a coordinate neighbourhood of p, denote by $(z_1, ..., z_n)$ the coordinates near p. Represent the quadratic transformation $\tilde{X}$ near $\pi^{-1}(U)$ by $W \subset U \times \mathcal{P}_{n-1}$ like in (1). Since $\pi^{-1}(p) = S \simeq \mathcal{P}_{n-1}$ then S is represented in $U \times \mathcal{P}_{n-1}$ by $z_1 = ... = z_n = 0$. The hyperplane $[t_1 = 0]$ is defined in the coordinate chart $V_\alpha = \{[t_1, ..., t_n] \in \mathcal{P}_{n-1} : t_\alpha \neq 0\}$ by the equation $[t_1/t_\alpha = 0]$. Therefore, the hyperplane section bundle $H \to S$ has transition functions

$$h_{\alpha\beta} = \left(\frac{t_1}{t_\alpha}\right) \cdot \left(\frac{t_1}{t_\beta}\right)^{-1} = \frac{t_\beta}{t_\alpha}. \text{ in } V_\alpha \cap V_\beta.$$

These are the same transition functions for the pullback bundle in $\sigma^* H$ in $U \times (V_\alpha \cap V_\beta) \cap W$. Notice that $S \cap (U \times V_\alpha) \cap W$ is determined by the equation $[z_\alpha = 0]$. Therefore, the line bundle $L \to S$ associated to the divisor S has transition functions

$$g_{\alpha\beta} = \frac{z_\alpha}{z_\beta} \text{ in } (U \times (V_\alpha \cap V_\beta) \cap W,$$

now by using the definition of W we see that $\frac{z_\alpha}{z_\beta} = \frac{t_\alpha}{t_\beta}$. Thus, $g_{\alpha\beta} = h_{\alpha\beta}^{-1}$, that means $L_{|W} = \sigma^* H^*$. □

For the rest of this section fix a compact complex manifold X, a quadratic transformation at p $\tilde{X} = Q_p(X)$ and $L_p \to Q_p(X)$ is the line bundle given in proposition 0.5.1.

Lemma 4.5.1. $K_{Q_p(X)} = \pi_p^* K_X \otimes L_p^{n-1}$

Proof. Begin by noticing that a frame f_1 for the canonical line bundle $K_{\tilde{X}}$ above $(U \times V_1) \cap W$ with respect to the local coordinates $(z_1, t_2/t_1, ..., t_n/t_1)$ is given by

$$f_1 = dz_1 \wedge d\left(\frac{t_2}{t_1}\right) \wedge ... \wedge d\left(\frac{t_n}{t_1}\right).$$

By using the definition of W we rewrite the above as follows

$$f_1 = (z_1)^{1-n} dz_1 \wedge dz_2 \wedge ... \wedge dz_n.$$

Thus, more generally a frame f_α above $(U \times V_\alpha) \cap W$ is given by

$$f_\alpha = (z_\alpha)^{1-n} dz_1 \wedge dz_2 \wedge ... \wedge dz_n.$$

A frame determines a system of trivializing section, therefore the relation $f_\beta = g_{\alpha\beta} f_\alpha$ holds. Hence, the transition function for $K_{\tilde{X}}$ in $U \times (V_\alpha \cap V_\beta) \cap W$ are given by

$$g_{\alpha\beta} = \left(\frac{z_\alpha}{z_\beta}\right)^{n-1}.$$

Consequently $K_{\tilde{X}|W} = L_{p|W}^{n-1}$ and since $K_{\tilde{X}}$ is trivial on U we have

$$K_{\tilde{X}|W} = L_p^{n-1} \otimes \pi_p^* K_{X|W}$$

.
Also $L_{|\tilde{X}\backslash W}$ is trivial and π_p is biholomorphic on $\tilde{X}\backslash W$. Hence $K_{\tilde{X}|X\backslash W} = K_X \otimes L_{p|\tilde{X}\backslash W}^{n-1}$, thus the claim holds. □

Let $p \in X$ and let $L_p \to Q_p(X)$ be the line bundle corresponding to the divisor $\pi_p^{-1}(p)$. If $q \neq p$ is another point on X, then it is clear that $Q_q Q_p X \simeq Q_p Q_q X$, since blowing up at the points p and q are local and independent operations. Let $\pi_{p,q} : Q_p Q_q X \to X$ be the composite projection and let $L_{p,q}$ be the line bundle corresponding to the divisor $\pi^{-1}(\{p\} \cup \{q\})$. Choose some positive integer μ, consider a complex line bundle $E \to X$ then we let

$$E^\mu = \underbrace{E \otimes ... \otimes E}_{\mu \text{ times}}$$

and

$$E^{-\mu} = (E^*)^\mu.$$

We let $E^0 = X \times \mathbb{C}$, the trivial line bundle over X, which is isomorphic to $E^\mu \otimes E^{-\mu}$ for all positive μ. As we have already used before, if $\{g_{\alpha\beta}\}$ is a set of transition function for E with respect to some locally finite set of trivializations, then $\{g_{\alpha\beta}\}^\mu$ is a set of transition functions for E^μ for all integers μ.

Proposition 4.5.2. *Let $E \to X$ be a positive holomorphic line bundle. There exists an integer $\mu_0 > 0$ such that if $\mu \geq \mu_0$, then for any points $p, q \in X, p \neq q$,*

(a) $\pi_p^* E^\mu \otimes L_p^* \otimes K_{Q_p(X)}^*$,

(b) $\pi_p^* E^\mu \otimes (L_p^*)^2 \otimes K_{Q_p(X)}^*$,

(c) $\pi_p^* E^\mu \otimes L_{p,q}^* \otimes K_{Q_p Q_q(X)}^*$,
are positive holomorphic line bundles.

Proof. In order to prove all the assertion we make use of the criterium expressed in Proposition 0.4.1 to establish wether if a certain holomorphic complex line bundle is positive. So, we shall find a hermitian metric such that the curvature form associated to (a) is a positive differential form. Firstly we prove the following result: Let F, G be two holomorphic complex line bundles on X, then

$$\Theta_{F \otimes G} = \Theta_F + \Theta_G.$$

To do so, let $\{\rho_\alpha\}$ and $\{r_\alpha\}$ be hermitian metrics of F and G respectively, then the changes of coordinates are given respectively by $\rho_\beta = |h_{\alpha\beta}|^2 \rho_\alpha$ and $r_\beta = |g_{\alpha\beta}|^2 r_\alpha$, where $\{h_{\alpha\beta}\}$ are the transition functions for F and $\{g_{\alpha\beta}\}$ are the transition functions for G. Then, the transition functions for the tensor bundle are given by $\{h_{\alpha\beta} \cdot g_{\alpha\beta}\}$, thus a hermitian metric for the same bundle is given by $\{\rho_\alpha \cdot r_\alpha\}$. The curvature form of F and G with respect to the chosen metrics are given by

$$\Theta_F = \bar{\partial}\partial \log \rho_\alpha,$$
$$\Theta_G = \bar{\partial}\partial \log r_\alpha.$$

Since the change of coordinates for the hermitian metric of the tensor bundle is given by

$$\rho_\beta r_\beta = |h_{\alpha\beta}|^2 |g_{\alpha\beta}|^2 \rho_\alpha r_\alpha,$$

Then,

$$\begin{aligned}\Theta_{F\otimes G} &= \bar{\partial}\partial \log(\rho_\alpha r_\alpha)\\ &= \bar{\partial}\partial \log \rho_\alpha + \bar{\partial}\partial \log r_\alpha\\ &= \Theta_F + \Theta_G.\end{aligned}$$

According to this last result, suppose we have found a hermitian metric for the tensor bundle $\pi_p^* E^\mu \otimes L_p^* \otimes K^*_{Q_p(X)}$, then

$$\Theta_{\pi_p^* E^\mu \otimes L_p^* \otimes K^*_{Q_p(X)}} = \mu\Theta_{\pi_p^* E} + \Theta_{L_p^*} + \Theta_{K^*_{\tilde{X}}}. \tag{4.2}$$

Now, using Lemma 4.5.1 and the above result we see that

$$\Theta_{K^*_{\tilde{X}}} = \Theta_{\pi_p^* K_X \otimes L_p^{n-1}} = \Theta_{\pi_p^* E} + (n-1)\Theta_{L_p} \tag{4.3}$$

Hence, substituting (3) into (2) we have the following

$$\Theta_{\pi_p^* E^\mu \otimes L_p^* \otimes K^*_{Q_p(X)}} = \mu\Theta_{\pi_p^* E} + n\Theta_{L_p^*} + \Theta_{\pi^* K_X}. \tag{4.4}$$

Now we shall find an hermitian metric such that (4) holds. We start by looking at L_p^* Consider a coordinate neighbourhood U of the point p, let $W \subset U \times \mathcal{P}_{n-1}$ be the local representation of $\tilde{X}$ and choose some $\rho \in \mathcal{D}(U)$ such that $\rho \geq 0$ in U and $\rho_{|U'} = 1$ where $U^{'} \subset U$ is a neighbourhood around the origin 0. By Proposition 0.5.1 we know that $L_{p|W} = \sigma^* H^*$, then $L^*_{p|W} = \sigma^* H$. Therefore, if $\tilde{h}_1$ is the natural hermitian metric for $H \to \mathcal{P}_{n-1}$ then $h_1 = \sigma^* \tilde{h}_1$ will be a hermitian metric for $L^*_{p|W}$. The curvature form on H with respect the natural metric $\tilde{h}_1$ is given by

$$\Theta_H = \bar{\partial}\partial \log \frac{|t_\alpha|^2}{|t_1|^2 + ... + |t_n|^2}.$$

Moreover, $(i/2)\Theta_H$ is the fundamental form associated with the standard Kähler metric on $\mathcal{P}_{n-1}$. Since $L^*_{p|\tilde{X}\setminus W}$ is trivial we can equipp this latter with a constant hermitian metric h_2. Then we define a hermitian metric on L_p^* by interpolating through the chosen section ρ the two above mentioned metrics, that is

$$h = \rho h_1 + (1 - \rho) h_2.$$

Observe that in $W^* = U^{'} \times \mathcal{P}_{n-1} \cap W$ $h = h_1$. Thus

$$\Theta_{L^*} = \Theta_{\sigma^* H} \text{ in } W^{'},$$

$$\Theta_{L^*} = 0 \text{ in } \tilde{X} \setminus W.$$

We can endow K_X with an arbitrary hermitian metric so that (4) holds. Now we shall prove the positivity of (4). Firstly, consider the sum

$$\mu \Theta_{\pi^* E} + \Theta_{\sigma^* H}$$

as differential forms on $U^{'} \times \mathcal{P}_{n-1}$, with the coordinates (z, t) as before. Then $\Theta_{\pi^* E}$ depends only on the variable z and $\Theta_{\sigma^* H}$ depends only on the variable t, and the coefficient matrix is positive definite in each of the respective directions, so their sum is a positive differential form in $U^{'} \times \mathcal{P}_{n-1}$ and the restriction to W is likewise positive. Moreover in $U \setminus U^{'}$ $\Theta_{\pi^* E}$ is positive definite, then there exists a $\mu_1(p)$ such that $\mu > \mu_1(p)$ implies that

$$[\mu \Theta_{\pi^* E}] + \Theta_{L^*} > 0. \tag{4.5}$$

Let μ_2 be chosen such that $\mu_2 \Theta_E + \Theta_{K_X^*} > 0$, which is possible since by assumption E is positive and X is compact. Then there is a $\mu_0(p) > 0$ such that (4) holds. Namely, let $\mu_0(p) = \mu_2 + n\mu_1(p)$ and consider the sum

$$\mu_2 \Theta_{\pi^* E} + \Theta_{\pi^* K_X^*}$$

that is positive definite everywhere besides at points of $S = \pi^{-1}(p)$ where is positive semidefinite. Suppose that $q \in U^{'}$. Then we claim that if $\mu \geq \mu_1(p)$, then the estimate (5) will hold for points q near p. This can be done by a continuity argument, namely we express the local representation in q centered at p by

$$W_q = \{(z, t) \in U \times \mathcal{P}_{n-1} : (z_i - q_i) t_j = (z_j - q_j) t_i\}$$

where $q = (q_1, ..., q_n)$ and $p = (0, ..., 0)$. By covering X with a finite number of such neighbourhoods we find that there is a μ_0 such that (4) holds for all $p \in X$ if $\mu \geq \mu_0$. Hence, (a) holds. Along the line of the proof of part (a) by using the same arguments the assertions (b) and (c) follows. $\square$

4.6 The Kodaira's Embedding Theorem

In this section we will prove the main result of this chapter, that is the Kodaira's Embedding theorem, which is a milestone in the field of complex algebraic and differential geometry. In the following we have the statement of the problem

Theorem 4.6.1. *(*Kodaira's Embedding Theorem*) Let X be a compact Hodge manifold. Then X is a projective algebraic manifold.*

In order to be able to prove the Kodaira's embedding theorem we require a sequence of lemmas. Consider the subsheaf of $\mathcal{O} = \mathcal{O}_X$ consisting of holomorphic sections that vanishes at points p and q of X, and denote it by m_{pq}. If $p = q$ then $m_{pp} = m_p^2$ consist of holomorphic sections that vanish at p to second order. From the assertion of the theorem if X is an Hodge manifold, then it admits an integral Kähler form Ω. Since for holomorphic complex line bundles hold $c_1(H^1(X, \mathcal{O}^*))$, then there is a complex line bundle $E \to X$ where Ω is a representative of $c_1(E)$, hence $E \to$ is a positive line bundle. Let $\mu \geq \mu_0$ from Proposition 4.5.2 and write $E^\mu = F$. Consider the short exact sequence of abelian sheaves:

$$0 \to m_{pq} \to \mathcal{O} \to \mathcal{O}/m_{pq} \to 0 \tag{4.6}$$

Since $\mathcal{O}(F)$ is a locally free sheaf, then we can tensor (6) with $\mathcal{O}(F)$ and get the following short exact sequence

$$0 \to \mathcal{O}(F) \otimes_{\mathcal{O}} m_{pq} \to \mathcal{O}(F) \to \mathcal{O}(F) \otimes_{\mathcal{O}} \mathcal{O}/m_{pq} \tag{4.7}$$

In (7) the quotient sheaf becomes

$$\begin{array}{ll} F_p \otimes_{\mathbb{C}} \mathcal{O}_p/m_p^2 & \text{if } x = p = q, \\ 0 & \text{if} \quad x \neq p. \end{array}$$

If $p = q$ note that m_p is a maximal ideal for the ring $\mathcal{O}_p$, therefore the quotient ring is a field, i.e. $\mathcal{O}_p/m_p = \mathbb{C}$, and because of the properties of the tensor product $F_p \otimes_{\mathbb{C}} \mathbb{C} \simeq F_p$, hence we have F_p when $x = p$, F_q when $x = q$, and 0 when $x \neq p$ or q.

Lemma 4.6.1. *$\mathcal{O}_p/m_p^2 \simeq \mathbb{C} \oplus T_p^* X$ and the quotient map is represented by $\mathcal{O}_p \ni f \mapsto [f(p), df(p)]$*

Proof. If $f \in \mathcal{O}_p$ then we can write it as power series near p, that is

$$f(z) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} D^\alpha f(p)(z-p)^\alpha.$$

Its class in the quotient is given by

$$[f]_p = \left[f(p) + \sum_{|\alpha|\geq 1} D^\alpha f(p)(z-p)^\alpha \right].$$

Therefore we have an induced mapping

$$\begin{aligned} \psi : \mathcal{O}_p &\longrightarrow \mathbb{C} \oplus T_p^* X \\ f &\longmapsto [f(p), df(p)] \end{aligned}$$

that factors through the quotient map, making the following diagram commute

$$\begin{tikzcd} \mathcal{O}_p \arrow[r, "\psi"] \arrow[dr] & \mathbb{C} \oplus T_p^* X \arrow[d, "\tilde{\psi}"] \\ & \mathcal{O}_p/m_p \end{tikzcd}$$

and $\tilde{\psi}$, because of the properties of composition of functions, is clearly an isomorphism. $\square$

From the exact sequence (7) we have an induced map

$$r : \ \mathcal{O}(X, F) \longrightarrow \mathcal{O}_p/m_p^2 \otimes F_p \simeq (\mathbb{C} \oplus T_p^* X) \otimes F_p$$

that can be specified as follows: let f be a local frame near the point p, if $\xi \in \mathcal{O}(X, F)$ then

$$r(\xi(f)) = (\xi(f)(p), d\xi(f)(p)) \in \mathbb{C} \oplus T_p^* X.$$

If r is surjective, then we can find some coordinates for X. Namely, we can have sections $\xi_j \in \mathcal{O}(X, F),\ j = 1, ..., m$ such that

$$\xi_0(p) = 1, \xi_j(p) = 0 \text{ for } j = 1, ..., m, \text{ and } d\xi_j(p) = dz_j. \tag{4.8}$$

In particular $d\xi_1(p) \wedge ... \wedge d\xi_m(p) \neq 0$, and $\xi_0(p) \neq 0$. Similarly, we can find from the sequence (7) the induced map:

$$s : \mathcal{O}(X, F) \longrightarrow F_p \oplus F_q$$

if s is surjective, then we can find sections $\xi_1, \xi_2 \in \mathcal{O}(X, F)$ such that

$$\xi_1(p) \neq 0,\ \xi_1(q) = 0,\ \xi_2(p) = 0,\ \xi_2(q) \neq 0. \tag{4.9}$$

Lemma 4.6.2. *If the maps r and s are surjective for all $p, q \in X$, then there exists an holomorphic embedding of X into $\mathcal{P}_m$, where* $\dim_{\mathbb{C}} \mathcal{O}(X, F) = m+1$.

Proof. Consider the map $r: \ \mathcal{O}(X,F) \to \mathcal{O}_p/m_p^2 \otimes F_p$, choose some basis $\varphi = \{\varphi_0, ..., \varphi_m\}$ of $\mathcal{O}(X,F)$. Then, if f is a holomorphic frame near the point p we have, for x near p

$$(\varphi_0(f)(x), ..., \varphi(f)(x)) \in \mathbb{C}^{m+1}.$$

We want to send the above vector onto $\mathcal{P}_m$, to do so we use surjectivity of r, namely we can find at least a φ_j, $j = 0, ..., m$ that is non zero at p. Therefore, we can have a mapping

$$\begin{aligned} \Phi_\varphi : X &\longrightarrow \mathcal{P}_m \\ x &\longmapsto \Phi_\varphi(x) := [\varphi_0(f)(x), ..., \varphi_m(f)(x)] \end{aligned}$$

The above map is by construction holomorphic, because the mapping

$$x \mapsto \varphi_j(f)(x)$$

is clearly holomorphic as a function of x. Furthermore, if $\tilde{f}$ is another frame near p, then $\forall j = 0, ..., m$ we get

$$\varphi_j(\tilde{f})(x) = c(x)\varphi_j(f)(x), \tag{4.10}$$

where c is a holomorphic function that does not vanish at p. Thus, (10) tells us that the basis represent the same class of homogeneous coordinates in $\mathcal{P}_m$. That means, the mapping Φ_φ is well defined. Suppose $\tilde{\varphi}$ is another basis of $\mathcal{O}(X,F)$ then there exists an invertible matrix with constant coefficients $C = (c_{ij}), c_{ij} \in \mathbb{C}$ such that $\forall j = 0, ..., m$ we get

$$\tilde{\varphi}_j = \sum_{ij} c_{ij}\phi_j.$$

Therefore, this induces a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Phi_\varphi"] \arrow[dr, "\Phi_{\tilde{\varphi}}"'] & \mathcal{P}_m \arrow[d, "\tilde{C}"] \\ & \mathcal{P}_m \end{tikzcd}$$

where $\tilde{C}$ is a biholomorphic mapping given by multiplication for the constant invertible matrix C. With this in mind, we see that Φ_φ is an embedding if and only if $\Phi_{\tilde{\varphi}}$ is an embedding, this means that Φ_φ would be an embedding independently of the choice of the basis φ. It remains to show that Φ_φ is an

embedding. We will prove that by using a basic fact of differential geometry, i.e. an injective immersion is an embedding. Thus we will prove firstly that Φ_φ is an immersion, namely Φ_φ ha maximal rank and then that Φ_φ is injective.
By assumption the map r is surjective, then it follows that we can find sections $\xi_0, ..., \xi_n \in \mathcal{O}(X, F)$ satisfying (8). It is clear that these sections are linearly independent in $\mathcal{O}(X, F)$, thus they extend to a basis $\tilde{\varphi}$, therefore if f is a frame near p like before we get

$$\Phi_{\tilde{\varphi}} = [\xi_0(f)(x), ..., \xi_n(f)(x), ...]$$

and using the local coordinates $(1, \zeta_1, ..., \zeta_n)$ we see that the Jacobian determinant

$$\det\left(\frac{\partial(\zeta_1, ..., \zeta_n)}{\partial(z_1, ..., z_n)}\right),$$

is given by

$$d\left(\frac{\xi_1(f)}{\xi_0(f)}\right) \wedge ... \wedge d\left(\frac{\xi_n(f)}{\xi_0(f)}\right) = (\xi_0(f)(p))^{-1} dz_1 \wedge ... \wedge dz_n \neq 0.$$

Hence, Φ_φ is an immersion, so $\Phi_{\tilde{\varphi}}$ is an immersion. The map s is surjective then we can find sections ξ_1, ξ_2 that satisfies (9), it is straighforward to see that are linearly independent, thus they can be extended to a basis $\tilde{\varphi} = \{\xi_1, \xi_2, ...\}$. Now, it is clear that $\Phi_{\tilde{\varphi}}$ is injective, and thus putting all together Φ_φ is an embedding. □

Lemma 4.6.3. *The maps r and s are surjective.*

Proof. We begin by proving surjectivity of $r: \mathcal{O}(X, F) \longrightarrow \mathcal{O}_p/m_p^2 \otimes F_p$. Let $\tilde{X} = Q_p(X)$ be the quadratic transformation of X at p with projection $\pi : \tilde{X} \to X$ and divisor $S = \pi^{-1}(p)$ and let $L \to S$ be the line bundle associated to that divisor. cal $\tilde{F} = \pi^*(F)$ the pullback bundle of F through the projection π, $\mathcal{O} = \mathcal{O}_X$ the structure sheaf of X and $\tilde{\mathcal{O}} = \mathcal{O}_{\tilde{X}}$. Consider $\mathcal{I}_S^2$ the subsheaf of $\tilde{\mathcal{O}}$ consisting of holomorphic sections vanishing to second order along S. Like before, we have a short exact sequence of abelian sheaves:

$$0 \to \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I}_S^2 \to \tilde{\mathcal{O}}(\tilde{F}) \to \tilde{\mathcal{O}}(\tilde{F}) \otimes \tilde{\mathcal{O}}/\mathcal{I}_S^2 \to 0$$

The pullback of π induces the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I}_S^2 & \longrightarrow & \tilde{\mathcal{O}}(\tilde{F}) & \longrightarrow & \tilde{\mathcal{O}}(\tilde{F}) \otimes \tilde{\mathcal{O}}/\mathcal{I}_S^2 \to 0 & \longrightarrow & 0 \\
 & & \pi_1^* \Big\uparrow & & \pi^* \Big\uparrow & & \pi_2^* \Big\uparrow & & \\
0 & \longrightarrow & \mathcal{O}(F) \otimes m_p^2 & \longrightarrow & \mathcal{O}(F) & \longrightarrow & \mathcal{O}(F) \otimes \mathcal{O}/m_p^2 & \longrightarrow & 0
\end{array}$$

where the map π^* is the restriction of π^* into the subsheaf $\mathcal{O} \times m_p^2$ and the map π_2^* is induced from the quotient map. Observe that if $f \in \Gamma(U, F)$ then vanishes to second order at p if and only if $\pi^* f \in \Gamma(\pi^{-1}(U), \tilde{F})$ vanishes along S. We rewrite the above diagram in the following way:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I}_S^2) & \longrightarrow & \Gamma(\tilde{\mathcal{O}}(\tilde{F})) & \longrightarrow & \Gamma(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \tilde{\mathcal{O}}/\mathcal{I}_S^2) & \longrightarrow & 0 \\
 & & \pi_1^* \Big\uparrow & & \pi^* \Big\uparrow & & \pi_2^* \Big\uparrow & & \\
0 & \longrightarrow & \Gamma(X, \mathcal{O}(F) \otimes m_p^2) & \longrightarrow & \Gamma(X, \mathcal{O}(F)) & \xrightarrow{r} & \Gamma(X, \mathcal{O}(F) \otimes \mathcal{O}/m_p^2) & \longrightarrow & 0
\end{array}$$

We claim that there exist isomorphisms

$$\beta : \Gamma(\tilde{\mathcal{O}}(\tilde{F})) \longrightarrow \Gamma(X, \mathcal{O}(F)),$$

$$\alpha : \Gamma(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I}_S^2) \longrightarrow \Gamma(X, \mathcal{O}(F) \otimes m_p^2),$$

that makes the above diagram commute. If this is true then to prove surjectivity of r it sufficies to prove that $H^1(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I}_S^2) = 0$. Indeed, considering the induced commutative diagram in cohomology and the below induced diagram in cohomolowy

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I}_S^2)) & \longrightarrow & H^0(\tilde{X}, (\tilde{\mathcal{O}}(\tilde{F}))) & \longrightarrow & H^0(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \tilde{\mathcal{O}}/\mathcal{I}_S^2) & \longrightarrow & 0 \\
 & & \pi_1^* \Big\uparrow & & \Big\uparrow & & \pi_2^* \Big\uparrow & & \\
0 & \longrightarrow & H^0(X, \mathcal{O}(F) \otimes m_p^2) & \longrightarrow & H^0(X, \mathcal{O}(F)) & \xrightarrow{r^*} & H^0(X, \mathcal{O}(F) \otimes \mathcal{O}/m_p^2) & \longrightarrow & 0
\end{array}$$

Both sequences are short exact, so r^* must be surjective. Therefore, by functoriality of cohomology the map r must be surjective. To explicitly construct the isomorphism β and α we would need to use Hartog's theorem, which asserts that a holomorphic function f defined on $U \setminus 0$ where U is a neighbourhood of the origin in $\mathbb{C}^n$, $n > 1$, can be analytically continued to all of U. Then, we shall define β and see that its restriction to the subspace $\Gamma(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I}_S^2)$ (which we shall call α) has the desired image. Suppose that $\xi \in \Gamma(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F})$, we know that π is biholomorphic on $\tilde{X} \setminus S$, then let

$$\tilde{\beta}(\xi) = (\pi^{-1})^*(\xi)$$

which is a well defined element of $\Gamma(X \setminus \{p\}, \mathcal{O}(F))$. Then by Hartog's theorem, there is a unique extension of $\tilde{\beta}(\xi)$ to a section $\mathcal{O}(F)$ of X which we call β. Clearly, we have $\beta^{-1} = \pi^*$ and hence β is an isomorphism. As we observed before $\beta^{-1}(\eta)$ will vanish to second order along S if and only if $\eta \in \Gamma(X, \mathcal{O}(F) \otimes m_p^2)$. Thus, it remains to show that $H^1(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I}_S) = 0$. Notice that $\mathcal{I}_S \simeq \tilde{\mathcal{O}}(L^*)$ since they have the same transition functions, then it follows that $\mathcal{I}_S^2 \simeq \tilde{\mathcal{O}}((L^*)^2)$. Therefore one has the following isomorphism

$$H^1(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I}_S^2) = H^1(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F} \otimes (L^*)^2).$$

By assertion (b) of Proposition 4.5.2

$$\tilde{F} \otimes (L^*)^2 \otimes K_X^* > 0,$$

therefore for assertion (1) of the Kodaira-Nakano vanishing theorem we see that $H^1(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I}_S^2) = 0$. To prove that $s : \mathcal{O}(X, F) \longrightarrow F_p \oplus F_q$ is surjective, we let $\tilde{X} = Q_p Q_q(X)$, $S = \pi^{-1}(\{p\} \cup \{q\})$, $\mathcal{I}_S$ the ideal sheaf of this divisor, let $\tilde{\mathcal{O}}$ be the structure sheaf for $\tilde{X}$, and let $\tilde{F} = \pi^* F$. We have the exact sequence

$$0 \to \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I} \to \tilde{\mathcal{O}}(\tilde{F}) \to \tilde{\mathcal{O}}(\tilde{F}) \otimes \tilde{\mathcal{O}}/\mathcal{I} \to 0,$$

and there exists isomorphism α and β constructed, using Hartog's theorem, like before that they make the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I}) \arrow[r] & \Gamma(\tilde{\mathcal{O}}(\tilde{F})) \arrow[r] & \Gamma(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \tilde{\mathcal{O}}/\mathcal{I}) \arrow[r] & 0 \\
0 \arrow[r] & \Gamma(X, \mathcal{O}(F) \otimes m_p q) \arrow[r] \arrow[u, "\pi^*_{p,q}"] & \Gamma(X, \mathcal{O}(F)) \arrow[r, "s"] \arrow[u, "\pi^*_{p,q}"] & \Gamma(X, \mathcal{O}(F) \otimes \mathcal{O}/m_{pq}) \arrow[r] \arrow[u, "\pi^*_{p,q}"] & 0
\end{tikzcd}$$

and thus we see that the vanishing of $H^1(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I})$ will ensure the surjectivity of s. But $\mathcal{I}_S \simeq \tilde{\mathcal{O}}(L_{pq}^*)$, and it follows from Proposition 0.5.2 that

$$\tilde{F} \otimes L_{pq}^* \otimes K_X^* > 0.$$

By applying assertion (1) of the Kodaira-Nakano vanishing theorem we see that $H^1(\tilde{X}, \tilde{\mathcal{O}}(\tilde{F}) \otimes \mathcal{I}_S) = 0$. $\square$

We can immediately observe that along the line of previous Lemmas, we obtain the proof of the Kodaira's Embedding Theorem.

Conclusions

We have seen in chapters 1,2 and 3 the basic concepts of complex geometry, sheaf theory and hermitian differential geometry in order to prepare the reader to the last chapter, that is the main target of this work. The Kodaira's embedding theorem plays a very important role into complex algebraic and differential geometry. In order to appreciate this, notice that as a consequence of the Kodaira's theorem, each of the examples of Hodge manifolds, seen in the last chapter, admits a projective algebraic embedding. In particular, any compact Riemann surface is projective algebraic. It follows immediately from the Kodaira's embedding theorem that any compact complex manifold X which admits a positive line bundle $L \to X$ is projective algebraic. Namely, in this case, the first Chern class $c_1(E)$ will have a Hodge form as a representative, and thus X will be projective algebraic. In the next and last section, we will give further motivations for the importance of the Kodaira's embedding theorem.

Beyond the Kodaira's Embedding Theorem

The essential tool for the proof of the Kodaira's embedding theorem is the Kodaira-Nakano vanishing theorem, that holds for compact complex manifolds. In case we consider a non compact complex manifold the vanishing theorem is not anymore valid. Andreotti and Vesentini in 1961 [20] proved, under some peculiar assumptions, an extension of the vanishing theorem in the case when the complex manifold is non compact. In the following we will give a very brief explanation of their work.

The non compact case of the Kodaira's embedding theorem

Let $E \to X$ be a complex line bundle and denote by $C^{p,q}(X, E)$ the space of the differential smooth forms of type (p, q) E-valued. Consider the space

$$L^{p,q}(X, E) := \{\varphi \in C^{p,q}(X, E) : \int_X A(\varphi, \varphi) dX < \infty\}$$

modulo the equivalence relation on the (p, q)- integrable function. In the above, A is a sesquilinear form on X and dX is the volume element of the metric. $L^{p,q}(X, E)$ is a Hilbert space. We say that a holomorphic complex line bundle $E \to X$ is complete and positive if the form

$$\omega = i\Theta\epsilon - 2\pi c_1(E)$$

where Ω is the fundamental form and $\epsilon > 0$ a real number, is the exterior form of a complete Kähler metric (in the sense of Hilbert) on X. We say that $E \to X$ is negative and complete if the dual bundle E^* is positive and complete. Then we have the following result

Theorem 4.6.2. *(Vanishing) If $E \to X$ is a negative complete holomorphic line bundle, then the natural morphism*

$$H_c^q(X, \Omega^p(E)) \longrightarrow H^q(X, \Omega^p(E))$$

vanish for $p + q < n$, where $n = \dim_{\mathbb{C}} X$.

The above theorem permits to prove the Kodaira's theorem in case when X is non compact. However, we shall stress the fact that not every non compact maifold admits an embedding into a finite dimensional complex projective space. In order to introduce the class of such non compact manifolds that admits such embedding, we some preliminary definitions.

Definition 4.6. *We call* upper bound form *a $(1, 1)$ differential form φ on the complex manifold X (not necessarily closed) a differential form that satisfies the following conditions:*

- *for all points $p \in X$ we can choose a form σ_p such that $\varphi - \sigma_p$ is positive definite on the complement of p.*
- *There exists a hermitian metric on the canonical bundle K such that, if $\Theta(K)$ is the curvature form of K, then $\varphi - \Theta(K)$ is positive definite.*
- *the hermitian metric defined on X from $\varphi - \Theta(K)$ is complete.*

We shall quote the following important result:

Lemma 4.6.4. *Every connected complex manifold (with numerable basis) admits at least one upper bound form.*

Definition 4.7. *We say that a holomorphic complex line bundle $E \to X$ over a complex manifold X is* uniformly positive *if we can choose an upper bound form φ on X, an integer $\mu > 0$ and a metric of the fibers of $E \to X$ such that, $\mu\Theta(E) - \varphi$ is positive definite at every point of X.*

We shall underline that not every complex manifold admits a holomorphic complex line bundle uniformly positive. We shall call the class of manifold that admits such line bundles as *reasonable* manifolds.
Hereby we can state the non compact version of the Kodaira's theorem:

Theorem 4.6.3. *Every connected reasonable manifold X can be embedded in a finite dimensional complex projective space $\mathcal{P}_n$.*

Isometric Embeddings and Symplectic Geometry

Let X be a compact Kähler manifold with fundamental Kähler form Ω and denote it by (X, Ω). Denote by Ω_{FS} the Fubini-Study form on $\mathcal{P}_n$, then for the Kodaira embedding theorem there exists an holomorphic embedding

$$k: \ (X, \Omega) \hookrightarrow (\mathcal{P}_n, \Omega_{FS}).$$

Observe that the form Ω is cohomologues to $k^*\Omega_{FS}$,i.e. $\Omega \sim k^*\Omega_{FS}$. Then, we can ask ourself when that embedding is also isometric, i.e. $k^*\Omega_{FS} = \Omega$. The answer to that question was provided in 1952 by Calabi in his Ph.D. dissertation *Isometric complex analytic imbedding of Kähler manifolds.* Further generalizations of the Kodaira's theorem can be found in the field of symplectic geometry. We briefly recall what a symplectic manifold is

Definition 4.8. *A symplectic manifold is a pair (M, ω) where M is a smooth manifold, and ω is a 2-form on M that satisfies:*

- $\forall x \in M, \ (T_xM, \omega_x)$ *is a symplectic vector space;*
- ω *is a closed form, i.e.* $d\omega = 0$.

The 2-form ω together with the above conditions is called symplectic structure.

One can think of a symplectic manifold as a family of symplectic vector spaces $\{(T_xM, \omega_x\}_{x \in M}$ "smoothly parametrized" by points of M. It follows that every Kähler manifold are symplectic manifolds. We can ask ourself under which condition there exists a smooth isometric embedding of a symplectic manifold into a complex projective space. A partial answer to that question can be found in a work of Tischler and Gromov in 1982 and the main result was

Theorem 4.6.4. *(Tischler-Gromov) Let (M, ω) be a compact symplectic manifold such that ω is integral. Then there exists a positive integer N and a symplectic embedding $\psi : (M, \omega) \to (\mathcal{P}_n, \Omega_{FS})$.*

The non compact case is in general false. By weaker hypothesis, it is licit to demand under which restrictive conditions a non compact symplectic manifold whose form is integral admits a symplectic embedding. A possible hypothesis is to consider reasonable manifolds, therefore the problem could be tackled as follows: consider a non compact symplectic manifold (M, ω) where M is also a reasonable manifold and ω is a integral form, then by the result of Andreotti and Vesentini, mentioned in the previous section, there exists a holomorphic embedding

$$\phi: \ M \longrightarrow \mathcal{P}_n.$$

Now, we shall prove that there exists a diffeomorphism $F : M \to M$ such that

$$F^*(\phi^*\Omega_{FS}) = \omega.$$

Observe that, in case M is compact and ω is also Kähler, such a diffeomorphism always exists, this can be shown using the Moser's trick tipically used in the literature to prove the Darboux-Weinstein theorem, which is one of the main results of symplectic geometry [11], [12]. In case M is non compact, then the Moser's trick is not anymore applicable. An approach to solve the problem can be found in [21].

The case of Geometric Quantization

All these frameworks have been so important to theoretical and modern mathematical physics. To give an insight we can see that all of these results have been useful in formulations of physical theories such as *string theory*, *quantum gravity* and *supersymmetric field theories*. In this section we want to briefly point out the case of *geometric quantization*. This latter theory, has been very important in mathematical physics as well as in some aspects of quantum theory [17], [18]. In order to appreciate this, geometric quantization is a very important procedure that permits to generalize the so called *canonical quantization* that can be performed only in flat symplectic manifolds, e.g. the canonical phase space $\mathbb{R}^{2N}$, into non flat symplectic manifolds by letting some peculiar conditions on the curvature 2-form. To have a geometric quantization one needs to consider a particular hermitian line bundle, called *quantum line bundle* on the underline symplectic manifold. Namely a quantum line bundle $L \to X$ is a hermitian complex line bundle above a symplectic manifold (X, ω) whose curvature 2-form Θ is a multiple of $\hbar/2\pi i$ of the symplectic form, i.e.

$$\Theta = -i\hbar\omega$$

Where $\hbar$ is the reduced Planck's constant.
A very important case for the geometric quantization is the Kähler case [19]. A geometric quantization of a Kähler manifold (M, ω) is a pair (L, h) where L is a holomorphic line bundle over M and h is a hermitian structure on L such that its curvature satisfies

$$\Theta = -2\pi i\omega.$$

Not all manifolds admits such a pair, e.g. a 2-sphere of radius $r > 0$ is quantizable if and only if the radius $r = n\hbar/2$ (this is a coherent results with quantum physics electrons). In terms of cohomology classes, a Kähler manifold X admits a geometric quantization if and only if the fundamental form Ω is integral, i.e. X must be a Hodge manifold. Because of the Kodaira's embedding theorem X is embeddable in a finite dimensional complex projective space $\mathcal{P}_n$. For further details the reader may read [19]. The Kähler case has been of crucial importance in particle physics in order to study supersymmetry (SUSY). Loosely speaking SUSY is a proposed type of space-time symmetry that relates two basic classes of elementary particles: *bosons*, which have integer valued spin, and *fermions*, which have half-integer spin. Each particle from one group is associated with a particle from the other, known as its superpartner, the spin of which differs by half-integer. In a theory with perfectly unbroken supersymmetry, each pair of superpartners would share the same mass and internal quantum numbers besides spin.

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