Rigidity properties of holomorphic isometries into homogeneous Kähler manifolds

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Aim of the talk: *provide an overview of the main rigidity phenomena on holomorphic isometries of Kähler manifolds into complex space forms and their extensions to homogeneous Kähler manifolds (hKm).*

Let (M, g) be and (S, g_s) be two Kähler manifolds. A *holomorphic isometry* $f : M \to S$ is a holomorphic map such that $f^* g_S = g$.

Remark. *No assumptions on the topology of the manifolds M; a holomorphic isometry $f : M \to S$ is not assumed to be injective or an embedding.*

Organization of the talk

1. Classification of hKm and rigidity phenomena

2. Known results on rigidity phenomena

3. Some recent results

1. Class. of hKm and rigidity phenomena

Classification of homogeneous Kähler manifolds

Theorem FC (J. Dorfmeister, K. Nakajima, 1988) *A hKm (S, g_S) is the total space of a holomorphic fiber bundle over a homogeneous bounded domain (hbd) (Ω, g_Ω). Moreover the fiber $\mathcal{E} \times \mathcal{C}$ is (with the induced Kähler metric) the Kähler product of a flat homogeneous Kähler manifold $(\mathcal{E}, g_\mathcal{E})$ and a generalized generalized flag manifold $(\mathcal{C}, g_\mathcal{C})$.*

$$\mathcal{E} \times \mathcal{C} \overset{\text{hol. isom.}}{\hookrightarrow} \begin{array}{c} S \\ \pi\downarrow \\ \Omega \end{array}$$

Remark. $S \overset{top}{=} \mathcal{E} \times \mathcal{C} \times \Omega$ as a complex manifold and

$(S, g_S) \cong [(\mathcal{E}, g_{\mathcal{E}}) \times (\mathcal{C}, g_{\mathcal{C}})] \rtimes (\Omega, g_\Omega)$ as a Kähler manifold.

Definition. $(\mathcal{E}, g_{\mathcal{E}}), (\mathcal{C}, g_{\mathcal{C}}), (\Omega, g_\Omega)$ are said hKm of different type.

Generalized flag manifolds

A generalized flag manifold $(\mathcal{C}, g_C)$ is a compact and *simply-connected* hKm.

Given a generalized flag manifold $(\mathcal{C}, g_C)$ there exists a homogeneous Kähler-Einstein metric g on M whose associated Kähler form is cohomologous to $\omega_{\mathcal{C}}$ (the Kähler form associated to $g_{\mathcal{C}}$).

The Hermitian symmetric spaces of compact type are special case of generalized flag manifolds.

Homogeneous bounded domains

A homogeneous bounded domain (Ω, g_Ω) (hbd) is a domain $\Omega \subset \mathbb{C}^N$ with a homogeneous Kähler metric g_Ω.

A bounded symmetric domain $\Omega \subset \mathbb{C}^N$ is a hbd (Ω, g_Ω) where the geodesic symmetry $\exp_x(v) \mapsto \exp_x(-v)$ is a holomorphic isometry. In this case $g_\Omega = g_B$ is the Bergman metric whose associated Kähler form $\omega_B = \frac{i}{2}\partial\bar{\partial}\log K$, where K is the reproducing kernel for the Hilbert space of holomorphic L^2-functions on Ω.

Remark. There exist hbd (Ω, g_B) which are not bounded symmetric domains (Pyatetskii-Shapiro, 1969).

A complex space form (S, g_c) is a Kähler manifold with constant holomorphic sectional curvature $H(g_c) = c$. Up to homotheties a complete and simply-connected complex space form is biholomorphically isometric to one of the following:

<u>Complex Euclidean space</u> $(\mathbb{C}^N, g_0)$

<u>Complex hyperbolic space</u> $(\mathbb{C}H^N, g_{hyp})$

<u>Complex projective space</u> $(\mathbb{C}P^N, g_{FS})$

Rigidity phenomena

R1. Rigidity of holomorphic isometries Given a Kähler manifold (M, g) and a hKm (S, g_s) how does the space of holomorphic isometries $\{f : M \to S \mid f^* g_S = g\}$ look like?

R2. Inducing canonical metrics. Assume that $f : M \to S$ is a holomorphic isometry and $g = f^* g_S$ is *canonical* (H(g)=const, homogeneous, KE, KRS) what can be said on M and on the map f?

R3. Classification of relatives hKm Two Kähler manifolds (S_1, g_1) and (S_2, g_2) are said to be relatives if there exist a (non trivial) Kähler manifold (M, g) and two holomorphic isometries $f_i : (M, g) \to (S_i, g_i)$, i.e. $f_i^*(g_i) = g$, $i = 1, 2$.

2. Known results on rigidity phenomena

R1 On the rigidity of holomorphic isometries

Theorem (Calabi, Ann. Math. 1953) *Let* (M, g) *be a Kähler manifold. Given two holomorphic isometries* $f_i : (M, g) \to (S, g_c)$, $i = 1, 2$ *into a complete and simply-connected complex space form* (S, g_c) *then there exists a congruence (i.e. a biholomorphic isometry)* $\mathcal{U}$ *of* (S, g_c), *such that* $f_2 = \mathcal{U} \circ f_1$.

Theorem* (M. Green, JDG 1978) *Let* (N, h) *and* (M, g) *be two real analytic Kähler manifolds. Assume that* (N, h) *is simply-connected and complete. Then a generic holomorphic isometry* $f : (M, g) \to (N, h)$ *is determined, up to a congruence in* (N, h).

*This theorem extends Griffiths' result, Duke Math. J. 1974, for maps into complex grassmannians.

Example (M. Green, JDG. 1978) *Consider the holomorphic maps*

$$f_1 : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^2, [z_0, z_1] \to \left([z_0, z_1], [z_0^4, z_0^2 z_1^2, z_1^4]\right)$$
$$f_2 : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^2, [z_0, z_1] \to \left([z_0^3, z_1^3], [z_0^2, z_0 z_1, z_1^2]\right)$$

In affine coordinates they read as
$f_1 : \mathbb{C} \to \mathbb{C}^3, z \to \left(z; z^2, z^4\right)$,
$f_2 : \mathbb{C} \to \mathbb{C}^3, z \to (z^3; z, z^2)$

$f_1^*(\omega_{FS} \oplus \omega_{FS}) = \frac{i}{2}\partial\bar{\partial}\log(1 + |z|^2) + \frac{i}{2}\partial\bar{\partial}\log(1 + |z|^4 + |z|^8) = \frac{i}{2}\partial\bar{\partial}\log\left[(1 + |z|^2)(1 + |z|^4 + |z|^8)\right] = \frac{i}{2}\partial\bar{\partial}\log(1+|z|^2+|z|^4+|z|^6+|z|^8 + |z|^{10})$

and

$f_2^*(\omega_{FS} \oplus \omega_{FS}) = \frac{i}{2}\partial\bar{\partial}\log(1 + |z|^6) + \frac{i}{2}\partial\bar{\partial}\log(1 + |z|^2 + |z|^4) = \frac{i}{2}\partial\bar{\partial}\log\left[(1 + |z|^6)(1 + |z|^2 + |z|^4)\right] = \frac{i}{2}\partial\bar{\partial}\log(1+|z|^2+|z|^4+|z|^6+|z|^8 + |z|^{10})$

Therefore $f_1^*(\omega_{FS} \oplus \omega_{FS}) = f_2^*(\omega_{FS} \oplus \omega_{FS})$

On the other hand in $H_2(\mathbb{C}P^1 \times \mathbb{C}P^2, \mathbb{Z}) \cong \mathbb{Z}^2$ *the homology classes are given by*

$$[f_1(\mathbb{C}P^1)] = (1, 4), \ \ [f_2(\mathbb{C}P^1)] = (3, 2)$$

hence there is not a biholomorphic isometry U *of* $\mathbb{C}P^1 \times \mathbb{C}P^2$ *such that* $f_2 = U \circ f_1$ *even at topological grounds*

Theorem *(Mok, J. Eur. Math. Soc. 2012)* Any holomorphic isometry $f : (B_1, g_1) \to (B_2, g_2)$ between bounded symmetric domains with B_1 irreducible of rank ≥ 2 is totally geodesic.

Example *(N. Mok, J. Eur. Math. Soc. 2012)* Let

$$\mathcal{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$$

be the upper half-plane with metric $g_{\mathcal{H}}$ whose associated Kähler form is given by

$$\omega_{\mathcal{H}} = -\frac{i}{2}\partial\bar{\partial}\log \Im z$$

The two holomorphic maps

$$f_1 : \mathcal{H} \to \mathcal{H}^2, z \mapsto (z, i)$$

and

$$f_2 : \mathcal{H} \to \mathcal{H}^2, z \mapsto (\sqrt{z}, i\sqrt{z})$$

are isometric embeddings, i.e. $f_i^*(g_{\mathcal{H}} \oplus g_{\mathcal{H}}) = g_{\mathcal{H}}$, $i = 1, 2$.

This is clear for f_1. The proof for f_2 is as follows:

$$f_2^*(\omega_{\mathcal{H}} \oplus \omega_{\mathcal{H}}) = -\frac{i}{2}\partial\bar{\partial} \log \Im(\sqrt{z}) - \frac{i}{2}\partial\bar{\partial} \log \Im(i\sqrt{z}) =$$

$$-\frac{i}{2}\partial\bar{\partial} \log \left(\Im(\sqrt{z})\Im(i\sqrt{z})\right) = -\frac{i}{2}\partial\bar{\partial} \log \left[\frac{\sqrt{z} - \sqrt{\bar{z}}}{2i} \cdot \frac{i\sqrt{z} + i\sqrt{\bar{z}}}{2i}\right] =$$

$$-\frac{i}{2}\partial\bar{\partial} \log \left[\frac{z - \bar{z}}{4i}\right] = -\frac{i}{2}\partial\bar{\partial} \log \left[\frac{z - \bar{z}}{2i}\right] = -\frac{i}{2}\partial\bar{\partial} \log \Im z = \omega_{\mathcal{H}}$$

Notice that f_1 is totally geodesic while f_2 is not. Hence they are not related by a biholomorphic isometry of $\mathcal{H}^2$.

Since $(\mathbb{C}H^1, g_{hyp}) \cong (\mathcal{H}, g_{\mathcal{H}})$ it follows that

$$f = (h \times h)^{-1} \circ f_2 \circ h : (\mathbb{C}H^1, g_{hyp}) \to (\mathbb{C}H^1 \times \mathbb{C}H^1, g_{hyp} \oplus g_{hyp})$$

is a non-totally geodesic holomorphic isometric embedding.

Conjecture A: *Complex space forms are characterized by Calabi's rigidity among complete and simply-connected hKm.*

2. Known results on rigidity phenomena

R2 On inducing canonical metrics

When (M, g) is a complex space form

Theorem (Calabi, Ann. Math. 1953) *Let* $f : (M, g) \rightarrow (S, g_c)$ *be a holomorphic isometry between a complex space form* (M, g) *and a complex space form* (S, g_c). *Then sign*$(H(g)) =$ *sign*(c). *If* $c \leq 0$ *then the immersion is totally geodesic.*

Example *Let* $k \in \mathbb{Z}^+$ *and Then the holomorpic map*

$$(\mathbb{C}P^1, kg_{FS}) \xrightarrow{V_k} (\mathbb{C}P^k, g_{FS}) : [z_0, z_1] \mapsto [z_0^k \dots, \sqrt{\binom{k}{j}} z_0^{k-j} z_1^j, \dots, z_1^k]$$

satisfies (in affine coordinates)

$$V_k^* \omega_{FS} = \frac{i}{2} \partial\bar{\partial} \log(\sum_{j=0}^{k} \binom{k}{j} |z|^{2j}) =$$

$$\frac{i}{2} \partial\bar{\partial} \log(1 + |z|^2)^k = k \frac{i}{2} \partial\bar{\partial} \log(1 + |z|^2) = k\omega_{FS}$$

and hence is a holomorphic isometric embedding, i.e.

$$V_k^* g_{FS} = k g_{FS}.$$

Moreover, one can construct a holomorphic isometric embedding

$$V_{k,n} : (\mathbb{C}P^n, k g_{FS}) \to (\mathbb{C}P^{N_k}, g_{FS})$$

by a suitable normalization of the Veronese embedding.

: :

When (M, g) is homogeneous

Theorem (-, A. J. Di Scala, H. Ishi, Asian J. Math. 2012) *Let $f : (M, g) \to (S, g_c)$ be a holomorphic isometry of a hKm (M, g) into a complex space form (S, g_c) with $c \leq 0$. Then f is totally geodesic.*

Theorem† *(-, A. J. Di Scala, H. Ishi, Asian J. Math. 2012)* Let (M, g) be a hKm such that its associated Kähler form is integral. There exists a holomorphic isometry embedding $f : (M, g) \to (\mathbb{C}P^N, g_{FS})$ iff (M, g) is a generalized flag manifold

†This theorem extends a classical result of H. Nakagawa and R. Takagi, J. Math. Soc. Japan 1976, where the authors prove that up to homotheties an Hermitian symmetric space of compact type admits a holomorphic isometry into a complex projective space.

When g is Kähler-Einstein (KE)

Theorem *(M. Umehara, Tohoku Math. J., 1987)* Let $f : (M, g) \to (S, g_c)$ be a holomorphic isometry with g KE into a complex space form with $c \leq 0$. Then f is totally geodesic.

Theorem (B. Smyth, Ann. Math. 1967) *A compact KE manifold of complex dimension n which admits a holomorphic isometric embedding into $(\mathbb{C}P^{n+1}, g_{FS})$ is totally geodesic or the complex quadric $Q = \{Z_0^2 + \cdots + Z_{n+1}^2 = 0\}$.*

Theorem (S. S. Chern, JDG 1967) *A KE manifold of complex dimension n which admits a holomorphic isometry into $(\mathbb{C}P^{n+1}, g_{FS})$ is totally geodesic or an open subset of the complex quadric.*

Theorem (K. Tsukada, Math. Ann. 1986) *A KE manifold of complex dimension n which admits a holomorphic isometry into $(\mathbb{C}P^{n+2}, g_{FS})$ is totally geodesic or an open subset of the complex quadric.*

Theorem (J. Hano, Math. Ann. 1975) *Let $M \subset \mathbb{C}P^N$ be a complete intersection. If the restriction of g_{FS} to M is Einstein then M is totally geodesic or the complex quadric.*

Theorem (D. Hulin, Journal of Geom. Anal. 2000) *A compact KE manifold which admits a holomorphic isometry into $(\mathbb{C}P^N, g_{FS})$ has positive scalar curvature.*

Open problem: *Drop the compactness assumption.*

Conjecture B: *A KE manifold which admits a holomorphic isometry into a complex projective space is an open subset of a generalized flag manifold (the immersion is an embedding and the Einstein constant is positive).*

When g is a Kähler-Ricci soliton[‡] (KRS)

Theorem (L. Bedulli and A. Gori, PAMS 2014) Let $M \subset \mathbb{C}P^N$ be a complete intersection. If the restriction g of g_{FS} to M is a KRS then g is KE.

Theorem (–, R. Mossa, PAMS 2021) *Let (g, X) be a KRS on complex manifold M. If there exists a holomorphic isometry $f : (M, g) \to (S, g_c)$ into a complex space form then g is KE.*

Theorem (-, R. Mossa, to appear in PAMS) *Let (g, X) be a KRS on complex manifold M. If there exists a holomorphic isometry $f : (M, g) \to (\Omega, g_\Omega)$ into a homogeneous bounded domain (Ω, g_Ω) then g is KE.*

[‡] A KRS is a Kähler metric g such that $Ric_g = \lambda g + L_X g$, where X (the *solitonic vector field*) is the real part of a holomorphic vector field.

Putting together the previous results we get:

Corollary Let (g, X) be a KRS on complex manifold M. If (M, g) admits a holomorphic isometry into one of the following hKm: a flat space $(\mathcal{E}, g_{\mathcal{E}})$, a generalized flag manifold of integral type $(\mathcal{C}, g_{\mathcal{C}})$ (in particular a KE generalized flag manifold), a homogeneous bounded domain (Ω, g_{Ω}). Then g is KE.

Q1. *Is it true that a KRS induced by any generalized flag manifold is KE?*

2. Known results on rigidity phenomena

R3 On the classification of relatives hKm

Definition *Two hKm* (S_1, g_1) *and* (S_2, g_2) *are said to be relatives if there exist a (non trivial) Kähler manifold* (M, g) *and two holomorphic isometries* $f_i : (M, g) \to (S_i, g_i)$.

Theorem (M. Umehara, Tokyo J. Math 1987) Two complex space forms of different type are not relatives.

Theorem *(–, A. J. Di Scala, Ann. Scuola Norm. Sup. Pisa 2010) A bounded homogeneous domain* (Ω, g_B) *equipped with its Bergman metric* g_B *is not relative to the complex projective space*. □

Theorem (R. Mossa, Mat. Univ. Parma 2013) *A bounded homogeneous domain (Ω, g_Ω) is not relative the the complex projective space.* □

Theorem (X. Cheng, Y. Hao, J. of Global Anal. and Geom. 2021) *A bounded homogeneous domain (Ω, g_B) equipped with its Bergman metric g_B is not relative to the flat space $\mathbb{C}^n$.*

Theorem (–, R. Mossa, to appear in PAMS) *A bounded homogeneous domain (Ω, g_Ω) is not relative the flat space $\mathbb{C}^n$.*

Putting together the previous results we get:

Corollary *Any two among a flat space, a generalized flag manifold of integral type and a homogeneous bounded domain are not relatives.*

Q2. *Can we extend the previous corollary to any generalized flag manifold? Equivalently is it true that two hKm of different type are not relatives?*

3. Some recent results

Q1. *Is it true that a KRS induced by a generalized flag manifold is KE?*

Theorem (–, R. Mossa, 2023) *Let (g, X) be a KRS on complex manifold M. If (M, g) admits a holomorphic isometry into a generalized flag manifold of classical type $(\mathcal{C}, g_{\mathcal{C}})$ then g is KE.*

Q1bis. *Is it true that a KRS induced by a hKm is KE?*

Example (negative answer to Q1bis) *The vector field X on $\mathbb{C} \times \mathbb{C}P^1$ (resp. $\mathbb{C} \times \mathbb{C}H^1$) defined by*

$$X(z, q) := -2\left(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}\right)$$

$$(\textit{resp. } X(z,q) := 2\left(z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}\right)$$

satisfies $Ric_g = 4g + L_X g$ *(resp.* $Ric_g = -4g + L_X g$ *and so it is a solitonic vector field for the metric* $g = g_0 \oplus g_{FS}$ *(resp.* $g = g_0 \oplus g_{hyp}$ *) and the metric* g *is not KE.*

Theorem *(–, R. Mossa, 2023)* Let (g, X) be a KRS on complex manifold M, $(\mathcal{C}, g_{\mathcal{C}})$ be a manifold either of integral type or classical type and (Ω, g_Ω) a homogeneous bounded domain . If (M, g) admits a holomorphic isometry into the Kähler product $(\mathcal{C}, g_{\mathcal{C}}) \times (\Omega, g_\Omega)$ then g is KE.

3. Some recent results

Q2. *Is it true that two hKm of different type are not relatives?*

Theorem (–, R. Mossa) *A generalized flag manifold of classical type $(\mathcal{C}, g_{\mathcal{C}})$ and a homogeneous bounded domain (Ω, g_{Ω}) are not relatives.*

Theorem (–, R. Mossa, 2023) *Let $(\mathcal{E}, g_{\mathcal{E}})$, $(\mathcal{C}, g_{\mathcal{C}})$ and (Ω, g_{Ω}) be respectively a flat manifold, a generalized flag manifold of integral or classical type and a homogeneous bounded domain. Then the following facts hold true:*
1. $(\mathcal{E}, g_{\mathcal{E}})$ is not relative to the Kähler product $(\mathcal{C}, g_{\mathcal{C}}) \times (\Omega, g_{\Omega})$;
2. $(\mathcal{C}, g_{\mathcal{C}})$ is not relative to the Kähler product $(\mathcal{E}, g_{\mathcal{E}}) \times (\Omega, g_{\Omega})$;
3. (Ω, g_{Ω}) is not relative to the Kähler product $(\mathcal{E}, g_{\mathcal{E}}) \times (\mathcal{C}, g_{\mathcal{C}})$.

Remark *Notice that* $(\mathbb{C}P^1, g_{FS})$ *and* $(\mathbb{C}P^1, 2g_{FS})$ *are not relatives On the other hand* $(\mathbb{C}P^1 \times \mathbb{C}P^1, g_{FS} \oplus g_{FS})$ *and* $(\mathbb{C}P^1, 2g_{FS})$ *are relatives (the holomorphic map* $\varphi : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1, q \mapsto (q, q)$ *satisfies* $\varphi^*(g_{FS} \oplus g_{FS}) = 2g_{FS}$*). Hence in the previous theorem we cannot restrict to single factors to prove the results.*

Recalling that

$$(S, g_S) \cong [(\mathcal{E}, g_{\mathcal{E}}) \times (\mathcal{C}, g_{\mathcal{C}})] \rtimes (\Omega, g_\Omega)$$

we have the following

Project: *Let* $(\mathcal{E}, g_{\mathcal{E}})$*,* $(\mathcal{C}, g_{\mathcal{C}})$ *and* (Ω, g_Ω) *be respectively a flat manifold, a generalized flag manifold and a homogeneous bounded domain. Then:*
1. $(\mathcal{E}, g_{\mathcal{E}})$ *is not relative to the twist product* $(\mathcal{C}, g_{\mathcal{C}}) \rtimes (\Omega, g_\Omega)$*;*
2. $(\mathcal{C}, g_{\mathcal{C}})$ *is not relative to the twist product* $(\mathcal{E}, g_{\mathcal{E}}) \rtimes (\Omega, g_\Omega)$*;*
3. (Ω, g_Ω) *is not relative to the Kähler product* $(\mathcal{E}, g_{\mathcal{E}}) \times (\mathcal{C}, g_{\mathcal{C}})$*.*

Sketch of some proofs

Theorem *(-, A. J. Di Scala, H. Ishi, Asian J. Math. 2012)* Let (M, g) be a hKm such that its associated Kähler form is integral. There exists a holomorphic isometry embedding $f : (M, g) \to (\mathbb{C}P^N, g_{FS})$ iff (M, g) is a generalized flag manifold

Sketch of the proof *The integrality of $\omega = f^*\omega_{FS}$ is immediate since ω_{FS} is integral. Moreover, M is compact by a result of M. Takeuchi (Japan. J. Math. 1978)*

Homogeneity+Calabi $\Rightarrow$ $\not{\mathbb{T}}^k \times (\mathcal{C}, g_{\mathcal{C}}) = (M, g) \stackrel{f}{\to} (\mathbb{C}P^N, g_{FS}) \Rightarrow$

$M = C$ is *simply-connected.*

Calabi's rigidity $\Rightarrow$ $f \circ g = \mathcal{U}_g \circ f$, $\forall g$ *biholomorphic isometry* $\Rightarrow$ $f(M)$ *is a hKm* $\Rightarrow$ $f(M) \subset \mathbb{C}P^N$ *is simply-connected.*

$f : M \to f(M)$ *is a local isometry* $\Rightarrow$ f *is a covering map* $\Rightarrow$ f *is injective.*

Now, let (M, g) *be a compact and simply-connected hKm (generalized flag) with* ω *integral we want to show that there esists a holomorphic isometric embedding* $(M, g) \to (\mathbb{C}P^N, g_{FS})$ *construct through the following steps.*

1. Let L *be a holomorphic line bundle with* $c_1(L) = [\omega]$*;* L *is very ample since* (M, g) *is a generalized flag manifold.*

2. *Let h be an Hermitian metric on L such that Ric$(h) = \omega$ (here Ric$(h) = -\frac{i}{2}\partial\bar{\partial} \log h(\sigma(x), \sigma(x))$, where $\sigma : U \to L$ is a trivialising holomorphic section of L).*

3. *Consider the smooth function on M given by:*

$$\epsilon(x) = \sum_{j=0}^{N} h(s_j(x), s_j(x)),$$

where $\{s_0, \ldots, s_N\}$ is an orthonormal basis of $(H^0(L), \langle\cdot,\cdot\rangle)$,

$$\langle s, t\rangle = \int_M h(s(x), t(x)) \frac{\omega^n}{n!}$$

This function is invariant by the biholomorphic isometry of M which lifts to L.

$\pi_1(M) = \{1\} \Rightarrow \epsilon(x)$ *is invariant by the biholomorphic isometries of* M

homogeneity $\Rightarrow \epsilon(x)$ *is a positive constant.*

4. Therefore the Kodaira map

$$\varphi : M \to \mathbb{C}P^N, x \mapsto [s_0(x) : \cdots : s_N(x)]$$

is an embedding (since L *is very ample) and satisfies* $\varphi^*\omega_{FS} = \omega + \frac{i}{2}\partial\bar{\partial}\log\epsilon = \omega$. $\square$

Theorem *(–, A. J. Di Scala, Ann. Scuola Norm. Sup. Pisa 2010) A bounded homogeneous domain (Ω, g_B) equipped with its Bergman metric g_B is not relative to the complex projective space*.

Idea of the proof (using Calabi's rigidity): assume by contradiction that there exist $U \subset \mathbb{C}$ and two holomorphic maps $f_1 : U \to (\Omega, g_B)$ and $f_2 : U \to \mathbb{C}P^N$ such that $f_1^* g_B = f_2^* g_{FS}$. Then by using a holomorphic isometry

$$f : (\Omega, g_B) \to (\mathbb{C}P^\infty, g_{FS})$$

one can construct the holomorphic map

$$h_1 = f \circ f_1 : U \to \mathbb{C}P^\infty$$

such that $h_1^* g_{FS} = f_2^* g_{FS}$ and show that h_1 is full. This contradicts Calabi's rigidity. $\square$

When g is cscK or extremal[§]

Theorem (Kobayashi, JDG 1967) Let M be a compact complex n-dimensional manifold equipped with a cscK Kähler metric g. If there exists a holomorphic isometric embedding[¶] of (M, g) nto into $(\mathbb{C}P^{n+1}, g_{FS})$ then g is KE (and hence totally geodesic or the complex quadric).

Conjecture C: Let M be a complex manifold equipped either with a cscK or extremal Kähler metric g. If (M, g) admits a holomorphic isometry into a finite dimensional complex space form (S, g_c) then g is KE.

[§]The $(1,0)$-part of the Hamiltonian vector field associated to the scalar curvature of g is holomorphic.

[¶]Extended to immersion by M. Kon, J. Math. Soc. Japan 1975

Remark *There exist complex manifolds equipped either with a cscK or extremal non-KE metric admitting holomorphic isometric embedding into infinite dimensional complex space forms (–, F. Salis, F. Zuddas, Journal of Geom. Anal. 2021)*

Thank you for your attention!