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Correspondences and selections

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One of the most interesting and important problem in topology is the extension problem, i.e. given two topological space X and Y, together with a closed set $A \subset X$, we would like to know whether every continuous function $g : A \to Y$ can be extended to a continuous function f from X into Y, or at least to some open $U \subset X$ which contains A. We may also add some additional conditions on f, for example a natural requirements could be that for every $x \in X$, $f(x)$ must be an element of a preassigned subset of Y. This is frequently the case in the theory of the fiber bundles. This kind of problem, called the *selection problem*, presents a challenge even when A is the null set or a one-point set.

Before Ernest Michael papers's [2], which appeared in Annals of Mathematics, only isolated and special cases of the selection problem have been considered, and excluding some isolated exception, no attempt has been made to obtain results under minimal hypotheses. The question of existence of selection in such a setting turns out to be the question about the unique choice of the solution of the problem under given initial condition. One could say that the key importance of Michael's theory is not so much in providing a comprehensive solution of different selection problems, but rather the immediate inclusion of the obtained results into the general contest of development of topology. However in a remarkable number of cases, results of Michael on solvability of the selection problems, turned out to be the final answers, that is they provided condition on which turned out to be necessary and sufficient.

For a large number of those working in topology, functional analysis, multivalued analysis, approximation theory, convex geometry, control theory, mathematical economics and several other areas find the result of Michael an indispensable tool for their studies. But they may find a detailed investigation on this pears some way discouraging. This theorems together with the same important application were described by Ernest Michael in his paper [2] to whom the present dissertation is based. The purpose of this thesis is to give a detailed proof of the principal results of Michael's papers together with some application, accessible to the casual reader. The thesis is organized in two chapters and two appendixes. In the first section of the first chapter we recall the basic definitions of correspondence and selections, and we state the main results of thee thesis: Theorem 1.1.3 and Theorem 1.1.4. The purpose of this thesis is to give a detailed proof of the principal results of Michael's papers together with some application. The thesis contains two chapters and two appendixes. In the first chapter, after recalling the definition of correspondences and selections, we state and proof the main results of the thesis (Theorem 1.1.3 and Theorem 1.1.4). In Chapter two we describe three applications of the main results, namely Theorem 2.1.2, Theorem 2.2.1 and Theorem 2.2.2. The two appendixes contain the basic results on topology and complex analysis needed for the applications.

Contents

1 Correspondences and selections **6**

1.1 Basic definitions and statements of the main results 6

1.2 Proof of the main results 7

2 Applications **15**

2.1 Bartle – Graves Theorem 15

2.2 Two others important applications 16

A Two open mapping theorems **18**

A.1 Maps between Banach spaces 18

A.2 Holomorphic maps 21

B Basic facts in topology **26**

B.1 Covering and paracompact spaces 26

B.2 Partitions of unity 28

Chapter 1

Correspondences and selections

1.1 Basic definitions and statements of the main results

Let X and Y be two topological space and let ϕ be a map from X to the powerset of Y, denoted by $P(Y)$ or by 2^y , we call ϕ a *correspondence* or multivalued function.

Definition 1.1.1. *Let $\phi : X \to P(Y)$ be a correspondence. A continuous map $f : X \to Y$ is called a selection for ϕ if $f \in \phi(x)$ for every $x \in X$.*

Definition 1.1.2. *A correspondence ϕ from X to Y is said to be lower semicontinuous at the point x if for any open set $V \subset Y$, such that $V \cap \phi(x) \neq \emptyset$, there exist a neighborhood U of x such that: $\phi(u) \cap V \neq \emptyset$ for all u in U.*

The main result we are interested in this thesis are represented by the following two theorems.

Theorem 1.1.3. *If X is a paracompact space, then every lower semi-continuous map ϕ from X to the non empty, closed , convex subsets of a Banach space Y admits a selection.*

Theorem 1.1.4. *Let X be paracompact and zero-dimensional topological space and let Y be a complete metric space. Then every lower semi continuous correspondence ϕ from X to the non-empty, closed subsets of Y admits a selection.*

1.2 Proof of the main results

In order to prove Theorem 1.1.3 we need the following lemma. In the sequel given a subset B of a metric space (Y, ρ) we denote $S_r(B) = \{y \in Y \mid \rho(y, B) < r\}$.

Lemma 1.2.1. *If X is a paracompact space, Y a normal linear space and ϕ a lower semi-continuous function from X to a non-empty and convex subset of Y, then there exist a continuous function f_r, with fixed $r > 0$, from X to Y such that $f(x) \in S_r(\phi(x))$ for every $x \in X$.*

Proof. We consider the two subsets:

$$U_y = \{x \in X \mid y \in S_r(\phi(x))\}$$

and

$$U'_y = \{x \in X \mid \phi(x) \cap S_r(y) \neq \emptyset\}.$$

We will proof that these sets are equal. The inclusion $U'_y \subset U_y$ is immediate.

On the other side chosen $y \in U_y$ it means that $y \in S_r(\phi(x))$ and so the distance ρ(y, $S_r(\phi(x))) < r$ consequently there exist at least one point of $\phi(x)$, that we can call y_0, such that $\rho(y, y_0) < r$ and this implies $\phi(x) \cap S_r(y) \neq \emptyset$. So $y \in U'_y$.

As U'_y and U_y define the same set, we can call it, from now, U_y and use one of the two definition freely. We observe that since ϕ is LSC then U_y= $\{ x \in X|\phi(x) \cap S_r(y) \neq \emptyset \}$ is an open set, indeed the subset $\{ x \in X|\phi(x) \cap U \neq \emptyset\}$ is an open set for every open subset $U \subset Y$ and $S_r(y)$ is obviously an open set of Y.

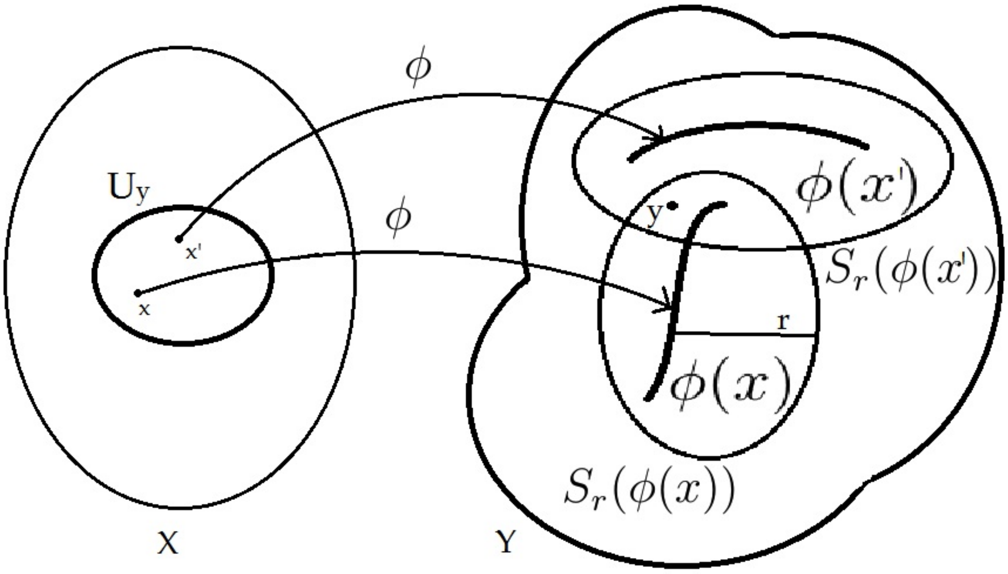

Figure 1.1: Representation of U_y. Here we represent the image of an element x throughout the function ϕ, which is a subset of Y, with a piece of curve in bold.

If we consider the family $\{U_y\}_{y \in Y}$ this is an open covering of X, and since X is paracompact, it follows that there exist an locally finite refinement ,that we will call $\{U_\alpha\}_{\alpha \in A}$. Since we lie in the hypothesis of proposition B.2.1 there exist a family of continuous function $\{p_\alpha\}_{\alpha \in A}$ such that every function goes from X to the unit interval $[0, 1] \subset \mathbb{R}$.

And since $\{U_\alpha\}_{\alpha \in A}$ is a refinement of $\{U_y\}_{y \in Y}$ we can choose for every $y \in Y : y(\alpha) \in V_\alpha \subset U_y(\alpha)$.

The desired continuous function f can be now defined by

$$f(x) = \sum p_\alpha(x) y(\alpha).$$

To see that f is the desired function, we can observe that every x has a neighborhood U intersecting only a finite number of V_α and so f is a finite

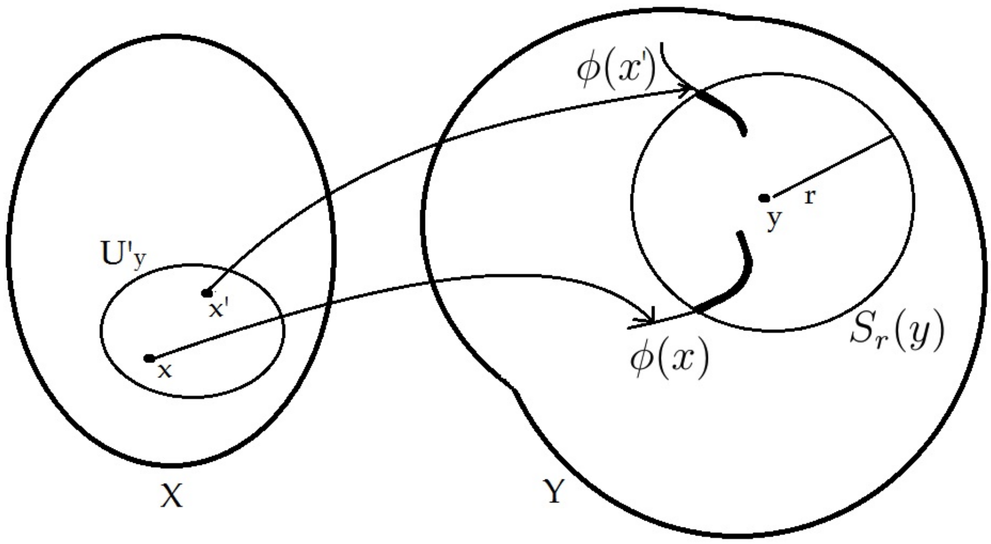

Figure 1.2: Representation of U'_y.

sum of continuous functions. Since f is continuous in a neighborhood of every $x \in X$ then f is continuous on X. Moreover, we notice that for every x, $f(x)$ is a convex, linear combination of finitely many $y(\alpha)$, and all of them lie in the convex set $S_r(\phi(x))$ and therefore also $f(x) \in S_r(\phi(x))$. This complete the proof of the lemma.

□

Proof of Theorem 1.1.3.

To proof Theorem 1.1.3 we will construct a sequence of continuous function $f_i : X \to Y$ such that:

(a) $f_i(x) \in S_{2^{-i+2}}(f_{i-1}(x))$, $i = 2, 3, ...$

(b) $f_i(x) \in S_{2^{-i}}(\phi(x))$, $i = 1, 2, 3, ...$

If we can build a such sequence $\{f_i\}$, it will be a Cauchy sequence since (a) is true and as the neighborhood $S_{2^{-i+2}}(f_{i-1}(x))$ has radius independent from the point x, then the sequence is an uniformly Cauchy sequence. As $\{f_i\}$ is a Cauchy sequence of continuous functions in a Banach space, it converges uniformly to a continuous function f which goes from X to Y and such that $f(x) \in \phi(x)$, by (b).

The function f will be the desired selection for ϕ.

Now we are now going to construct the sequence $\{f_i\}$ for induction. As we lie in the previous lemma's hypothesis, we know that there exist f_1 such that (b) is valid. In other worlds, by Lemma 1.2.1, we know that there exist f such that:

$$f_r : X \to Y \mid f(x) \in S_r(\phi(x)), \quad \forall x \in X$$

where we choose $r = 2^{-1}$.

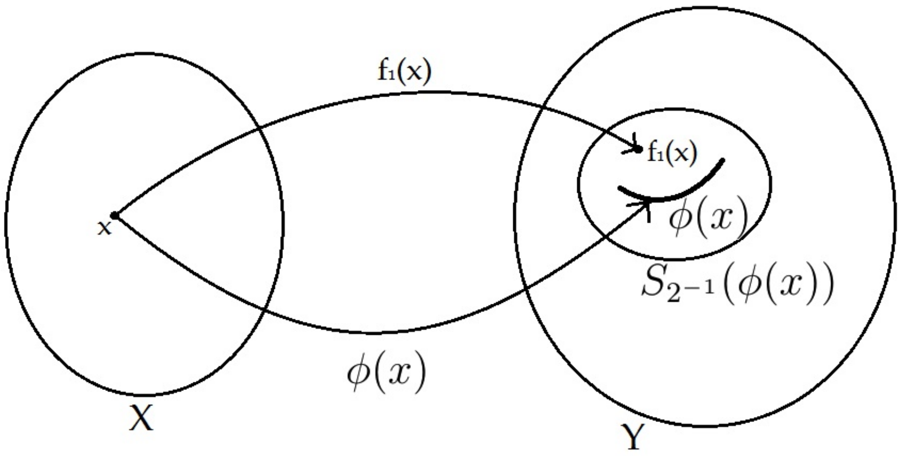

Figure 1.3: Representation of f_1.

We use now the induction hypothesis and we suppose to have already built $f_1, \ldots, f_n$, and we will show that we can build also f_{n+1}. Let us build the function ϕ_{n+1} :

$$\phi_{n+1}(x) := \phi(x) \cap S_{2^{-n}}(f_n(x)).$$

Since $f_n(x)$ lies in a neighborhood of $\phi(x)$ of radius 2^{-n}, we are sure that the mentioned intersection is not empty. Now we want to show that also ϕ_{n+1} is lower semi-continuous. In other worlds we want to show that for

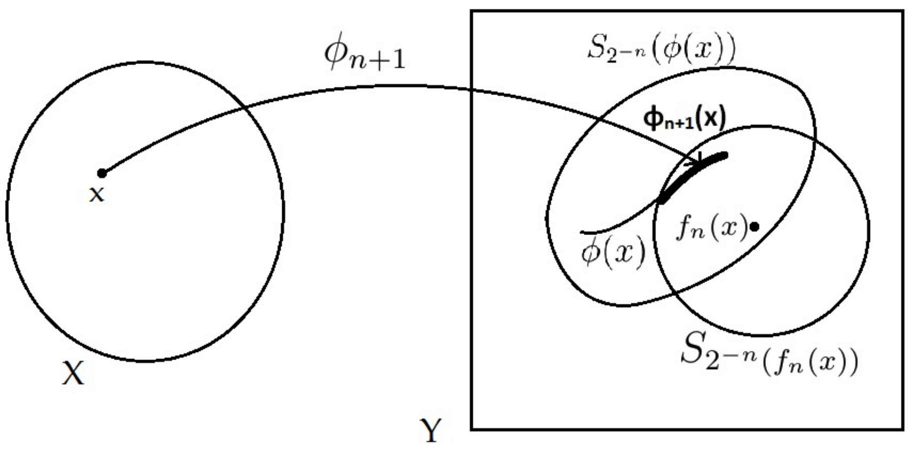

Figure 1.4: Representation of ϕ_{n+1}.

every $U \subset Y$, the set $V = \{x \in X \mid \phi_{n+1}(x) \cap U \neq \emptyset\}$ is open. We will show that V contains a neighborhood of all his points. Let x_0 in V and choose $y_0 \in \phi_{n+1}(x_0)$ i.e. such that $y_0 \in S_\lambda(f_n(x_0))$, where $\lambda < 2^{-n}$. Let us consider now the two sets:

$$W_1 = \{x \in X \mid \phi(x) \cap S_\lambda(f_n(x_o)) \cap U \neq \emptyset\}$$

and

$$W_2 = \{x \in X \mid f_n(x) \in S_{2^{-n}-\lambda}(f_n(x_0))\}\,.$$

We want to show that $W_1 \cap W_2$ is a neighborhood of x_0. Let see that W_1 is non empty. In fact at least $x_0 \in W_1$ since : $\phi(x_0) \cap S_\lambda(f_n(x_0)) \cap U$ is equivalent to say $\phi_{n+1}(x_0) \cap U$ but for all $x \in V$ we know that this intersection is non empty.

Shall we see now that W_1 is also open. Indeed for the lower semi-continuity of ϕ we have that $\{x \in X \mid \phi(x) \cap \widetilde{U} \neq \emptyset\}$ is an open set in X for every open set $\widetilde{U}$, and we observe that $S_\lambda(f_n(x_0)) \cap U$ is an open set in Y. So W_1 is open in X. We are going to show that also W_2 open and nonempty. W_2 is open since the continuity of f_n. Because $S_{s^{-n}-\lambda}(f_n(x_0))$ is an open set of Y and his inverse image, since f_n is continuous, is an open in

X. On the other hand we see that W_2 is also non empty because at least x_0 is in W_2, since $f_n(x_0) \in S_{1/2^n-\lambda}(f_n(x_0))$.

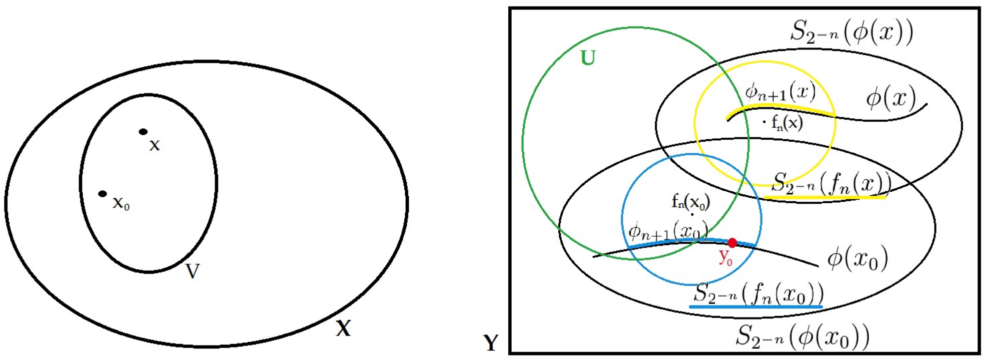

Figure 1.5: Representation of ϕ_{n+1}.

Now that we have shown that W_1 and W_2 are non empty, open and that $x_0 \in W_1 \cap W_2$, we will proof that $W_1 \cap W_2$ is contained in V. In other words we have to proof that

$$[\phi(x) \cap S_\lambda(f_n(x_0))] \cap \left[U \cap S_{1/2^n-\lambda}(f_n(x_0)) \subset \phi_{n+1}(x) \cap U\right].$$

We have that

$$S_\lambda(f_n(x_0)) \cap S_{1/2^n-\lambda}(f_n(x_0)) \subset S_{1/2^n}(f(x_0)).$$

This implies

$$\phi(x) \cap \left[S_\lambda(f_n(x_0)) \cap S_{1/2^n-\lambda}(f_n(x_0))\right] \cap U \subset \phi(x) \cap S_{1/2^n}(f(x_0)) \cap U.$$

By definition of ϕ_{n+1} we have

$$\phi(x) \cap S_{1/2^n}(f(x_0)) \cap U = \phi_{n+1}(x) \cap U.$$

So we have that $W_1 \cap W_2 \subset V$ and as we choose x_0 generically it follows that V is open and we shown that ϕ_{n+1} is lower semicontinuous. By Lemma 1.2.1, we can find a function that we will call f_{n+1}, such that $f_{n+1}(x) \in$

$S_{2^{-n-1}}(\phi_{n+1}(x))$, as we pick $r = 1/2^{n+1}$. By definition of ϕ_{n+1} and by definition of S_n, we see now that if $f_{n+1}(x)$ belongs to $S_{2^{-n-1}}(\phi_{n+1}(x))$ then $f_{n+1}(x) \in S_{2^{-n-1}}(f_n(x))$. But $f_{n+1}(x) \in S_{2^{-n-1}}(f_n(x))$ is condition (a) and $f_{n+1} \in S_{2^{-n-1}}(\phi_{n+1}(x))$ is condition (b).

We have shown how to build f_{n+1} by induction, and so the Cauchy sequence $\{f_i\}$ that is convergent to a selection f for ϕ.

Lemma 1.2.2. *Let X be paracompact and zero-dimensional, Y a metric space and ϕ a lower semicontinuous function from X to the non-empty subset of Y. Then, for every $r > 0$, there exist a continuous function f such that*

$$f(x) \in S_r(\phi(x)) \quad \forall x \in X.$$

Proof. For every $y \in Y$ let us define $U_y := \{x \in X \,|\, y \in S_r(\phi(x))\}$. We have shown in Lemma 1.2.1 since ϕ is lower semi continuous, U_y is open in X. For y varying in Y, $\{U_y\}_{y \in Y}$ is an open covering of X. Since X is paracompact $\{U_y\}_{y \in Y}$ admits an open locally finite refinement $\{W_\alpha\}_{\alpha \in A}$. By proposition B.2.1 in the appendix below, as X is zero-dimensional, so this refinement is also disjoint.

Now consider the function $y : X \to Y$, where $y(W_\alpha) \in Y$ is such that $W_\alpha \subset U_{y(W_\alpha)}$. This function associate to every point of a certain W_α a point $y \in Y$ which is the same that induced the open set U_y, in which W_α is contained. We observe that W_α is contained in $U_{y(W_\alpha)}$ as $\{W_\alpha\}_{\alpha \in A}$ is a refinement for $\{U_y\}_{y \in Y}$. Now we define $f(x) := y(W)$ if $x \in W$. The function f just defined satisfy our requirements. As a matter of facts f is a continuous map. By construction of f, we see that all point from a same W_α are sent in a same $y \in Y$. Then the inverse image of a point $y \in Y$ is or the empty set or union of some W_α. Therefore the inverse image of every set in Y is an open set, since is union of some open sets. Moreover by construction we have that $f(x) \in S_r(\phi(x))$. Since

$$f(x) := y(W_\alpha) \quad if \quad x \in W_\alpha \quad where \quad W_\alpha \subset U_{y(W_\alpha)} := \{x \in X \,|\, y \in S_r(\phi(x))\},$$

and we are done.

$\square$

Proof of Theorem 1.1.4.

In order to prove Theorem 1.1.4 we will need, as in the proof of Theorem 1.1.3, to build an uniformly Cauchy sequence convergent to our desired function f which is a selection for ϕ. By Lemma 1.2.2 we can build a sequence as follow:

(a) $f_i(x) \in S_{2^{-i+2}}(f_{i-1}(x))$ $(i \in 2, 3, ...\ )$

(b) $f_i(x) \in S_{2^{-i}}(\phi(x))$ $(i \in 1, 2, 3, ...\ )$.

Where the existence of the sequence is assured by Lemma 1.2.2, meanwhile the convergence of the saucy sequence is assured by the completeness of Y. The full proof repeat step by step the proof of Theorem 1.1.3 by using Lemma 1.2.2 instead of Lemma 1.2.1.

Chapter 2

Applications

2.1 Bartle – Graves Theorem

This chapter is dedicated to three consequences (Theorem 2.1.2, Theorem 2.2.1 and Theorem 2.2.2) of the results proved in the first chapter. In order to prove them we need the following lemma dealing with an important class of correspondences, namely those arising as preimage of maps.

Proposition 2.1.1. *Let $u : Y \to X$ be surjective and $\phi : X \to P(Y)$ defined as $\phi = u^{-1}(x)$. The correspondence ϕ is lower semi continuous if and only if u is an open function.*

Proof. Let us remember that under this hypothesis, ϕ is lower semicontinuous if for each open $V \subset Y$, the set $W := \{x \in X | u^{-1}(x) \cap V \neq \emptyset\} \subset X$ is an open set.

We first proof, that if u is an open function then ϕ is lower semicontinuous. In order to show that W is an open set, we show first that $W = u(V)$.

Let $x \in u(V)$, then we have that $u^{-1}(x) \cap V \neq \emptyset$, since we know that there exist at least one y such that $x = u(y)$, and this y belongs to this intersection.

Let $x \in W$, hence $u^{-1}(x) \cap V \neq \emptyset$, then there exist $\omega \in W$ such that $u(w) = x$, and therefore: $x \in u(V)$. In this way we just proof that $W = u(V)$

where V is open and u is an open function and it implies that W is also open. To proof that if ϕ is lower semicontinuous then u is an open function, is sufficient remember that $W = u(V)$ is open and this implies that $u(V)$ is open since V is open. $\square$

Theorem 2.1.2. *(Bartle–Graves)*

Let X and Y be two Banach spaces, and let $u : Y \to X$ be a continuous, linear and surjective map. Then there exists a continuous function $f : X \to Y$ which is a continuous right inverse for u.

Proof. We want to apply Theorem 1.1.3 , then we define the correspondence ϕ as follows:

$$\phi : X \to P(Y) \quad such \quad that \quad \phi(x) = u^{-1}(x).$$

Now we want to be sure that we lie in the Theorem 1.1.3 hypothesis. By A.H. Stone theorem[9], every metric space is paracompact and since X is a Banach space then is also a metric space. Moreover, the image of a point in X through ϕ is closed as it is the inverse image through the continuous function u of a point, which is closed. The image through ϕ is also non-empty by the subjectivity of u. The convexity of the image comes from the linearity of u: let v_1 and $v_2 \in u^{-1}(x)$ and $t \in [0, 1]$ then

$$u(tv_1 + (1 - t)v_2) = tx + (1 - t)x = x.$$

The lower semicontinuity of ϕ arise by Proposition 2.1.1. As we lie in theorem 1.1.3 's hypothesis, there exist a continuous function f such that $f(x) \in \phi(x)$. By construction of ϕ, f is a continuous right inverse of u. $\square$

2.2 Two others important applications

Theorem 2.2.1. *Let X be a zero-dimensional and paracompact space, and let Y be a complete metric space. If $u : Y \to X$ is a continuous, open and surjective map, then there exist a continuous $f : X \to Y$ which is a continuous right inverse for u.*

Proof. Our aim is to apply theorem 1.1.4. For first let us define the correspondence ϕ as follows:

$$\phi : X \to P(Y) \quad \text{and} \quad \phi(x) = u^{-1}(x).$$

By proposition 2.1.1, ϕ is lower semi continuous. Moreover the image trough ϕ is non-empty and closed, by construction as we proof in Theorem 2.2.1. Then we lie in theorem 1.1.4's hypothesis and there exist a selection for ϕ, which is also a continuous right inverse for u.

□

Theorem 2.2.2. *Let X be a zero-dimensional, paracompact space; let $\mathbb{C}$ be the complex numbers and let $g : X \to \mathbb{C}$ be continuous. If ω is any polynomial with complex coefficients, then there exist a continuous function $f : X \to \mathbb{C}$ such that $\omega(f(x)) = g(x)$ for every $x \in X$.*

Proof. Let us define a correspondence ϕ as follows:

$$\phi : X \to P(\mathbb{C}) \quad \text{and} \quad \phi(x) = \omega^{-1}(g(x)).$$

If the hypotheses of Theorem 1.1.4 are satisfied, it would exist a continuous function f which is a selection for ϕ and therefore such that

$$f(x) \in \phi(x) \to f(x) = \omega^{-1}(g(x)) \to \omega(f(x)) = g(x)$$

as wished. Hence we have to proof that the image trough ϕ is closed and non empty and that ϕ is lower semicontinuous. The lower-semicontinuity of ϕ follows by Proposition 2.1.1. Moreover the image trough ϕ is closed as it is inverse image trough a continuous function of a closed set. The non emptiness of $\phi(x)$ is assured by the fundamental theorem of algebra. □

Appendix A

Two open mapping theorems

A.1 Maps between Banach spaces

Definition A.1.1. *X is a Banach space if it is a complete, normed vector space. In other words X is a Banach space if it is a normed vector space in which every Cauchy sequence is convergent to an element of X.*

Definition A.1.2. *Let X be a topological space, we say that $A \subset X$ is a nowhere dense set if the closure of A has empty interior.*

Theorem A.1.3. *Let X and Y be two Banach spaces, and let A be a linear, continuous, surjective map from X in to Y. Then A is an open map.*

In order to proof Theorem A.1.3 we first need a lemma and its corollary.

Lemma A.1.4. *Let X be a complete metric space and that $A_1, A_2, A_3, \ldots$ is a sequence of dense open sets. Then $\cap_{n\in\mathbb{N}} A_n$ is nonempty.*

Proof. In this proof we will frequently use the fact that an open ball $B_\epsilon(x)$ contains the closure of another open ball, for example $B_{\epsilon/2}(x)$.

Let $B_{\epsilon 1}(x_1)$ be an open ball such that $\overline{B_{\epsilon 1}(x_1)}$ is contained in A_1. Since A_2 is open and dense, the intersection between $\overline{B_{\epsilon 1}(x_1)}$ and A_2 is still an open set. Therefore there exists an open ball $B_{\epsilon 2}(x_2)$ contained in $\overline{B_{\epsilon 1}(x_1)} \cap A_2$. In this way, we can construct a sequence of open ball, in which every open ball has its closure contained in the previous one. The sequence $B_{\epsilon 1}(x_1), B_{\epsilon 2}(x_2), ...$ can be choose such that ϵ_n vanish when n approaches to zero (for example, it is sufficient to choose $\epsilon_{j+1} < \epsilon_{j/2}$). In a such case the sequences $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is a complete metric space, this sequence converges to a certain $x \in X$. Note that $B_{\epsilon n}(x_n)$ contains all x_j with $j \geq n$, then x is contained in $\overline{B_{\epsilon n}(x_n)}$. Thus $x \in A_n$. Therefore $x \in \cap_{n\in\mathbb{N}} A_n$, then the intersection is nonempty.

$\square$

Corollary A.1.5. *Let X be a complete metric space and let $F_1, F_2, F_3, ...$ be closed and nowhere dense sets. Then $\cup_{n\in\mathbb{N}} F_n \neq X$.*

Proof. Set $A_j = X \setminus F_j$. By Lemma A.1.4:

$$\cap_{n\in\mathbb{N}} A_n \neq \emptyset \Rightarrow \cap_{n\in\mathbb{N}}(X \setminus F_j) \neq \emptyset \Rightarrow X \setminus \cup_{n\in\mathbb{N}} F_n \neq \emptyset \Rightarrow X \neq \cup_{n\in\mathbb{N}} F_n,$$

as wished. $\square$

Proof of Theorem A.1.3.
We want to show that $A(U)$ is open in Y when U is open in X. By the linearity of A it is sufficient to prove that A send the open unitary ball centered in $0 \in X$ in one neighborhood of $0 \in Y$.

Let $U = B_1(0) \subset X$, by the surjectivity of the map A we have:

$$Y = A(X) = A(\cup_{k=1}^{\infty} kU) = \cup_{k=1}^{\infty} A(kU).$$

By Lemma A.1.4 and its corollary, we know that a Banach space cannot be union of a numerable number of nowhere dense sets. As Y is a Banach space we deduce that there exist a $k > 0$ such that $\overline{A(kU)}$ has nonempty interior. Therefore there exist $c \in \overline{A(kU)}$ and $r > 0$ such that $B(c, r) \subset \overline{A(kU)}$.

Choose $v \in V := B_1(0) \subset Y$, then both c and $c + rv$ are interior point of $A(kU)$. As the translation is an open map, $rv = c + rv - c$ is an internal point of $\overline{A(kU)} - \overline{A(kU)} := \{z = x - y \,|\, x, y \in \overline{A(kU)}\}$. In particular rv is an interior point of $\overline{A(2kU)}$. As $rv \in \overline{A(2kU)}$ then $v \in \overline{A(\frac{2k}{r}U)}$, by linearity of A. Thus, for any $\epsilon > 0$, $B_\epsilon(v) \cap A\left(\frac{2k}{r}U\right) \neq \emptyset$. Therefore for any $v \in \overline{V}$ and

$\epsilon > 0$ there exists x such that $\|x\|_X < \frac{2k}{r}$ and $\|v - A(x)\|_Y < \epsilon$. It follow, by the linearity of A, that for any $y \in Y$ and any $\epsilon > 0$, there is an $x \in X$ such that

$$\|x\|_X < \delta^{-1}\|y\|_Y \quad \text{and} \quad \|y - A(x)\|_Y < \epsilon, \tag{A.1}$$

where $\delta = \frac{r}{2k}$.

Fix $y_1 \in \delta U$ and $\epsilon = \frac{\delta}{2}$. By (A.1) there exists x_1 such that $\|x_1\| < \delta^{-1}\|y\| < 1$ and $\|y - A(x_1)\| < \frac{\delta}{2}$. We define by induction the sequence $\{x_j\}$ as follow. Assume $\|x_{n-1}\| < 2^{-(n-2)}$ and $y_n = y - A(x_1 + \cdots + x_{n-1})$ with $\|y_n\| < \delta 2^{-(n-1)}$. By (A.1), choosing $y = y_n$ and $\epsilon = \delta 2^n$, there exist x_n such that

$$\|x_n\| < 2^{-(n-1)} \quad \text{and} \quad \|y - A(x_1 + ... + x_n)\| < \delta 2^{-n}. \tag{A.2}$$

Consider the sequence $\{s_n\}$ with $s_n = x_1 + \cdots + x_n$. $\{s_n\}$ is a Cauchy sequence. Assume $n > m$, we have

$$\|s_n - s_m\| = \|x_{m+1} + ... + x_n\| \leq \|x_{m+1}\| + ... + \|x_n\|$$
$$< 2^{-m} + ... + 2^{-(n-1)} = 2^{-(n-1)}\left(1 + \frac{1}{2} + \cdots + 2^{n-m-1}\right) \leq 2^{-n+2}.$$

So $\{s_n\}$ is a Cauchy sequence, since X is complete, s_n converge to point $x \in X$, moreover by (A.2) the sequence $A(s_n)$ converge to a point $y \in Y$. By the continuity $A(x) = y$. Note that

$$\|x\| = \lim_{n\to\infty} \|s_n\| \leq \sum_{n=1}^{\infty} \|x_n\| = 2.$$

Hence $x \in 2U$ and $y \in A(2U)$. By the arbitrariness of y we see that $\delta V \subset A(2U)$, equivalently $\frac{\delta}{2} V \subset A(U)$. We proved that $A(U)$ is a neighborhood of the origin in Y.

Now let U be an open set in X, $y_0 \in A(U) \subset Y$ and x_0 such that $y_0 = A(x_0)$. Since U is open there exists $B_\epsilon(x_0) \subset U$. We consider a translation of $B_\epsilon(x_0)$ in the origin, then $A(B_\epsilon(x_0) - x_0)$ is a neighborhood of 0 in Y. Therefore $A(B_\epsilon(x_0) - x_0) + y_0$ is a neighborhood of y_0 in Y. Our aim is to show that $A(B_\epsilon(x_0) - x_0) + y_0$ is contained in $A(U)$. In this way since y_0 is arbitrary, every point in $A(U)$ is an internal point and then $A(U)$ is open. By linearity of A:

$$A(B_\epsilon(x_0) - x_0) + y_0 = A(B_\epsilon(x_0)) - A(x_0) + y_0 = A(B_\epsilon(x_0)).$$

But since $B_\epsilon(x_0) \subset U$ then $A(B_\epsilon(x_0)) \subset A(U)$, as wished.

A.2 Holomorphic maps

Definition A.2.1. *A function f is meromorphic on an open subset D of the complex plane, if f is a function that is holomorphic on all D except a set of isolated points, which are poles for the function.*

Definition A.2.2. *An homotopy map between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous map $H : X \times [0, 1] \to Y$ from the product of the space X with the unit interval $[0, 1]$ to Y such that, if $x \in X$ then:*

$$H(x, 0) = f(x) \quad \text{and} \quad H(x, 1) = g(x).$$

In other words two continuous map from one topological space to another are called homotopic, if one can be "continuously deformed" into the other, and such deformation being called a homotopy between the two functions.

Definition A.2.3. *A topological space X is contractible if the identity map on X is null-homotopic, i.e. if it is homotopic to some constant map.*

Theorem A.2.4. *Any non constant holomorphic function on an open and connected set is an open map.*

In order to proof the theorem we need some lemma.

Lemma A.2.5. *Let $f(z)$ be a meromorphic function on an open set Ω in the complex plane and that C is a closed, simple, counter-clockwise oriented curve in Ω , and f has no zeros or poles on C. Then*

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i \,(N - P)\,,$$

where N and P denote respectively the number of zeros and poles of $f(z)$ inside the domain delineated by the curve C, with each zero and pole counted as many times as its multiplicity, respectively order.

Proof. Let z_N be a zero of f, with multiplicity k. We can write:

$$f(z) = (z - z_N)^k g(z),$$

where $g(z_N) \neq 0$. We have

$$f'(z) = k(z - z_N)^{k-1} g(z) + (z - z_N)^k g'(z)$$

and then:

$$\frac{f'(z)}{f(z)} = \frac{k}{z - z_N} + \frac{g'(z)}{g(z)}.$$

Since $g(z_N) \neq 0$, it follows that $\frac{g'(z)}{g(z)}$ has no singularities at z_N, and ths is analitic at z_N, which implies that the residue of $\frac{f'(z)}{f(z)}$ at z_N is k. Now let z_P be a pole for f. We have

$$f'(z) = -m(z - z_P)^{-m-1} h(z) + (z - z_P)^{-m} h'(z),$$

and

$$\frac{f'(z)}{f(z)} = \frac{-m}{z - z_P} + \frac{h'(z)}{h(z)}.$$

It follows that $\frac{h'(z)}{h(z)}$ has no singularities at z_P since $h(z_P) \neq 0$ and thus it is analytic at z_P. We find that the residue of $\frac{f'(z)}{f(z)}$ at z_P is $-m$. Putting these together, each zero z_N of multiplicity k of f generates a simple pole for $\frac{f'(z)}{f(z)}$ with the residue being k, and each pole z_P of order m of f generates a simple pole for $\frac{f'(z)}{f(z)}$ with the residue being $-m$. By Residue Theorem [7] we have that the integral about C is the product of $2\pi i$ and the sum of the residues. As N is the number of zeroes counted with their multiplicity and P the number of the poles counted with their orders we have our result.

$\square$

Lemma A.2.6. *Let $K \subset \mathbb{C}$ be a bounded region with continuous boundary. Two holomorphic functions f and g have the same number of roots in K, if the strict inequality:*

$$|f(z) - g(z)| < |g(z)| \quad (z \in \partial K)$$

holds on the boundary ∂K.

Proof. Note that by hypotheses both f and g do not have any roots on the boundary ∂K and that $\frac{f(z)}{g(z)}$ is not a negative real number for $z \in \partial K$. Thus the homotopy map:

$$I(t) := \frac{1}{2\pi i} \int_{\partial K} \frac{F'(z)}{F(z) + t} dz,$$

where $F(z) := \frac{f(z)}{g(z)}$, is well defined for $t \geq 0$. Clearly $I(t)$ tends to zero as t increases indefinitely. As $I(t)$ is continuous and integer valued, it folows that $I(0) = 0$. By Lemma A.2.5:

$$0 = \frac{1}{2\pi i} \int_{\partial K} \frac{F'(z)}{F(z)} dz = N_F(K) - P_F(K),$$

where $N_F(K)$ is the number of zeroes of F inside K and $P_F(K)$ is the number of poles inside K counted with they multiplicity and order respectively. Hence $N_F(K) = P_F(K)$. As F is the ratio of two holomorphic functions f and g inside K, the zeroes are those of f and the poles are the zeroes of g, that is :

$$0 = N_F(K) - P_F(K) = N_f(K) - N_g(K),$$

as wished.

□

Observation A.2.7. *Lemma A.2.6 can be used also to give a short proof of the Fundamental Theorem of Algebra. Let p be the polynomial:*

$$p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$$

and let R be positive and such that:

$$|a_0+a_1z+a_2z^2+...+a_{n-1}z^{n-1}| \leq \sum_{j=0}^{n-1}|a_j|R^{n-1} < |a_n|r^n = |a_nz^n| \quad for \quad |z| = R.$$

since a_nz^n has n zeros inside the disk $|z| < R$ (as $R > 0$), it follows from Lemma A.2.6 that p has the same number of zeros inside the disk.

Proof of Theorem A.2.4.

Let $U \subset \mathbb{C}$ be an open set and $f : U \subset \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic map. We want to show that every point in $f(U)$ is an interior point in $f(U)$. Let w_0 be an arbitrary point in $f(U)$. Then there exists a point z_0 in U such that $w_0 = f(z_0)$. Since U is an open set we can find an open ball $B_d(z_0)$ such that its closure is fully contained in U. Consider the function $g(z) = f(z) - w_0$. Note that z_0 is a root for $g(z)$. By definition, $g(z)$ is also non constant and holomorphic. Then the roots of $g(z)$ are isolated. Indeed if v_0 is a root of order k for g, we have $g(z) = (z - v_0)^k h(z)$, where $h(z)$ does not vanish in v_0. Therefore $|h(v_0)| \neq 0$ and then there exists an ϵ such that $|h(v_0)| \neq 0$ for z in $B_\epsilon(v_0)$. Let $z \in B_\epsilon(v_0)$ and $z \neq z_0$, then $g(z) = (z - z_0)^k h(0) \neq 0$. Thus z_0 is an isolated root. So, by further decreasing the radius of the image disk d, we can assume that $g(z)$ has only a single root in B (although this single root may have multiplicity greater than 1), in fact it is sufficient to pick $d = \epsilon/2$ and we have $B_d \subset B_\epsilon(z_0)$. The boundary of B_d is a circle and hence a compact set, and $|g(z)|$ is a continuous function, so $g(z)$ reach a minimum in the boundary of B_d. Let e be this minimum $e = \min_{\partial B_d}(|g(z)|)$. Now let $D := B_e(w_0)$ be an open disk around w_0. By Lemma A.2.6 the function $g(z) = f(z) - w_0$ has the same number of roots (counted with multiplicity) in B as $f(z) - w$ for any $w \in D$. Indeed we can pick (by abuse of notation) as $f(z)$ the function $f(z) - w$ and as $g(z)$ the function $g(z) = f(z) - w_0$. We have in fact

$$|f(z) - g(z)| = |w - w_0| < |g(z)|$$

on the boundary of B_d as $e := \min_{\partial B_d}(|g(z)|)$ and $w \in D$. Thus for every

$w \in D$ there exists at least one z_1 in B_d such that $f(z_1) = w$. This means that the disk D is contained in $f(B)$ which is contained in $f(U)$, as wished.

Appendix B

Basic facts in topology

B.1 Covering and paracompact spaces

In the following definitions X will denote a topological space.

Definition B.1.1. *A collection* $\{V_\alpha\}_{\alpha\in A}$ *of subsets is said to be a covering of* X *if their union equal* X*.*

Definition B.1.2. *We say that a family* $\{V_\alpha\}_{\alpha\in A}$ *is an open covering of* X *if it is a covering in which every member is an open set.*

Definition B.1.3. *If* $\{V_\alpha\}_{\alpha\in A}$ *and* $\{W_\beta\}_{\beta\in B}$ *are covering of* X*, then* $\{V_\alpha\}_{\alpha\in A}$ *is a refinement of* $\{W_\beta\}_{\beta\in B}$ *if every* V_α *is a subset of some* W_β *.*

Definition B.1.4. *A covering* $\{V_\alpha\}_{\alpha\in A}$ *of* X *is said to be locally finite if every point of* X *has a neighborhood which intersects only finitely many* V_α *of the covering.*

Definition B.1.5. *A covering* $\{V_\alpha\}_{\alpha\in A}$ *of* X *is said to be an open disjoint covering if for every of* V_α *of the family we have that* $V_i \cap V_j = \emptyset$ *if* $i \neq j$*, in other words, if all the sets in the family are two by two disjoint.*

Definition B.1.6. X *is paracompact if every open finite cover has an open refinement that is locally finite.*

Definition B.1.7. *A vector space X is said to be convex if the segment joining each of its points is contained in the space.*

Definition B.1.8. *X is a $T1$ space if for every x and y belonging to X there exist two open neighborhood U and V respectively of x and y such that $y \notin U$ and $x \notin V$.*

Definition B.1.9. *X is zero-dimensional if every open cover of the space admits a finite open refinement such that any point in the space is contained in exactly one set of the refinement.*

Definition B.1.10. *X is a normal space if for every two closed and disjoint subset of X, there exists two disjoint open neighborhoods which separate the two closed.*

Proposition B.1.11. *The topological space X is a normal space if and only if for every closed subset $F \subset X$ and every open set U, which contains F, there exist an open set V such that $F \subset V$ and $\overline{V} \subset U$.*

Proof. Given a closed set $F \subset X$ and an open set U which contains F, we consider the two closed set F and U^c the complement U. By the normality of X we can find two open and disjoint set V and V' such that $F \subset V$ and $U^c \subset V'$. As V and V' are disjoint, and as V and V' are open set, also $\overline{V}$ and V' are disjoint. By construction U^c is contained in V' so $\overline{V}$ and U^c are also disjoint.

On the other hand, given two disjoint and closed sets F and F', we have to find two open and disjoint sets U and U' such that $F \subset U$ and $F' \subset U'$, only by using that given a closed set and an open which contains it, then there exist an open set with the closure contained in the open, and which contains the closed. Let we consider the set $(F')^c$, the complement of F' in X, which is an open set. As F and F' are disjoint, F is contained in $(F')^c$, then there exist an open set V such that contains F and with its closure contained in $(F')^c$. Now let we consider the open set $(\overline{V})^c$, which contains F', then there exist some open set W which contains F', but with its closure contained in $(\overline{V})^c$. The two sets V and W separate F and F', as wished. $\square$

B.2 Partitions of unity

Proposition B.2.1. *If $\{V_\alpha\}_{\alpha\in A}$ is a locally finite, open covering of a normal space X, then there exist a family $\{p_\alpha\}_{\alpha\in A}$ of continuous function from X to the closed unit interval with the following properties:*

(a) p_α vanishes outside V_α

(b) $\sum_{\alpha\in A} p_\alpha(x) = 1$ for every $x \in X$

In order to prove the proposition we need some lemma.

Lemma B.2.2. *Let $\{V_\alpha\}_{\alpha\in A}$ be a locally finite open covering of a normal space X, there exist an open covering $\{W_\alpha\}_{\alpha\in A}$ of X such that $\overline{W}_\alpha \subset V_\alpha$.*

Proof. Let $\mathbb{A}$ be the collection of all open sets A such that $\overline{A}$ is contained in $V_\beta \in \{V_\alpha\}_{\alpha\in A}$, for some β in A. As X is normal, $\mathbb{A}$ is a covering of X. Indeed taken $p \in X$, there exist V_α such that $p \in V_\alpha$. Now let us consider the two closed subset $\{p, X \smallsetminus V_\alpha\}$. Since X is a normal space, then there exists two open sets A and B such that $A \cap B = \emptyset$, $p \in A$ and $X \smallsetminus V_\alpha \subset B$. By construction $\overline{A}$ is contained in V_α. By the paracompactness of X, we can choose a refinement $\mathbb{B} = \{B_\alpha\}_{\alpha\in J}$ of $\mathbb{A}$. Consider f defined as follows:

$$f : J \to A$$

such that

$$f(j) = \alpha \Rightarrow \overline{B}_j \subset V_\alpha$$

.

Let us define $W_\alpha = \cup_{j|f(j)=\alpha} B_j$. Note that W_α satisfies our condition. First, $\overline{W}_\alpha \subset V_\alpha$ by construction. Second, it is locally finite: pick a point $p \in M$, then there exist a neighborhood U of p such that U intersect only a finite number of element of $\mathbb{B}$, we call them $\{B_{\alpha_1}, ..., B_{\alpha_n}\}$. Then, by construction U intersect exactly the element $\{W_{f(\alpha_1)}, ..., W_{f(\alpha_n)}\}$ of $\{W_\alpha\}_{\alpha\in A}$. $\square$

Lemma B.2.3. *If X is a normal space, then for every two closed and disjoint sets A and B of X there exists a continuous function $f : X \to [0,1]$, such that*

$$f(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \in B \end{cases} \tag{B.1}$$

Proof. Let $\mathbb{D}$ be the set of the dyadic rational fraction between 0 and 1, i. e. all the fraction of the form $\frac{k}{2^n}$ with $k \in [0, 2^n]$. For each $r \in \mathbb{D}$ we will build a sequence of sets $U(r)$ such that:

1. $B \subset U(r)\,, \quad \forall\ r \in \mathbb{D}$
2. $U(r) \cap A = \emptyset\,, \quad \forall\ r \in \mathbb{D}$
3. if $\ r < s \Rightarrow \overline{U}(r) \subset U(s)\,, \quad \forall\ r\,,\, s\, \in \mathbb{D}$.

First we define $U(0)$ as that open set which has its closure contained in A^c, contains B and which existence is assured by the normality of X and Proposition B.1.11, let we call this set V, then $U(0) := V$. Moreover we define $U(1) := A^c$. We suppose, as induction hypothesis, that we have already build the sequence of $U(r)$ until $r = \frac{k}{2^n}$ where k is odd. We observe that as k is odd, $k-1$ is even and then $\frac{k-1}{2^n}$ can be simplify, becoming of the form $\frac{m}{2^{n-1}}$ where m is odd. Then we assured the existence of the set $U(\frac{k-1}{2^n})$ with the inductive hypothesis. The same holds for the set $U(\frac{k+1}{2^n})$. By the inductive hypothesis and by (3) we have: $\overline{U(\frac{k-1}{2^n})}$ is contained in $U(\frac{k+1}{2^n})$. On the other hand, by normality of X there exist an open set, that we can call $U(\frac{k}{2^n})$, such that :

$$\overline{U\left(\frac{k-1}{2^n}\right)} \subset U\left(\frac{k}{2^n}\right) \subset \overline{U\left(\frac{k}{2^n}\right)} \subset U\left(\frac{k+1}{2^n}\right).$$

Then we proof the existence of a such sequence. Now we will finally able to define a continuous function f as wished:

$$f(x) = \begin{cases} 1 \text{ if } x \in A \\ \inf\{r \,|\, x \in U(r)\} \text{ if } x \notin A \end{cases} \tag{B.2}$$

Let we observe that if x belongs to $U(0)$ then $f(x) = 0$. We now have to show that a such f is continuous, and we will use the definition of continuity in a point. Let t, s, r be in $\mathbb{D}$ with $r - \epsilon < s < r < t < r + \epsilon$. The set $U(t) \smallsetminus \overline{U(s)}$ is an open set and it is such that:

$$r - \epsilon < s \leq f(U(t) \smallsetminus \overline{U(s)}) \leq t < r + \epsilon$$

and then f is continuous in the point x. By arbitrarily of x it follows that f is continuous. □

Remark B.2.4. *The previous lemma can be proof also by using two generical sets A and B instead of two closed sets. Indeed it is sufficient require that the two closures of the sets has empty intersection.*

Proof. of Proposition B.2.1. By Lemma B.2.2 there exists a covering $\{W_\alpha\}_{\alpha \in A}$ such that $\overline{W}_\alpha \subset V_\alpha$ for every $x \in X$. By the normality of X and by to lemma B.2.3 we can find a continuous function q_α such that:

$$(a) \quad q_\alpha = 0 \quad x \in X \smallsetminus V_\alpha$$

.

$$(b) \quad q_\alpha = 1 \quad x \in \overline{W}_\alpha$$

Now define $p_\alpha(x) = \frac{q_\alpha(x)}{\sum_{\beta \in A} q_\beta(x)}$. The family $\{p_\alpha\}_{\alpha \in A}$ satisfies by construction all our requirements.

□

Proposition B.2.5. *Let X be a normal space, then the following sentences are equivalent:*

(1) *Every open finite covering of X admits a disjoint open finite refinement. (i.e. X is zero dimensional.)*

(2) *every local finite open covering of X admits an open disjoint refinement.*

(3) *if $V \subset X$ is an open set and $A \subset V$ is a closed set , then there exists a set W which is open and closed such that : $A \subset W \subset V \subset X$.*

Proof. (1) $\Rightarrow$ (3)

Let consider as a open finite covering of X the family: $\{X \setminus A, V\}$. By (1) we know that there exists an open disjoint finite refinement of this covering, lets call it $U := \{U_i\}_{i\in I}$. Lets define $W := \bigcup_{j\in J} U_j$ for all $U_i \subset V$. By definition of W, we have that $W \subset V$. Furthermore $A \subset W$. As a matter of fact taken $x \in A$, since U is also a covering of X, there exist U_i such that $x \in U_i$. As U is also a refinement of the covering: $\{X \setminus A, V\}$, we have that every element of U must be contained or in $X \setminus A$ or in V. But $U_i \in \bigcup_{j\in J} U_j$ cannot be all contained in $X \setminus A$, because $x \in A$. Therefore $U_i \subset V$ implies, by construction of W, $U_i \subset W$. So we have just shown that if x is in A then it is contained in W.

Now we will show that W is an open and closed subset of X. W is clearly an open set as it is union of some open sets U_i. To proof that W is also closed we will show that every U_i is also closed, showing that every complement of every U_i is open. Fixed U_i , let x be in $X \setminus U_i$. As U is a covering of X, then there exists U_j such that $x \in U_j$ with $U_j \cap U_i = \emptyset$. In fact there exist an open neighborhood of x contained in $X \setminus U_i$. Since x is arbitrary, we proof that $X \setminus U_i$ is open. So $W := \bigcup_{j\in J} U_j$ is also closed.

(3) $\Rightarrow$ (2)

Let U_α be an open covering of X. By to the Lemma B.2.2 we know that there exist an open covering $\{V_\alpha\}_{\alpha\in A}$ of X such that $\overline{V_\alpha} \subset U_\alpha$ for each $\alpha \in A$. As we know by (3) we can find for each $\alpha \in A$ also a set W_α such that $\overline{V_\alpha} \subset W_\alpha \subset U_\alpha$. Using $Well-ordering\ theorem$ we can provide A of a well-order, so we can define $R_\alpha := W_\alpha \setminus \cup_{\beta<\alpha} W_\beta$. We observe that, for every $\alpha \in A$, $\cup_{\beta<\alpha} W_\beta$ is the union of a locally finite collection of closed sets, therefore it is closed. In other words we want to show that $\cup_{\beta<\alpha} W_\beta$ is a closed set, so so it remains to verify that its complement contains an open neighborhood of every points, and therefore is open. Let $p \notin \cup_{\beta<\alpha} W_\beta$ then there exists at least one open ball $B_r(p)$ intersecting only a finite number of W_j, in particular, we can find one such that intersect only a finite number of W_β with $\beta < \alpha$ and therefore such that $B(p) \setminus \cup_{\beta<\alpha} W_\beta$ is still an open set, as $\cup_{\beta<\alpha} W_\beta$ is a finite number of closed set. Then by generality of p $\cup_{\beta<\alpha} W_\beta$ is open. Therefore R_α is open. Moreover we observe that the family $\{R_\alpha\}_{\{\alpha\in A\}}$ is disjoint. As a matter of fact let R_α and $R_\beta \in \{R_\alpha\}$ with $\beta < \alpha$. By definition $R_\alpha = W_\alpha \setminus \cup_{\gamma<\alpha} W_\gamma$ and since $\beta < \alpha$ in R_α there are no point of W_β. But as we know $R_\beta \subset W_\beta$. Then $R_\alpha \cap R_\beta = \emptyset$.

R_α is also a refinement for U_α . As $R_\alpha \subset W_\alpha \subset U_\alpha$ we know that every R_α in contained in one U_α. Now we will proof that R_α is also a covering for X. Let $p \in X$ we know that there exists a ball $B(p)$ such that $B(p) \cap W_\beta \neq \emptyset$ with $\beta \in \{1, ..., n\}$. More precisely p is just in a finite number of W_β. Where the W_β are a covering of X and therefore p is in at least one of them. Let $W_{\overline{\beta}}$ be such that $\overline{\beta}$ is the lowest in $\{1, ..., n\}$ for which $p \in W_\beta$; $(\beta \in \{1, ..., n\})$. Then $R_{\overline{\beta}} = W_{\overline{\beta}} \setminus \cup_{\beta<\overline{\beta}} W_\beta$ contains p. As by construction none of W_β with $\beta < \overline{\beta}$ does contains p meanwhile $W_{\overline{\beta}}$ does. By generality of p R_β is a covering.

(2) $\Rightarrow$ (1) immediate. $\square$

Bibliography

[1] C. Kosniowski, *Introduzione alla topologia algebrica*, Zanichelli (1988).

[2] E. Michael *Continuous selections. I*, Ann. of Math. (2) 63 (1956), 361–382.

[3] E. Michael *Continuous selections. II*, Ann. of Math. (2) 64 (1956), 562–580.

[4] E. Michael *Continuous selections. III*, Ann. of Math. (2) 65 (1957), 375–390.

[5] E. Michael *Selected Selection Theorems*, Amer. Math. Monthly 63 (1956), no. 4, 233–238.

[6] J. Milnor, *Topology from the differentiable viewpoint*, Princeton University Press, Princeton, NJ, 1997

[7] C.D. Pagani, S.Salsa, *Analisi matematica, Volume 1*, Masson S.p.A, Milano.

[8] A. Sen, *Fundamental Theorem of Algebra – Yet Another Proof*, Amer. Math. Monthly 107 (2000), no. 9, 842–843.

[9] E. Sernesi, *Geometria 2*, Bollati Boringhieri (1994).

[10] A. Loi, *Topologia generale*,