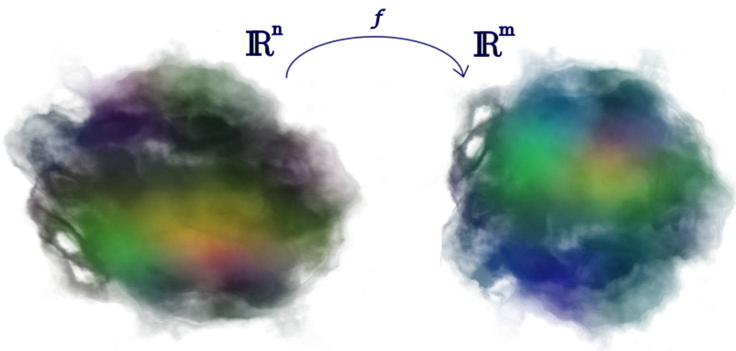
University of Cagliari
Mathematics department

A topological proof of the Invariance of dimension Theorem



Thesis supervisor
Prof. Andrea Loi

Thesis of
Andrea F.Schirru

26 February 2015

A.Y. 2013/2014

Contents

Introduction

In this thesis we are going to speak about one important result of algebraic topology, the *Invariance of dimension theorem*:
There is no continuous one-to-one map $f : \mathbb{R}^n \to \mathbb{R}^m$*, for* $m < n$.
In the first chapter we start recalling all preliminary notions that we need, before proceeding. Then we dwell on another important brick, necessary to build the structure of our aim: *Brouwer fixed point theorem.*
Going on with the statement, we come to the third chapter, dedicated to our target.
There we propose three ways to solve our problem: Brouwer's original proof, a standard proof (with the help of *Homology theory)* and finally a topological demonstration made by Wladislaw Kulpa, that is easily deduced as a corollary of another important principle, the *Invariance of domain theorem.*
It is important to underline that this work is inspired by a publication of Terence Tao[1].
It will be not hard to see that instead of the first two demonstrations, the last one, that gives the name at this work, is the easiest and maybe the most brilliant among all that we showed, because is purely topological.

[1] [Tao]

Chapter 1

Preliminary notions

Before proceeding to the *Invariance of dimension theorem*, we recall the basic notions that we are going to use[1]. We start with

Definition 1.1 (Topology and open set).
Let X be a set and let $\mathcal{U}$ be a collection of subsets of X satisfying:

(i) $\emptyset \in \mathcal{U}, X \in \mathcal{U}$*;*

(ii) the intersection of two members of $\mathcal{U}$ *is in* $\mathcal{U}$*;*

(iii) the union of any number of members of $\mathcal{U}$ *is in* $\mathcal{U}$.

*Such a collection of subsets of X is called a **topology** for X.*
*The set X together with $\mathcal{U}$ is called a **topological space** and is denoted by $(X, \mathcal{U})$ which is often abbreviated to T or just X.*
*The members $U \in \mathcal{U}$ are called the **open sets** of T.*
*Elements of X are called **points** of T.*

Open sets are really important in Topology, because on them are built almost all the notions. From this definition, we can consider now a family of open sets.

[1] the references of this chapter are [Kosniowsky],[Morris],[Weisstein], [Joshi],[Sernesi], [Do Carmo],[Kosn],[Kim], [Kreyszig], [Estep], [Cohen].

Definition 1.2 (**Open covering**).
*Let X be a topological space, and $E \subseteq X$. We say that the family of open sets $\{G_\alpha\}_{\alpha \in A}$ is an **open covering** of E if and only if*

$$E \subseteq \bigcup_{\alpha} G_\alpha.$$

And as its direct consequence, we have

Definition 1.3 (**Compact set**).
*A subset S of a topological space X is **compact**, if for every open covering of S there exists a finite subcover of S.*

We remark that from an open set we define **closed set** too, as its complement. So we can look at

Definition 1.4 (**Normal space**).
*A **normal space** is a topological space in which, for any two disjoint closed sets C,D, there are two disjoint open sets $\mathcal{U}$ and $\mathcal{V}$ such that $C \subseteq \mathcal{U}$ and $D \subseteq \mathcal{V}$.*

There could exist sets there are open and closed at the same time, the **closed-open** sets.
If in a topological space X, the only closed-open subsets of X are X and $\emptyset$, then we say that the topological space X is called **connected**.
Another important concept is

Definition 1.5 (**Continuous map**).
*Let X, Y be topological spaces. Then a map $f : X \to Y$ is said to be **continuous** if if for every open set $U \in Y$, the inverse image $f^{-1}(U)$ is open in X.*

From the last definition we can now deal with

Definition 1.6 (**Path connected set**).
*Let X be topological space. Then it is said to be **path connected** if for each pair of distinct points $a, b \in X$, there exists a continuous mapping*

$$f : [0,1] \to X, \qquad s.t. \quad f(0) = a,\ f(1) = b.$$

Later, in the third chapter of this paper, we are going to use an example of the next concept, the $\mathbf{n-ball}\ \mathbf{B}^n$, that is the interior of the sphere $\mathbf{S}^{n-1}$.

Definition 1.7 (**Interior of a set**).
*Let S be a subset of a topological space X. Then we define the **interior of a set**, the union of all open sets $\mathcal{U} \subset S$:*

$$\boldsymbol{int}(S) \stackrel{def}{=} \bigcup \mathcal{U},\ \mathcal{U} \textit{ open set of } S.$$

To continue with considerations about definitions linked to open sets, we need to speak about distance and metric space

Definition 1.8 (**Distance**).
*We define as **distance**, the map*

$$\begin{aligned} d : A \times A \ &\to\ \mathbb{R} \\ (\mathbf{x}, \mathbf{y})\ &\mapsto\ d(\mathbf{x}, \mathbf{y}) \stackrel{def}{=} \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} = \|\mathbf{x} - \mathbf{y}\| \end{aligned}$$

that satisfies the properties:

$$d(\mathbf{x}, \mathbf{y}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{y}; \tag{1.1}$$

$$d(\mathbf{x}, \mathbf{y}) + d(\mathbf{x}, \mathbf{z}) \geq d(\mathbf{y}, \mathbf{z}), \quad \forall\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in A. \tag{1.2}$$

*The couple (A, d) is called **metric space**. When $A = \mathbb{R}^n$, d is called **Euclidean distance**.*

With the notion of metric space we can consider

Definition 1.9 (**Bounded set**).
A set S in a metric space (S, d) is ***bounded****, if it has a finite generalized diameter, i.e., there is an $R < \infty$, such that*

$$d(x, y) \le R, \qquad \forall\ x, y \in S.$$

Remember that if we are in a Euclidean space, the closed and bounded sets are compact set.
A very important concept is the next one, widely used in every branch of mathematics

Definition 1.10 (**Homeomorphism**).
Let X and Y be topological spaces. We say that X and Y are ***homeomorphic*** *if there exist inverse continuous functions $f : X \to Y$, $g : Y \to X$.*
We write $X \cong Y$ and say that f and g are a ***homeomorphism*** *between X and Y.*

So if two topological spaces are homeomorphic, they are, using the techniques of topology, equivalent.

From homeomorphisms and continuous maps we give

Definition 1.11 (**Embedding**).
Let (X, τ) and (Y, τ') be topological spaces.
We say that (X, τ) can be embedded in (Y, τ'), if there exists a continuous mapping

$$f : X \to Y$$

such that

$$f : (X, \tau) \to (f(X), \tau")$$

is a homeomorphism, where $\tau"$ is the subspace topology on $f(X)$ from (Y, τ').
The mapping

$$f : (X, \tau) \to (f(X), \tau')$$

is said to be an ***embedding****.*

In differential geometry and particularly related with regular surfaces, there is the

Definition 1.12 (Differential of a map).
Given a map

$$F : \mathcal{U} \subseteq \mathbb{R}^m \quad \rightarrow \quad \mathcal{V} \subseteq \mathbb{R}^n \tag{1.3}$$

we call **differential** *of F at $p \in \mathcal{U}$, the linear map*

$$dF_p : \mathcal{U} \subseteq \mathbb{R}^m \quad \rightarrow \quad \mathcal{V} \subseteq \mathbb{R}^n \tag{1.4}$$

$$w_p \quad \mapsto \quad dF_p(w) = v. \tag{1.5}$$

Since F is defined as a mapping of points and has no notion of vectors, the differential is defined by examining a curve $\alpha(t)$ passing through p and having velocity at p equal to w , that is

$$\alpha(0) = p \quad \text{and} \quad \alpha'(0) = w. \tag{1.6}$$

One of the last concept that we are going to recall is the stereographic projection.
We will consider both cases, from the North pole and from the South pole.
It could be described as a bijection between the points of a hypersphere divested of a point and the points of a hyperplane.

Formally we have

Definition 1.13 (**Stereographic projection**). *Let* $N = (0, 0, ..., 1) \in \mathbb{R}^{n+1}$ *and let* $H = \{ x_{n+1} = 0 \}$ *be the hyperplane of* $\mathbb{R}^{n+1}$ *that does not contain* N*. The* **stereographic projection** (**from the north pole**) *is defined by:*

$$\pi_N : \mathbf{S}^n \setminus N \rightarrow H \tag{1.7}$$
$$\mathbf{x} \mapsto \pi_N(\mathbf{x}) = \left(\frac{x_1}{1 - x_n}, \frac{x_2}{1 - x_n}, ..., \frac{x_{n-1}}{1 - x_n}, 0 \right). \tag{1.8}$$

The inverse of this application is defined by:

$$\pi_N^{-1}((x_1, x_2, ..., x_n)) = \left(\frac{2x_1}{\|\mathbf{x}\|^2 + 1}, ..., \frac{2x_n}{\|\mathbf{x}\|^2 + 1}, \frac{\|\mathbf{x}\|^2 - 1}{\|\mathbf{x}\|^2 + 1} \right). \tag{1.9}$$

Now we consider the antipodal point of the north pole.
Let $S = (0, 0, ..., -1) \in \mathbb{R}^{n+1}$ *and let* $H = \{ x_{n+1} = 0 \}$ *be the hyperplane of* $\mathbb{R}^{n+1}$ *that does not contain* S*.*
The **stereographic projection** (**from the south pole**) *is defined by:*

$$\pi_S : \mathbf{S}^n \setminus S \rightarrow H \tag{1.10}$$
$$\mathbf{x} \mapsto \pi_S(\mathbf{x}) = \left(\frac{x_1}{1 + x_n}, \frac{x_2}{1 + x_n}, ..., \frac{x_{n-1}}{1 + x_n}, 0 \right). \tag{1.11}$$

And its inverse is:

$$\pi_S^{-1}((x_1, x_2, ..., x_n)) = \left(\frac{2x_1}{\|\mathbf{x}\|^2 + 1}, ..., \frac{2x_n}{\|\mathbf{x}\|^2 + 1}, \frac{1 - \|\mathbf{x}\|^2}{\|\mathbf{x}\|^2 + 1} \right). \tag{1.12}$$

At this point, we introduce a definition mainly related to set theory and algebra than topology

Definition 1.14 (**Cardinality**). *Sets* X *and* Y *have the same cardinality if there is a one-to-one and onto function (a bijection) from* X *to* Y*. Symbolically, we write* $|X| = |Y|$*.*

The cardinality of the real numbers, known as the cardinality of the continuum and denoted by $\mathfrak{c}$, was one of the most important Cantor's results.

The cardinality of the continuum is strictly greater then the cardinality of the natural numbers, indicated as $\aleph_0$: $\mathfrak{c} > \aleph_0$.

The last one preliminary notion is a theorem, an essential instrument in polynomial interpolation and complex analysis

Theorem 1.1 (**Weierstrass approximation theorem**).
Assume that f is a continuous map on a closed bounded interval $I = [a, b]$. Given any $\varepsilon > 0$, there is a polynomial P_n, with sufficiently high degree n, such that

$$|f(x) - P_n(x)| < \varepsilon, \quad for\ a \leq x \leq b.$$

Chapter 2

Brouwer fixed point theorem

Theorem 2.1 (**Brouwer fixed point theorem**[1]**).**
Every continuous mapping f from the disk $\mathcal{D}^n$ to itself, possesses at least one fixed point.

Proof.
First we want to define a non-zero vector field on $\mathcal{D}^n$ and we can call it $\boldsymbol{v}$. If we suppose that

$$f(x) \neq x, \qquad \forall x \in \mathcal{D}^n \tag{2.1}$$

then we can define $\boldsymbol{v}$ as

$$\boldsymbol{v} = x - f(x) \tag{2.2}$$

remarking that this vector field points away from the boundary in each point of it: that means

$$s \cdot \boldsymbol{v}(s) > 0 \qquad \forall s \in \mathbf{S}^{n-1}. \tag{2.3}$$

To narrow the field on $\mathcal{D}^n$ we say that $\boldsymbol{w}$ is a non-zero vector field that heads outwards on the boundary, if

$$\boldsymbol{w}(u) = u \qquad \forall u \in \mathbf{S}^{n-1}. \tag{2.4}$$

[1]The complete proof was made by Milnor. For more details, see [Milnor]

as it is shown in the following picture

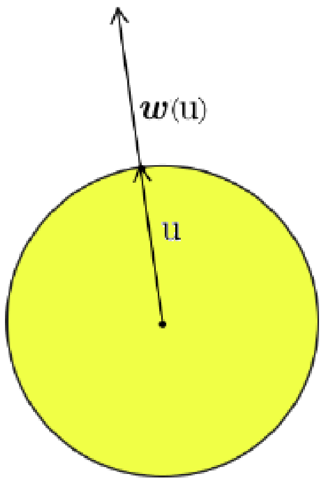


Figure 2.1: The vector field $\boldsymbol{w}$ points outward in $\mathbb{R}^2$.

As example, we could take

$$\boldsymbol{w}(x) = x - y\frac{(1 - x \cdot x)}{(1 - x \cdot y)} \qquad \forall x, y \in \mathbb{R}^n \tag{2.5}$$

where is easy to show that verifies the property 2.4 every time that[2] $x \in \mathbf{S}^{n-1}$, and even varying the value of x, the denominator could not be zero.
It is important to underline that the definition of $\boldsymbol{w}$ in 2.5 does not vanish obviously when x and y are independent, but it works even if they are dependent, in fact:

$$x = \lambda y \Rightarrow (x \cdot x)y = (x \cdot y)x, \tag{2.6}$$

then

$$\boldsymbol{w}(x) = \frac{(x - y)}{(1 - x \cdot y)} \neq 0. \tag{2.7}$$

Let us consider the unit n-sphere $\mathbf{S}^n \subseteq \mathbb{R}^{n+1}$ and we try to bring that vector field previously defined on $\mathbf{S}^n$; we also bethink of the hyperplane $x_{n+1} = 0$

[2]namely $x \cdot x = 1$

that divides $\mathbf{S}^n$ into two equal parts.
Now we take the stereographic projection previously defined [3], its inverse[4], and the definition of $\boldsymbol{w}$ in equation 2.5 .
Derivating π_N in x and applying it to $\boldsymbol{w}(x)$, we accomplish a tangent vector $\mathbf{V}(u)$ to $\mathbf{S}^n$ at the image point $\pi_N^{-1}(x) = u$, as it is showed in the following picture.

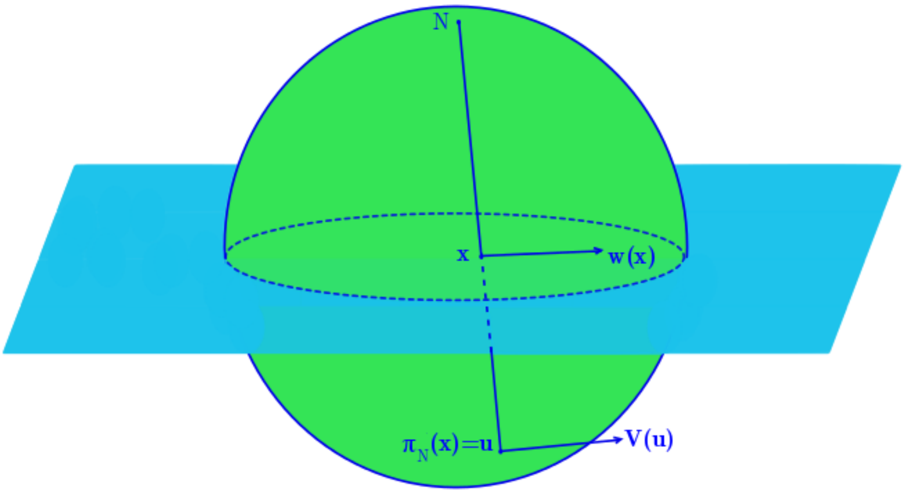


Figure 2.2: Construction of the tangent vector field $\mathbf{V}(u)$ on $\mathbf{S}^2 \subseteq \mathbb{R}^3$.

To figure out what this tangent vector field is, we think about the curve γ on the sphere, defined by

$$\begin{aligned} \gamma : \mathbb{R} &\rightarrow \mathbf{S}^n && (2.8)\\ t &\mapsto \gamma(\mathbf{t}) = \pi_N(x + t\boldsymbol{w}(x)). && (2.9) \end{aligned}$$

Then we can define $\mathbf{V}(u)$ as

$$\mathbf{V}(u) = \frac{d}{dt}\pi_N\big(x + t\boldsymbol{w}(x)\big)_{t=0} \quad . \tag{2.10}$$

Now we want show that this tangent vector field is non-zero for all points of $\mathbf{S}^n$: first we will look at it on the northern hemisphere, and then on the

[3] See equation 1.11
[4] See equation 1.9

southern.

If we get a point $\mathbf{y}$ of the equator of $\mathbf{S}^n$ we have that

$$\pi_N^{-1}(\mathbf{y}) = \pi_N^{-1}((y_1, y_2, ..., y_n, 0)) = \left(\frac{(2y_1, 2y_2, ..., 2y_n, \|\mathbf{y}\|^2 - 1)}{\|\mathbf{y}\|^2 + 1}\right), \quad (2.11)$$

but the point belongs to the sphere too, so $\|\mathbf{y}\|^2 = 1$ and then

$$\pi_N^{-1}(\mathbf{x}) = \left(\frac{(2x_1, 2x_2, ..., 2x_n, 0)}{2}\right) = (x_1, x_2, ..., x_n, 0) = \mathbf{x}. \quad (2.12)$$

Additionally, we know that $\boldsymbol{w}(\mathrm{y})$ points outward, but what about the vector field $\mathbf{V}$?

Remembering definition 1.12, we want calculate the differential of $\mathbf{V}$:

$$d\pi_N^{-1}(\mathbf{x}) = \frac{d}{dt}\pi_N^{-1}\big(\mathbf{x} + t\mathbf{V}(\mathbf{x})\big)_{t=0} =$$

$$= \frac{d}{dt}\left(\frac{\big(2(x_1 + t\mathbf{V}(\mathbf{x})), \dots, 2(x_n + t\mathbf{V}(\mathbf{x})), \|(\mathbf{x} + t\mathbf{V}(\mathbf{x}))\|^2 - 1\big)}{\big(\|(\mathbf{x} + t\mathbf{V}(\mathbf{x}))\|^2 + 1\big)}\right)_{t=0} =$$

$$\left(\frac{\Big(2\mathbf{V}(\mathbf{x})\big(\|(\mathbf{x} + t\mathbf{V}(\mathbf{x}))\|^2 + 1\big) - 2(\mathbf{x} + t\mathbf{V}(\mathbf{x}))[2\mathbf{x}\cdot\mathbf{V}(\mathbf{x})],\ 2(2\mathbf{x}\cdot\mathbf{V}(\mathbf{x}))\Big)}{\big(\|(\mathbf{x} + t\mathbf{V}(\mathbf{x}))\|^2 + 1\big)^2}\right)_{t=0} =$$

$$\left(\frac{\Big(2\mathbf{V}(\mathbf{x})\big(\|(\mathbf{x})\|^2 + 1\big) - 2\mathbf{x}[2\mathbf{x}\cdot\mathbf{V}(\mathbf{x})],\ 4(\mathbf{x}\cdot\mathbf{V}(\mathbf{x}))\Big)}{\big(\|(\mathbf{x})\|^2 + 1\big)^2}\right),$$

where in the second line of the calculus we used the fact

$$\frac{\big(\|(\mathbf{x} + t\mathbf{V}(\mathbf{x}))\|^2 - 1\big)}{\big(\|(\mathbf{x} + t\mathbf{V}(\mathbf{x}))\|^2 + 1\big)} = \frac{\big(\|(\mathbf{x} + t\mathbf{V}(\mathbf{x}))\|^2 + 1 - 2\big)}{\big(\|(\mathbf{x} + t\mathbf{V}(\mathbf{x}))\|^2 + 1\big)} =$$

$$= 1 - \frac{2}{\big(\|(\mathbf{x} + t\mathbf{V}(\mathbf{x}))\|^2 + 1\big)}.$$

Considering that $\mathbf{x}$ belongs to the equator of $\mathbf{S}^n$,

$$\mathbf{V}(\mathbf{x}) = \mathbf{x} \tag{2.13}$$

then

$$\mathbf{V}(\mathbf{x}) \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2 \tag{2.14}$$

and also

$$\|\mathbf{x}\|^2 = 1. \tag{2.15}$$

So we obtain

$$d\pi_N^{-1}(\mathbf{x}) = \left(\frac{\Big(2\mathbf{V}(\mathbf{x})\big(\|(\mathbf{x})\|^2 + 1\big) - 2(\mathbf{x})[2\mathbf{x} \cdot \mathbf{V}(\mathbf{x})],\ 4(\mathbf{x} \cdot \mathbf{V}(\mathbf{x}))\Big)}{\big(\|(\mathbf{x} + t\mathbf{V}(\mathbf{x}))\|^2 + 1\big)^2} \right) =$$

$$= \left(\frac{\Big(2\mathbf{x} \cdot \big(\|\mathbf{x}\|^2 + 1\big) - 2(\mathbf{x})[2\mathbf{x} \cdot \mathbf{x}],\ 4(\mathbf{x} \cdot \mathbf{x})\Big)}{\big(\|\mathbf{x}\|^2 + 1\big)^2} \right) =$$

$$= \left(\frac{\big(2\mathbf{x}(1+1) - 2\mathbf{x}(2\|\mathbf{x}\|^2), 4\|\mathbf{x}\|^2\big)}{(1+1)^2} \right) = \frac{(4\mathbf{x} - 4\mathbf{x}, 4)}{4} =$$

and definitely we have

$$d\pi_N^{-1}(\mathbf{x}) = (0, \ldots, 0, 1). \tag{2.16}$$

The value in 2.16 means that the corresponding vector of a point of the equator of $\mathbf{S}^n$ tips toward the north and then, away from the southern hemisphere.

Likewise, taking the inverse of the stereographic projection from the south pole [5],

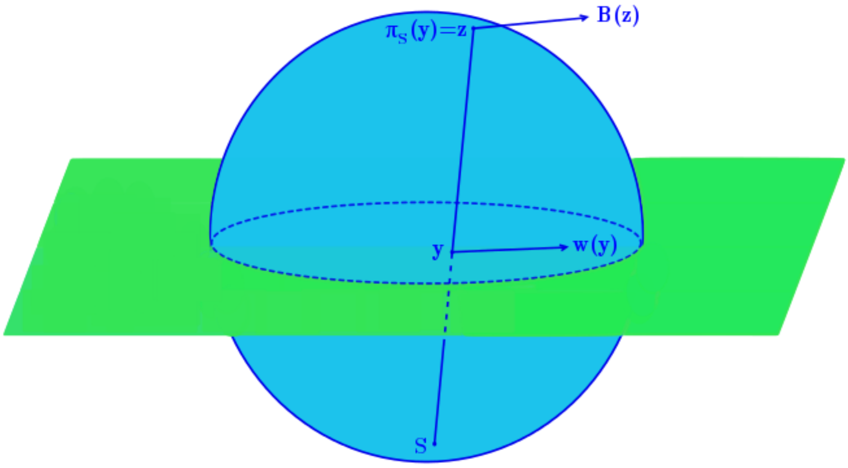


Figure 2.3: Construction of the tangent vector field $\mathbf{B}(z)$ on $\mathbf{S}^2 \subseteq \mathbb{R}^3$.

calculating the differential of the vector field $\mathbf{B}$:

$$\mathbf{B}(z) = \frac{d}{dt}\pi_S\big(y + t\boldsymbol{w}(y)\big)_{t=0} \quad . \tag{2.17}$$

[5]see equation 1.12

Its differentials will be found on the same way we did for $\mathbf{V}(x)$:

$$d\pi_S^{-1}(\mathbf{y}) = \frac{d}{dt}\pi_S^{-1}\big(y + t\mathbf{B}(y)\big)_{t=0} =$$

$$= \frac{d}{dt}\left(\frac{\big(2(y_1 + t\mathbf{B}(\mathbf{y})), \ldots, 2(y_n + t\mathbf{B}(\mathbf{y})), 1 - \|(\mathbf{y} + t\mathbf{B}(\mathbf{y}))\|^2\big)}{\big(\|(\mathbf{y} + t\mathbf{B}(\mathbf{y}))\|^2 + 1\big)}\right)_{t=0} =$$

$$\left(\frac{\Big(2\mathbf{B}(\mathbf{y})\big(\|(\mathbf{y} + t\mathbf{B}(\mathbf{y}))\|^2 + 1\big) - 2(\mathbf{y} + t\mathbf{B}(\mathbf{y}))[2\mathbf{y}\cdot\mathbf{B}(\mathbf{y})], \ \ -2(2\mathbf{y}\cdot\mathbf{B}(\mathbf{y}))\Big)}{\big(\|(\mathbf{y} + t\mathbf{B}(\mathbf{y}))\|^2 + 1\big)^2}\right)_{t=0} =$$

$$\left(\frac{\Big(2\mathbf{B}(\mathbf{y})\big(\|(\mathbf{y})\|^2 + 1\big) - 2\mathbf{y}[2\mathbf{y}\cdot\mathbf{B}(\mathbf{y})], \ \ -4(\mathbf{y}\cdot\mathbf{B}(\mathbf{y}))\Big)}{\big(\|(\mathbf{y})\|^2 + 1\big)^2}\right),$$

where in the second line of the calculus we used the fact

$$\frac{\big(1 - \|(\mathbf{y} + t\mathbf{B}(\mathbf{y}))\|^2\big)}{\big(\|(\mathbf{y} + t\mathbf{B}(\mathbf{y}))\|^2 + 1\big)} = \frac{\big(2 - 1 - \|(\mathbf{y} + t\mathbf{B}(\mathbf{y}))\|^2\big)}{\big(\|(\mathbf{y} + t\mathbf{B}(\mathbf{y}))\|^2 + 1\big)} =$$

$$= \frac{2}{\big(\|(\mathbf{y} + t\mathbf{B}(\mathbf{y}))\|^2 + 1\big)} - 1.$$

Considering that $\mathbf{y}$ belongs to the equator of $\mathbb{S}^n$,

$$\mathbf{B}(\mathbf{y}) = \mathbf{y} \tag{2.18}$$

then

$$\mathbf{B}(\mathbf{y}) \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{y} = \|\mathbf{y}\|^2 \tag{2.19}$$

and also

$$\|\mathbf{y}\|^2 = 1. \tag{2.20}$$

So we obtain

$$d\pi_S^{-1}(\mathbf{y}) = \left(\frac{\Big(2\mathbf{B}(\mathbf{y})\big(\|(\mathbf{y})\|^2+1\big) - 2(\mathbf{y})[2\mathbf{y}\cdot\mathbf{B}(\mathbf{y})], \quad -4(\mathbf{y}\cdot\mathbf{B}(\mathbf{y}))\Big)}{\big(\|(\mathbf{y}+t\mathbf{B}(\mathbf{y}))\|^2+1\big)^2} \right) =$$

$$= \left(\frac{\Big(2\mathbf{y}\cdot\big(\|\mathbf{y}\|^2+1\big) - 2(\mathbf{y})[2\mathbf{y}\cdot\mathbf{y}], \quad -4(\mathbf{y}\cdot\mathbf{y})\Big)}{\big(\|\mathbf{y}\|^2+1\big)^2} \right) =$$

$$= \left(\frac{\big(2\mathbf{y}(1+1) - 2\mathbf{y}(2\|\mathbf{y}\|^2), -4\|\mathbf{y}\|^2\big)}{(1+1)^2} \right) = \frac{(4\mathbf{y}-4\mathbf{y},-4)}{4} =$$

and definitely we have

$$d\pi_S^{-1}(\mathbf{z}) = (0,\dots,0,-1). \tag{2.21}$$

If we take the vector field -***w(y)*** instead of ***w(y)*** in 2.17, we obtain in 2.21

$$d\pi_S^{-1}(\mathbf{z}) = (0,\dots,0,1) \tag{2.22}$$

and we can associate to it, by the stereographic projection from the south pole, a vector field on the northern hemisphere that points north too on the equator.
Assembling together the vector fields ***w*** and -***w*** we have a tangent vector field **BV** on the entire $\mathbf{S}^n$, and it is non-zero.
Now we need the

Theorem 2.1. *An even dimensional sphere does not admit any continuous field of non-zero tangent vectors.*

which help us to claim that if n of $\mathbf{S}^n$ is even, the construction of $\mathbf{S}^n$ is not realizable and this contradiction proves theorem 2.1 for this values.
In a more general way, we can take odd values of n too: from generic map

$$f : \mathcal{D}^{2k-1} \;\to\; \mathcal{D}^{2k-1} \tag{2.23}$$

where $\mathcal{D}^{2k-1}$ has no fixed points, we can build another map

$$\begin{array}{rcl} \mathbf{F} : \mathcal{D}^{2k} & \to & \mathcal{D}^{2k} \\ (x_1, \ldots, x_n) & \mapsto & \mathbf{F}(x_1, \ldots, x_n) = f\big((x_1, \ldots, x_n), 0\big), \end{array}$$

which has no fixed point. $\square$

Chapter 3

Topological invariance of dimension theorem

3.1 Theorem and historical background

The[1] concept of dimension was a very hard notion to manage in mathematics.
Before the second part of XIX century no-one among mathematicians denied or has showed before that every point x of $\mathbb{R}^n$ could be uniquely identified from n coordinates, (i.e. n real numbers) but they just supposed it as true.
In fact, until 1870, there was a conviction, that taking arbitrary $X \subset \mathbb{R}^n$, it would be possible to create a (at least local) correspondence between them and some open $Y \subset \mathbb{R}$ by injective maps.
We can not imagine how wonderful in 1877 it were seemed the Cantor's discover of an application between $\mathbb{R}$ and $\mathbb{R}^n$ for some values of n, and in the same manner, the curve of Peano in 1890: both results, seemed to be able to help in the search to formalize the concept of the size.
However in the first case it was a non-continuous function, and in the second a continuous but not injective function.
The correspondence between Dedekind and Cantor, started in 1872, gave us

[1]see [Dieudonne] and [Nastasi]

in a letter in 1877, the formulation of the theorem that will still use[2] and from that date to the first decade of XX century, a lot of mathematicians tried to manage specific cases, assigning a specific value at n and m, considering $X \subset \mathbb{R}^n$ and $X \subset \mathbb{R}^m$, as we have in statement of the theorem.
Only Brouwer in 1911, found the solution at this problem. After this date, it started a diatribe between him and Lebesgue, because the last one published a lot of demonstrations about this theorem, but no-one of them was as correct as the Brouwer's one.
In 1924 finally the disputations ended and Lebesgue recognized to Brouwer the fatherhood of the discovering.

3.2 Just two simple cases

We now recall the[3]

Theorem 3.1 (**Invariance of dimension theorem).**
Let $n, m \in \mathbb{Z}$. Then $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.

Supposing that $n < m$, if we consider two very simple cases, we could easily show, in a not very rigorous way, what it is happening.
First we consider the case $n = 0$, where we have $\mathbb{R}^0$ and $\mathbb{R}^m$ where $m > 0$.
It is obvious to see that can not exist a bijection, between two set with different cardinality:

$$|\mathbb{R}^0| = 1 \neq \mathfrak{c} = |\mathbb{R}^m| \tag{3.1}$$

Instead of the previous case, let us take now $\mathbb{R}^1$ and $\mathbb{R}^m$ with $m > 1$.
For absurd, if there exists a homeomorphism

$$f : \mathbb{R} \to \mathbb{R}^m \tag{3.2}$$

then the restriction

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\} \tag{3.3}$$

[2]see 3.1
[3]see for more details [Hatcher]

has to be a homeomorphism too.
But $\mathbb{R} \setminus \{0\}$ is not connected, conversely $\mathbb{R}^m \setminus \{f(0)\}$ is path connected, then the restriction of f can not be a homeomorphism and also f.
At the increasing of n it is really hard to find a simple way to argue our aim, and so we can not use this approach to demonstrate a general case.

3.3 Brouwer's proof

We want to demonstrate the following

Theorem 3.2 (**Theorem of topological invariance of dimension**). *Let $n, m \in \mathbb{Z}, n > m$ and let U be an open subspace of $\mathbb{R}^n$ then there are no injective and continuous maps from U to $\mathbb{R}^m$. Particularly $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic.*

The idea of Brouwer was to take first the continuous map

$$f : [-1, 1]^n \quad \to \quad \mathbb{R}^n$$

which satisfies

$$|f(x) - x| < \frac{1}{2} \qquad \forall x \in [-1, 1]^n \tag{3.4}$$

and then show

$$f([-1, 1]^n) \supset \left[-\frac{1}{2}, \frac{1}{2}\right]^n. \tag{3.5}$$

We turn now into details of Brouwer's proof.
We assume by a contradiction, that exists an injective continuous map

$$\chi : [-1, 1]^n \quad \to \quad \mathcal{C} \subset \mathbb{R}^n$$

where $\mathcal{C}$ is a rare set, and also we can find another continuous function that comes back on the n-cube

$$\psi : \mathcal{C} \subset \mathbb{R}^n \to [-1, 1]^n \tag{3.6}$$

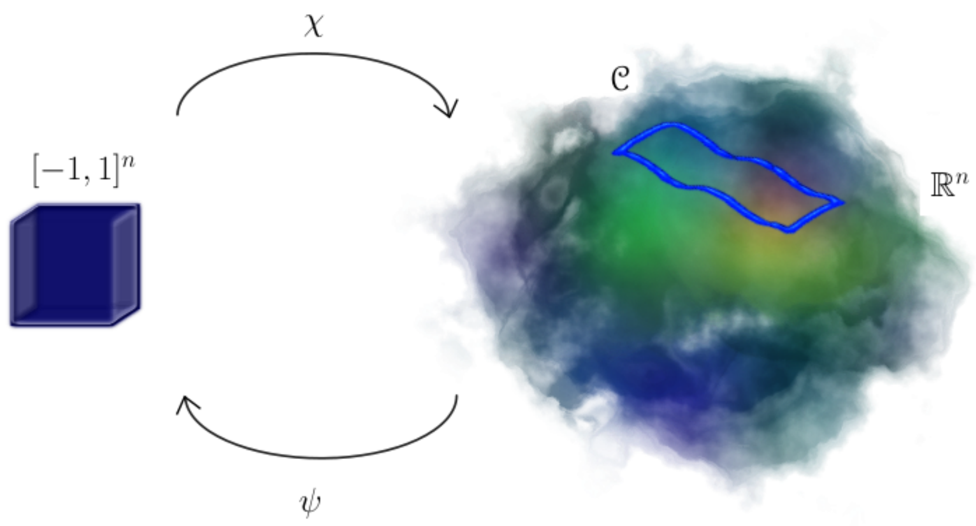


Figure 3.1: Representation of the maps ψ and χ in $\mathbb{R}^n$ where $n = 3$.

such that

$$|\psi(\chi(x)) - x| < \frac{1}{2}, \qquad \forall x \in [-1,1]^n. \tag{3.7}$$

How is it possible to build ψ?

Suppose now to take a cube ρ that contains $\mathcal{C}$ and to create a triangulation T of ρ.

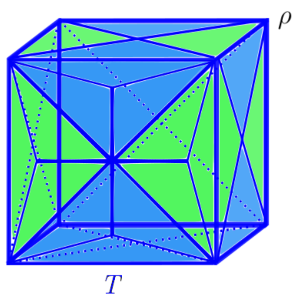


Figure 3.2: Triangulation T on a cube ρ of $\mathbb{R}^3$

We define

$$\mathcal{F} \stackrel{def}{=} \left\{ \bigcup_{n>2} \frac{\{3,\ldots,3\}}{n-1} \;\middle|\; \frac{\{3,\ldots,3\}}{n-1} \in T,\ \frac{\{3,\ldots,3\}}{n-1} \bigcap \mathcal{C} \neq \emptyset \right\} \tag{3.8}$$

Let us consider a n-simplex $\sigma \subset \mathcal{F}$, let $c \in \sigma$ a vertex which satisfies

$$\psi_0(c) \in [-1,1]^n \text{ and } \chi(\psi_0(c)) \in \sigma \tag{3.9}$$

as it is shown in the following picture

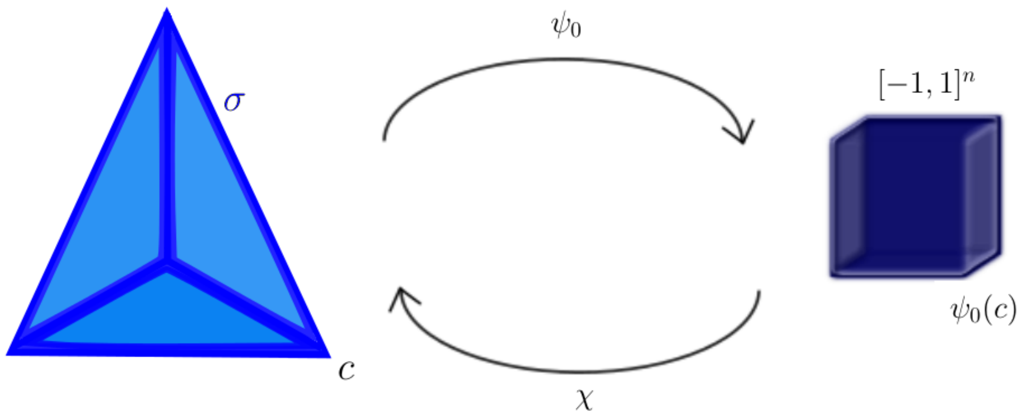


Figure 3.3: Action of the map ψ_0

where we have chosen a piecewise affine map

$$\psi_0 : \mathcal{F} \to [-1,1]^n. \tag{3.10}$$

Supposing to see the function 3.6 as

$$\psi : \mathcal{C} \to [-1,1]^n \tag{3.11}$$

so

$$\psi = (\psi_0)_{|\mathcal{C}} \tag{3.12}$$

hence we can affirm that the set

$$S = (\sigma \cap \mathcal{C}), \quad \forall \sigma \subset \mathcal{F} \tag{3.13}$$

is rare, and definitely, T, by definition of ψ, it has been as fine as we needed. After all these considerations, we come back now at the statement of theorem 3.2 and consider the case $n > m$.
At this point Brouwer considered a continuous injection

$$\upsilon : [-1, 1]^n \to Q \subset \mathbb{R}^m, \tag{3.14}$$

where Q is a cube that contains a rare image Q' of the n-cube $[-1, 1]^n$ by the map υ. But if exists another continuous injection

$$\varphi : Q \to [-1, 1]^n, \tag{3.15}$$

the composition $\varphi \circ \upsilon$, go against the theorem.
In the other case, if $n < m$, he considered another continuous injection

$$\xi : [-1, 1]^m \to Q' \subset \mathbb{R}^n, \tag{3.16}$$

where Q' is a cube that contains a rare image Q of the n-cube $[-1, 1]^m$ by the map ξ.
But if exists a continuous injection

$$\zeta : Q \to [-1, 1]^n, \tag{3.17}$$

the composition $\zeta \circ \xi$ goes again against the theorem.

3.4 Nods of standard proof: homological approach

Premise To proceed now, we have to precise that all the next part is based on *Homology theory*, which is a very important part of Algebraic Topology. Just to introduce and explain the preliminary notions that we are going to use it will be necessary to do a specific course and handle these concepts should be a very good competence.
Unfortunately, we could not bridge this gap, but we retained to report the structure of the proof, to show to the reader how the modern technique of solving this mathematical theorem is.

Proof of the invariance of dimension theorem.
So we assume that all discernment that we are to treat, are familiar and clear. After this small premise, we can start.
Suppose there exists homeomorphism f between two non-empty spaces $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, and let x be a point of X.
From this point x, we build the homology groups

$$H_i(X, X \setminus \{x\}), \text{ where } i \in \mathbb{N} \tag{3.18}$$

and because for definition of X, it is open, we can use the following

Theorem 3.3 (Excision Theorem). *Let $U \subset A \subset X$ be subspaces such that the closure $\bar{U}$ of U lies in the interior $\mathring{A}$ of A. Then the inclusion*

$$(X \setminus U, A \setminus U) \to (X, A) \tag{3.19}$$

induces isomorphisms on relative homology groups:

$$H_n(X \setminus U, A \setminus U) \xrightarrow{\cong} H_n(X, A) \text{ where } n \geq 0. \tag{3.20}$$

to find the

$$H_i(X, X \setminus \{x\}) \cong H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}). \tag{3.21}$$

From

$$(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \tag{3.22}$$

we look for the associated long right sequence, to gain

$$H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \tag{3.23}$$

that is isomorphic to

$$\tilde{H}_i(\mathbb{R}^n \setminus \{x\}). \tag{3.24}$$

We know the topological fact that, if we make a hole in a sphere, it deflates to a sphere of lower dimension, so 3.24 contracts to the sphere S^{n-1}.
Remembering the important property of homology, or rather that preserves homotopy type, then we have found

$$H_i(X, X \setminus \{x\}) \cong H_i(S^{n-1}) \tag{3.25}$$

Recalling that homology groups provides the invariance of dimension of spheres, by properties of homology we can prove that

$$H_i(S^{n-1}) = 0, \text{ if } i \neq 0, n-1 \tag{3.26}$$

and $\mathbb{Z}$ diversely.
Doing the same work on Y, if X and Y are homeomorphic, then we will will find isomorphic homology groups for every $i \in \mathbb{N}$.
Precisely,

$$\mathbb{Z} \cong H_{m-1}(S^{m-1}) \cong H_{m-1}(Y, Y - \{f(x)\}) \cong H_{m-1}(X, X - \{x\}) \cong H_{m-1}(S^{n-1}). \tag{3.27}$$

Then $m = 1$ which implies $n = 0, 1$ as $n \leq m$, or $m = n$. But if $n = 0$, we have $m = n$.
So we can extend the result for all m, n. □

3.5 Into topological proof

We start this section enunciating and demonstrating the following

Theorem 3.4 (**Brouwer invariance of domain theorem**).
Let $f : B^n \to \mathbb{R}^n$ be a continuous and injective map. Hence $f(0) \in \boldsymbol{int}(f(B^n))$.

The *Invariance of Dimension Theorem* will be deduced as a corollary of the previous one.
Before starting the proof, we need some preliminary conditions: first we have to show that the translation and the rescaling are invariants.
Let $p \in U$, we need to point out that $f(0) \in \mathbf{int}(f(U))$ and let $p \in B$, where B is the closed unitary ball of $\mathbb{R}^n$.
Therefore exists a homeomorphism ϕ such that $\phi(B) = B^n$.
So proving that $f(0) \in \mathbf{int}(f(U))$ is the same of establishing that

$$(f \circ \phi^{-1})(0) \in \mathbf{int}(f \circ \phi^{-1})(B^n). \tag{3.28}$$

Now we consider the map $\tilde{f} : B^n \to \mathbb{R}^n$ and let $f : B^n \to f(B^n) \subseteq \mathbb{R}^n$.
The last one is an homeomorphism[4] between compact and Hausdorff spaces, then the map

$$f^{-1} : f(B)^n \subset \mathbb{R}^n \to B^n$$

is continuous too.
In addition to this, we recall the

Theorem 3.5 (**Tietze extension theorem**). *The topological space X is normal if and only if, for all closed subsets C of X, every continuous function*

$$h : C \to \mathbb{R}$$

can be extended to a continuous function

$$H : X \to \mathbb{R}.$$

[4]because of the Closed map Lemma. See [Lee], *An introduction to smooth manifolds*, page 610, for more explanations.

And we are going to use it to extend f^{-1} with $G : \mathbb{R}^n \to B^n$: this map has a zero on $f(B^n)$ by construction.

Before proceeding is necessary to argue the following

Lemma 1 (Lemma of stability of a zero).
Let $\tilde{G} : f(B^n) \to \mathbb{R}^n$ *a continuous map that satisfy the condition*

$$\|G(y) - \tilde{G}(y)\| \leq 1, \qquad \forall\ y \in f(B^n). \tag{3.29}$$

Therefore $\tilde{G}$ *has a zero, that is there exists* $y \in f(B^n)$ *such that* $\tilde{G}(y) = 0$.

Proof (of the Lemma).
Let H be the map such that

$$x \mapsto x - \tilde{G}(f(x))$$

and consider the extension G of the map f^{-1} as we has defined it before.
So we can say that

$$x \mapsto (f^{-1} \circ f)(x) - \tilde{G}(f(x))$$

applying the Tietze extension theorem first

$$x \mapsto (G \circ f)(x) - \tilde{G}(f(x)) = G(f(x)) - \tilde{G}(f(x)).$$

We know that for construction, $\forall y \in f(B^n)$ and from the properties of the norm

$$0 \leq \|G(y) - \tilde{G}(y)\| \leq 1.$$

So we can consider two cases:

(I) $G(y) = \tilde{G}(y) \Rightarrow \|G(y) - \tilde{G}(y)\| = 0 \Rightarrow G = \tilde{G}$

(II) $\tilde{G}(y) = 0 \Rightarrow \|G(y)\| \leq 1 \Rightarrow G = f^{-1}$.

In the second case $G(y) \in B^n$, then all points such that $\tilde{G}(y) = 0$ stay on the same circle and, therefore exists one zero of the map. ●

Finally we can start to prove theorem 3.4.

Proof (of Brouwer invariance of domain theorem).
We suppose a contradiction, that is

$$f(0) \notin f(B^n) \tag{3.30}$$

Let ε be as small as we need, thus for the continuity of the map G we affirm that

$$\|G(y)\| \leq 0,1 \Leftrightarrow \forall y \in \mathbb{R}^n, \qquad \|y - f(0)\| \leq 2\varepsilon. \tag{3.31}$$

In other way the condition 3.30 implies

$$\exists\, c \in \mathbb{R}^n, \quad c \notin f(B^n) \qquad s.t. \qquad \|c - f(0)\| < \varepsilon. \tag{3.32}$$

Translating c in 0 we obtain that

$$\|f(0)\| < \varepsilon$$

and consequently we have a new continuity condition

$$\|G(y)\| \leq 0,1 \Leftrightarrow \forall y \in \mathbb{R}^n, \qquad s.t. \quad \|y\| \leq \varepsilon. \tag{3.33}$$

Let's regard now the set

$$\Sigma \overset{def}{=} \Sigma_1 \cup \Sigma_2$$

where

$$\Sigma_1 \overset{def}{=} \{\, y \in f(B^n) \mid \|y\| \geq \varepsilon \,\} \qquad and \qquad \Sigma_2 \overset{def}{=} \{\, y \in f(B^n) \mid \|y\| = \varepsilon \,\}\,.$$

This set is compact as it has built and the point $f(0) \notin \Sigma$ because $\|f(0)\| < \varepsilon$.
Now we define the map

$$\begin{aligned} \Phi : f(B^n) \ &\to \ \Sigma \\ y \ &\mapsto \ \Phi(y) \overset{def}{=} \left[max \left(\frac{\varepsilon}{\|y\|}, 1 \right) \right] \cdot y \end{aligned}$$

which is well defined[5].

[5] Φ is called "pushing map" because bring the point far from 0.

The restriction on Σ_1 of the map G does not reach zero and because of the compactness of Σ_1 we say it is confined by a positive number δ and let $\delta < 0,1$.
Recalling the Weierstrass approximation theorem for polynomials[6], we can affirm that exists a polynomial

$$P(y) : \mathbb{R}^n \to \mathbb{R}^n$$

such that

$$\|P(y) - G(y)\| < \delta \qquad \forall y \in \Sigma.$$

$P(y)$ is non-zero on Σ_1 as it has built, but what happens on Σ_2?
A polynomial is a smooth map and the Lebesgue measure[7] of Σ_2 is zero; both imply that the Lebesgue measure of the set $P(\Sigma_2)$ is zero too.
So without losing general case we can suppose $P = constant$ and then P does not set to zero on Σ_2 because it lies on an hyper-surface which can not be influenced by a perturbation.
Let's consider now the continuous function[8] $\tilde{G} \stackrel{def}{=} P(\Phi(y))$. Remembering the limitations of Σ_1 we can apply the theorem 1.1 again

$$\|G(y) - \tilde{G}(y)\| < \delta \qquad and \qquad \|y\| > \varepsilon \Leftrightarrow y \in f(B^n).$$

But supposing $y \notin f(B^n)$ we have

$$\|y\| \leq \varepsilon$$

hence

$$\|G(y)\| \leq 0,1$$

and from the definition of Φ

$$\|G(\Phi(y))\| \leq 0,1$$

[6] theorem 1.1.
[7] for explanation see [Meisters]
[8] this map is non-zero for the definition of Φ.

that can let us say

$$\|G(y)\| = \|G(\Phi(y))\|.$$

Applying another time the theorem 1.1 and because of triangular inequality[9]

$$\|a\| - \|b\| \leq \|a - b\| \leq \|a + b\| \leq \|a\| + \|b\|,$$

we can affirm that

$$\|G(y) - \tilde{G}(y)\| \leq 0,2 + \delta \leq 0,3 \qquad for \quad \forall\, y \notin f(B^n).$$

Because of $\tilde{G}(y) \nrightarrow 0$ for construction, we conclude that the Lemma of stability of a zero is not preserved and obviously must be $y \in f(B^n)$. □

Now that the theorem 3.4 has been proved, we can go on the final steps of our path.

Theorem 3.6 (**Invariance of dimension theorem**). [10]
There is no continuous one-to-one map

$$f : \mathbb{R}^n \to \mathbb{R}^m \tag{3.34}$$

for $m < n$.

Proof (of Invariance of dimension theorem).
Let us take an embedding

$$\begin{array}{rcl} \iota : \mathbb{R}^m & \to & \mathbb{R}^n \\ (x_1, \ldots, x_m) & \mapsto & \iota(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0), \end{array}$$

and then we define the map

$$\eta \stackrel{def}{=} \iota \circ f. \tag{3.35}$$

For construction of $\eta(\mathbb{R}^n)$, it is a boundary subset of $\mathbb{R}^n$, but because of theorem 3.4 is an open subset of $\mathbb{R}^n$ too. And obviously, this is impossible, so finally we found the result we were looking for. ■

[9] property 1.2
[10] [Kulpa]

Bibliography

[Cohen] Paul J. Cohen, *Theory and the Continuum Hypothesis*,Addison-Wesley, (August 1966).

[Dieudonne] J.Dieudonne, *A History of Algebraic and Differential Topology, 1900-1960*, Modern Birkhäuser Classics, Ch.II, 2009.

[Do Carmo] Manfredo P. Do Carmo, *Differential Geometry of Curves and Surfaces*, Pearson, 1976.

[Estep] Estep Donald, *Practical Analysis in One Variable*, Springer eBooks, *ISBN 978-0-387-22644-6*, XX, pp 509-524.

[Hatcher] Allen Hatcher, *Algebraic Topology,http://www.math.cornell.edu/ hatcher/AT/AT.pdf* Cambridge University Press, Cambridge, 2002.

[Joshi] K.D. Joshi, *Introduction to General Topology*, Wiley Eastern Ltd., 1983.

[Kim] K.H. Kim, F.W. Roush, Ellis Horwood Series, *Applied Abstract Algebra*, ISBN 0-85312-612-7 (student edition), ISBN 0-85312-563-5 (library edition), 1983.

[Kosn] Czes Kosniowsky, *A first course in algebraic topology*, Cambridge University Press, 1980.

[Kosniowsky] Czes Kosniowsky, *Introduzione alla topologia algebrica*, Zanichelli, 1988.

[Kreyszig] Erwin Kreyszig, *Introductory Functional Analysis with Applications*, Wiley Classic Library,1989.

[Kulpa] Wladislaw Kulpa, *Poincaré and domain invariance theorem*, *http://dml.cz/dmclz/702050*, Acta Universitatis Carolinae, Mathematica et Physica, Vol.39 (1998), No.1-2, pp. 127-136.

[Lee] John Lee, *Introduction to smooth manifolds*, Springer, 2011.

[Meisters] G. H. Meisters, *Lebesgue Measure on the real line*, *http://www.math.unl.edu/ gmeisters1/papers/Measure/measure.pdf*, January, 1997.

[Milnor] John Milnor, *Analytic Proofs of the "Hairy Ball Theorem" and the Brouwer Fixed Point Theorem*, *http://www.jstor.org/stable/2320860*, The American Mathematical Monthly, Vol.85, No.7(Aug.-Sep., 1978), pp 521-524.

[Morris] Sidney A.Morris, *Topology with no tears*, *http://www.topologywithouttears.net/*, January, 2011, page 68, page 70, page 101, page 224.

[Nastasi] Pietro Nastasi, *Note di matematica, Storia, Cultura 6*, Springer, 2002.

[Sernesi] Edoardo Sernesi, *Geometria 2*, Bollati Boringhieri, 1994.

[Tao] Terence Tao, *Brouwer's fixed point and invariance of domain theorems, and Hilbert's fifth problem*, June, 2011, *https://terrytao.wordpress.com/2011/06/13/brouwers-fixed-point-and-invariance-of-domain-theorems-and-hilberts-fifth-problem/*

[Weisstein] Eric W.Weisstein, *Bounded Set*, MathWorld, *http://mathworld.wolfram.com/BoundedSet.html.*