A SYMPLECTIC VERSION OF NASH C^1-ISOMETRIC EMBEDDING THEOREM

joint work with:
Giusi D'Ambra

Problem:

Let (V, ω) e (W, Ω) be two symplectic manifolds endowed with Riemannian metrics g and G respectively.

We look for a map $f : V \to W$ such that:

$$f^*(\Omega) = \omega, \tag{1}$$

$$f^*(G) = g. \tag{2}$$

ABOUT THE EQUATION $f^*(\Omega) = \omega$

Theorem 1 (Gromov, 1986) *Let* (V, ω) *and* (W, Ω) *be two symplectic manifolds. Let* $f_0 : V \to W$ *be an embedding satisfying the following two conditions:*

(i) the forms ω *and* $f_0^*(\Omega)$ *are cohomologous;*

(ii) the differential $Df_0 : T(V) \to T(W)$ *is homotopic to a symplectic homomorphism between bundle* $F : T(V) \to T(W)$ *(i.e.* $F^*(\Omega) = \omega$ *on the fibers) throughout a family of monomorphisms.*

Then in the following two cases f_0 *is isotopic to a symplectic embedding,* $f : V \to W$*, arbitrarly* C^0*-close to* f_0*:*

(a) $dim\, W \geq dim\, V + 4$*;*

(b) $dim\, W > dim\, V$ *and* V *is open.*

Corollary 1 *Let* (V, ω) *be a symplectic manifold with* V *contractible and* $\dim V < 2N$. *Then every embedding* $f_0 : V \to \mathbb{R}^{2N}$ *is isotopic to a symplectic embedding* $f : (V, \omega) \to (\mathbb{R}^{2N}, \Omega_{can})$. *In particular, if* V *is diffeomorphic to* $\mathbb{R}^{2n-2}$ *then there exists a symplectic embedding*

$$(V, \omega) \to (\mathbb{R}^{2n}, \Omega_{can}).$$

Corollary 2 *(Tischler, 1977) Let* (V, ω) *a compact symplectic embedding with* ω *integral. Then for* N *sufficiently large there exists a symplectic embedding* $f : (V, \omega) \to (\mathbb{C}P^N, \Omega_{FS})$.

ABOUT THE EQUATION $f^*(G) = g$

Theorem 2 (Nash, 1954) *Let (V, g) be a compact Riemannian manifold which admits a smooth immersion (resp. embedding) $f_0 : V \to \mathbb{R}^q$ and $dim V \leq q - 2$, then there exists a C^1 isometric immersion (resp. embedding) $f : (V, g) \to (\mathbb{R}^q, G_{can})$. Moreover if f_0 is g-short, i.e. $g - f_0^*(G_{can}) > 0$ then f can be chosen arbitrarly C^0-close to f_0.*

Corollary 3 *Every Riemannian manifold (V, g) can be isometrically C^1-embedded in $(\mathbb{R}^q, G_{can})$ for $q = 2\, dim V$.*

Remark

The equation $g = f^*(G_{can})$ is equivalent to the system of PDE

$$\sum_{\alpha=1}^{q} \frac{\partial f_\alpha}{\partial x_i} \frac{\partial f_\alpha}{\partial x_j} = g_{ij}, \quad i, j = 1, \ldots n,$$

where $f = (f_1, f_2, \ldots, f_q)$.

IDEAS OF THE PROOF OF NASH'S THEOREM

- **The decomposition Lemma** Let $V = \cup_j B_j$, B_j contractible. Any metric g on V can be written as:

$$g = \sum_j d\phi_j^2$$

where $\phi_i : V \to \mathbb{R}$ and $\operatorname{supp} \phi_i \subset B_j$, $j = j(i)$.

- **The normal extension Lemma** Let $B \subset V$ be contractible and $q \geq \dim V + 2$. For every embedding $f_0 : V \to \mathbb{R}^q$ there exists *a normal extension* f_0 *on* B, namely a smooth map $\tilde{f}_0 : B \times \mathbb{R}^2 \to \mathbb{R}^q$ such that:

1. $\tilde{f}_0|_{B\times\{(0,0)\}} = f_0|_B$;

2. the vector fields $\partial_1 = \frac{\partial \tilde{f}_0}{\partial x}|_{B\times\{(0,0)\}}$ and $\partial_2 = \frac{\partial \tilde{f}_0}{\partial y}|_{B\times\{(0,0)\}}$ are G-ortogonal to B and mutually G-orthonormal.

Proof of Nash's theorem

- Let $f_0 : V \longrightarrow \mathbb{R}^q$ be a g-short map, i.e. $g - f_0^*(G) > 0$.
- It follows by the decomposition Lemma that:

$$g_1 = g - f_0^*(G) = \sum_j d\phi_j^2$$

where $\phi_i : V \longrightarrow \mathbb{R}$ and $\operatorname{supp}\phi_i \subset B_j, j = j(i)$.

- Fix $\epsilon_1 > 0$ and consider the <u>perturbation</u> (called the "twisting perturbation") $f_{\epsilon_1} : V \longrightarrow \mathbb{R}^q$ of f_0 on $B_{j(1)}$:

$$f_{\epsilon_1} = \tilde{f}_0 \circ s_{1|_{B_{j(1)}}} \tag{3}$$

and $f_{\epsilon_1} = f_0$ outside $B_{j(1)}$, where $s_1 : V \longrightarrow V \times \mathbb{R}^2$ is a section of the trivial bundle $V \times \mathbb{R}^2 \longrightarrow V$ (constructed by ϕ_1) such that $\operatorname{supp} s_1 \subset B_{j(1)}$.

[More precisely, $s_1 : V \longrightarrow V \times \mathbb{R}^2$ is defined by the composition:

$$s_1 : V \xrightarrow{\Gamma_{\phi_1}} V \times \mathbb{R} \xrightarrow{\alpha_{\epsilon_1}} V \times \mathbb{R}^2, \tag{4}$$

where $\Gamma_{\phi_1} : V \to V \times \mathbb{R} : v \mapsto (v, \phi_1(v))$ and $\alpha_{\epsilon_1}(v,t) = (v, \frac{\epsilon_1}{2\pi}\sin\frac{t}{\epsilon_1}, \frac{\epsilon_1}{2\pi}\cos\frac{t}{\epsilon_1})$.]

- One does the same woth the map f_{ϵ_1}, namely, given $\epsilon_2 > 0$ one perturbs f_{ϵ_1} on $B_{j(2)}$ using ϕ_2 instead of ϕ_1 and one obtains $f_{\epsilon_2} : V \longrightarrow \mathbb{R}^q$.

- The sequence of maps $f_{\epsilon_1}, f_{\epsilon_2}, \ldots$ converges to an immersion $f_1 : V \longrightarrow \mathbb{R}^q$ g-short which satisfies the following inequalities:

$$\| g - f_1^*(G) \|_0 < \frac{2}{3} \| g - f_0^*(G) \|_0,$$

$$\| f_1 - f_0 \|_1 < c(n) \| g - f_0^*(G) \|_0^{\frac{1}{2}}.$$

- Starting from f_1 one builds f_2 in a similar way. One then gets a sequence of g-shorts maps $f_j : V \longrightarrow \mathbb{R}^q$ satisfying:

$$\| g - f_j^*(G) \|_0 < \frac{2}{3} \| g - f_{j-1}^*(G) \|_0,$$

$$\| f_j - f_{j-1} \|_1 < c(n) \| g - f_{j-1}^*(G) \|_0^{\frac{1}{2}}.$$

- It follows that $\{f_j\}$ C^1-converges to a map $f : V \longrightarrow \mathbb{R}^q$ such that $f^*(G) = g$.

- By choosing the ϵ_j small enough the map f can be taken C^0-close to f_0.

Theorem 3 (D'Ambra and Loi, DGA 2002)
Let (V, ω) and (W, Ω) be two symplectic manifolds endowed with Riemannian metrics g and G respectivelty and let $f_0 : V \to W$ be a C^∞ symplectic embedding, i.e. $f_0^*(\Omega) = \omega$.
Let us suppose that the following three conditions are satisfied:

(i) (W, Ω, G) *is almost Kähler namely there exists an almost complex structure* J *such that:*

$$\Omega(X, Y) = G(X, JY), \quad G(JX, JY) = G(X, Y);$$

(ii) $\dim W \geq \max(2 \dim V + 4, 3 \dim V)$*;*

(iii) the map f_0 *is* g*-short, namely*

$$g - f_0^*(G) > 0.$$

Then there exists a C^1-symplectic and isometric embedding, $f : (V, \omega, g) \to (W, \Omega, G)$, arbitrarly C^0-close to f_0.

Remarks

1. If also the manifold (V, ω, g, j) is almost Kähler then it cannot exist and embedding $f_0 : V \to W$ g-short (i.e. $g - f_0^* G > 0$) and symplectic (i.e. $f_0^*(\Omega) = \omega$).

2. If (V, ω, g, j) and (W, Ω, G, J) are Kähler then a map $f : V \to W$ which satisfies:

$$f^*(\Omega) = \omega, \quad f^*(G) = g$$

is necessarily <u>holomorphic</u>.

3. The hypothesis that f_0 is g-short is <u>necessary</u>.

Corollary 4 *Let g be a metric on $\mathbb{R}^{2n}$ such that $g > g_{can}$. If $N \geq max(2n+2, 3n)$, then there exists a C^1-embedding, $f : \mathbb{R}^{2n} \to \mathbb{R}^{2N}$, which satisfies:*

$$f^*(G_{can}) = g, \;\; f^*(\Omega_{can}) = \omega_{can}.$$

Corollary 5 *Let (V, ω, g) be a compact symplectic manifold with ω integral. Then there exist N,λ sufficiently large and a C^1-embedding, $f : V \to \mathbb{C}P^N$, such that:*

$$f^*(G_{FS}) = \lambda g, \;\; f^*(\Omega_{FS}) = \omega.$$

MAIN INGREDIENTS OF THE PROOF OF THEOREM 3

Let us consider the perturbation $f_{\text{pert}} : V \to W$ of f_0 on B defined by

$$f_{\text{pert}} = \tilde{f}_0 \circ s_{|B}$$

and $f_{\text{pert}} = f_0$ outside B, where $s : V \to V \times \mathbb{R}^2$ is a section of the trivial bundle $V \times \mathbb{R}^2 \to V$ with $\operatorname{supp} s \subset B$.

Key question: Under which conditions

$$f^*_{\text{pert}}(\Omega) = \omega? \tag{5}$$

Answer: If the normal extension $\tilde{f}_0$ di f_0 is "symplectic", manely

$$\tilde{f}^*_0(\Omega) = p^*(\omega) \tag{6}$$

(where $p : B \times \mathbb{R}^2 \to B$ is the projection on the first factor) then the equation (5) is satisfied.

We then have to look for maps $f_0 : V \to W$ admitting *a normal symplectic extension* namely a smooth map $\tilde{f}_0 : B \times \mathbb{R}^2 \to W$ such that:

- $\tilde{f}_0$ is a normal extension of f_0 on B;
- $\tilde{f}_0^*(\Omega) = p^*(\omega)$, where $p : B \times \mathbb{R}^2 \to B$ is the projection in the first factor.

An embedding $f : V \to (W, J)$ is *totally real* if $(J \circ df(TV)) \cap df(TV) = 0$.

Normal symplectic extension Lemma
Let $f_0 : V \to W$ be a totally real and symplectic embedding, $B \subset V$ a contractible subset of V. If $dim\, W \geq 2\, dim\, V + 4$ then there exists a normal symplectic extension of f_0 on B.
Proof: Assume $B \subset W$.

- Since $B \subset W$ is totally real in W then

$$\Theta = TB^{\Omega} \cap TB^{G}$$

  is a vector bundle on B and rank $\Theta = dimW - 2\dim V \geq 4$.

- There exists two vector fields ∂_1 and ∂_2 on W along B which are G-ortogonali to B, G-orthonormal, Ω-ortogonali to B and mutually Ω-orthogonal.
- Extend ∂_1 to an Hamiltonian vector field $\tilde{\partial}_1$ on W and consider the one parametergroup of diffeomorphism generated by $\tilde{\partial}_1$ which we denote by $\varphi_{1,x}$, $x \in [-\epsilon, \epsilon]$.
- The vector field $(\bar{\partial}_2)_{\varphi_{1,x}(b)} = (D\varphi_{1,x})_b(\partial_2)_b$. is Ω- orthogonal to $\varphi_{1,x}(B)$.
- Extend $\bar{\partial}_2$ to an Hamiltonian vector field $\tilde{\partial}_2$ on W.
- Let $\varphi_{2,y}$, $y \in [-\epsilon, \epsilon]$, the corresponding one parameter group of diffeomorphisms.

  Then the map

$$\tilde{f}_0 : B \times (-\epsilon, \epsilon)^2 \cong B \times \mathbb{R}^2 \to W :$$

$$(b, x, y) \mapsto \varphi_{2,y}(\varphi_{1,x}(b)).$$

is a normal symplectic extension of f_0 on B.

$\square$

We have then proved our Theorem 3 for the totally real map without the assumption $\dim W \geq 3 \dim V$:

Theorem 4 *Let (V, ω) and (W, Ω) two symplectic manifolds equipped with Riemannian metrics g and G respectively and let $f_0 : V \to W$ be a C^∞ symplectic and totally real. embedding. Let us suppose that the following three conditions are satisfied:*

(i) (W, Ω, G) is almost Kähler namely there exists an almost complex structure J such that:
$\Omega(X, Y) = G(X, JY), \quad G(JX, JY) = G(X, Y);$

(ii) dim $W \geq 2$ dim $V + 4$ (without the assumption dim $W \geq 3$ dim V);

(iii) the map f_0 is g-short, namely

$$g - f_0^*(G) > 0.$$

Then there exists a C^1-symplectic and isometric embedding, $f : (V, \omega, g) \to (W, \Omega, G)$, arbitrarly C^0-close to f_0.

The next step is to show that if $\dim W \geq 3 \dim V$ then there are a "lot " of totally rela maps.

Proposition 1 *If* $\dim W \geq 3 \dim V$*. Then the generic map* $f : V \to W$ *is totally real.*

Proof:

- The idea of the proof is to interpret the fact that a map $f : V \to W$ is not totally real as a singularity of the space $\mathrm{J}^1(V, W)$ of 1-jets.
- The space $J^1(V, W)$ is a bundle over $V \times W$ and the fibers $J^1_{v,w}$ are the linear maps $L : T_vV \to T_wW$. The 1-jet J^1_f of a map $f : V \to W$ in a point $v \in V$ is given by the differential in v of f, i.e. $J^1_f(v) = df_v : T_vV \to T_{f(v)}W \in J^1_{v,f(v)}$.
- The set $\Sigma_{v,w} \subset J^1_{v,w}$ given by the 1-jets of non totally real maps is a stratified manifold of codimension $\dim W - 2\dim V + 2$.

  Indeed $\Sigma_{v,w}$ can be identified with the set of homomorphisms in $\mathrm{Hom}(\mathbb{R}^m, \mathbb{C}^n)$ which take $(e_1, \ldots, e_m)$ to

m $\mathbb{C}$-dependent vectors. Moreover $\Sigma_{v,w} = \cup_{i=0}^{m-1}\Sigma_i$, where $\Sigma_i \subset \mathrm{Hom}(\mathbb{R}^m, \mathbb{C}^n)$ is the set of linear maps which takes $(e_1, \ldots, e_i)$ to i $\mathbb{C}$-independent vectors and $(e_1, \ldots, e_{i+1})$ to $i+1$ $\mathbb{C}$-dependent vectors. Σ_i is a complex submanifold of $M_{\mathbb{C}}(n,m)$ of complex codimension $(n-i)(m-i)$. Therefore $\Sigma_{u,v} \subset M_{\mathbb{C}}(n,m)$ is a stratified manifold of complex codimension $(n-(m-1))(m-(m-1)) = n-m+1$.

- The set $\Sigma = \cup_{(v,w)\in V\times W}\Sigma_{v,w} \subset \mathrm{J}^1(V,W)$ fibers on $V \times W$ and so it is a stratified manifold in $\mathrm{J}^1(V,W)$ of codimension $\dim W - 2\dim V + 2$.
- By Thom's transversality theorem a generic map $f : V \to W$ satisfies $J^1_f(V) \cap \Sigma = \emptyset$ (i.e. is totally real) iff $\dim W - 2\dim V + 2 \geq \dim V + 1$ which is equivalent to $\dim W \geq 3\dim V$ being V and W even dimensional.

$\square$

Conclusion of the proof

In order to proof Theorem 3 we must get rid of the hypothesis that f_0 is totally real by using the inequality $\dim W \geq 3 \dim V$.

The idea is to sobstitute to the map $f_0 : V \to W$ (which is g-short and such that $f_0^*(\Omega) = \omega$) a map $f_0^{tot} : V \to W$ totally real, g-short and such that $f_0^{tot*}(\Omega) = \omega$.

The main steps are the following:

- By Proposition 1 there exists an homotopy $\varphi_t : V \to W$ between $\varphi_0 = f_0$ and a totally real map $\varphi_1 = f_1 : V \to W$ such that all the $\varphi_t : V \to W$ are C^∞-close to f_0.
- By applying Moser's theorem to the forms $\omega_t = \varphi_t^*(\Omega)$, $t \in [0, 1]$, we find a diffeomorphism $\delta : V \to V$, C^∞-close to Id_V and such that $\delta^*(\omega_1) = \omega$.
- The map $f_0^{tot} = f_1 \circ \delta : V \to W$ is an embedding totally real, C^∞-close to f_0 (and so g-short) such that $f_0^{tot*}(\Omega) = \omega$.