Institut Fourier de Mathématiques
24 Janvier 2012, Grenoble

Kähler immersions of homogeneous Kähler manifolds into complex space forms

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Aim. *Classify all homogeneous Kähler manifolds which admit a Kähler immersion into a given finite or infinite dimensional complex space form.*

Kähler manifolds

Let (M, g, ω, J) be a Kähler manifold of complex dimension n.

$\omega(X, Y) = g(X, JY)$, $X, Y \in \mathfrak{X}(M)$, $d\omega = 0$.

The form ω is called the *Kähler form associated to the metric* g.

On a contractible open set $U \subset M$

$$\omega = \frac{i}{2}\partial\bar{\partial}\Phi = \frac{i}{2}\sum_{j=1}^{n}\frac{\partial^2\Phi}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k,$$

where $\Phi : U \to \mathbb{R}$ is a strictly PSH function called a *Kähler potential* for the metric g.

Complex space forms

A *complex space form* $(S, g_S) = (S, g_S, \omega_S, J_S)$ is a finite or infinite dimensional Kähler manifold of constant holomorphic sectional curvature.

Classification of complex space forms

Euclidean space $\mathbb{C}^{N\leq\infty} := (\mathbb{C}^{N\leq\infty}, g_0)$

$\mathbb{C}^\infty := \ell^2(\mathbb{C})$ $(z = \{z_j\} \in \ell^2(\mathbb{C})$ iff $\sum_{j=1}^{\infty} |z_j|^2 < \infty)$

$\omega_0 = \frac{i}{2}\partial\bar{\partial}|z|^2 = \frac{i}{2}\sum_{j=1}^{N} dz_j \wedge d\bar{z}_j,\ |z|^2 = |z_1|^2 + \cdots + |z_N|^2.$

Hyperbolic space $\mathbb{C}H^{N\leq\infty} := (\{z \in \mathbb{C}^N \mid |z|^2 < 1\}, g_{hyp})$

$\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial}\log(1 - |z|^2).$

Projective space $\mathbb{C}P^{N\leq\infty} = (\mathbb{C}^{N+1} \setminus \{0\}/z \sim \lambda z, g_{FS})$

$\omega_{FS}|_{U_0} = \frac{i}{2}\partial\bar{\partial}\log(1+|z|^2),\ z_j = \frac{Z_j}{Z_0},\ j = 1, \ldots, N,\ \ U_0 = \{Z_0 \neq 0\}.$

Kähler immersions into complex space forms

Let (M, g, ω, J) be a Kähler manifold. A Kähler immersion

$$f : (M, g, J) \to (S, g_S, J_S)$$

is a holomorphic map (i.e. $df \circ J = J_S \circ df$) which is isometric (i.e. $f^* g_S = g$).

Remark The "starting" manifold M will be always finite dimensional.

Remark A Kähler immersion $f : (M, g, J) \to (S, g_S, J_S)$ is symplectic, i.e. $f^* \omega_S = \omega$. Viceversa a holomorphic and symplectic map $f : (M, \omega, J) \to (S, \omega_S, J_S)$ is isometric, i.e. $f^* g_S = g$.

Calabi's results on Kähler immersions (Ann. Math. 1953)

Theorem (Calabi's rigidity) *If* $f : (M, g) \to (S, g_S)$ *is a Kähler immersion then* g *is real analytic. Moreover, any other Kähler immersion of* (M, g) *into* (S, g_S) *is given by* $\mathcal{U} \circ f$ *where* $\mathcal{U}$ *is a unitary transformation, i.e.* $\mathcal{U} \in$ *Aut*$(S) \cap$ *Isom*(S, g_S).

Theorem (local immersions vs global immersions) *A simply-connected real-analytic Kähler manifold* (M, g) *admits a Kähler immersion into a given complex space form* (S, g_S) *iff there exists an open set* $U \subset M$ *such that* $(U, g_{|U})$ *can be Kähler immersed into* (S, g_S).

Complex space forms into complex space forms

$\mathbb{C}H^n \not\to \mathbb{C}^{N<\infty}, \mathbb{C}P^{N<\infty}$

$\mathbb{C}^n \not\to \mathbb{C}H^{N\leq\infty}, \mathbb{C}P^{N<\infty}$

$\mathbb{C}P^n \not\to \mathbb{C}H^{N\leq\infty}, \mathbb{C}^{N\leq\infty}$

$$\mathbb{C}H^n \to \ell^2(\mathbb{C}) : z \mapsto (\ldots, \sqrt{\frac{(|j|-1)!}{j!}}\; z^j, \ldots),\ \ |j| \geq 1$$

$$\mathbb{C}H^n \to \mathbb{C}P^\infty : z \mapsto (\ldots, \sqrt{\frac{|j|!}{j!}}\; z^j, \ldots),\ \ |j| \geq 0$$

$$\mathbb{C}^n \to \mathbb{C}P^\infty : z \mapsto (\ldots, \sqrt{\frac{1}{j!}}\; z^j, \ldots),\ \ |j| \geq 0$$

$z^j = z_1^{j_1} \cdots z_n^{j_n},\ |j| = j_1 + \cdots + j_n,\ j! = j_1! \cdots j_n!$

Multiplying g_{hyp} by some constant

Let $\mathbb{C}H^n_\lambda = (\mathbb{C}H^n, \lambda g_{hyp}),\ \lambda > 0,\ \mathbb{C}H^n := \mathbb{C}H^n_1 = (\mathbb{C}H^n, g_{hyp})$

$\mathbb{C}H^n_\lambda \not\to \mathbb{C}^{N<\infty}, \mathbb{C}P^{N<\infty}$

$\mathbb{C}H^n_\lambda \to \mathbb{C}H^{N\leq\infty} \ \Leftrightarrow\ \lambda = 1,\ n \leq N$

$$\mathbb{C}H^n_\lambda \to \ell^2(\mathbb{C}) : z \mapsto \sqrt{\lambda}(\ldots, \sqrt{\frac{(|j|-1)!}{j!}}\ z^j, \ldots),\ \ |j| \geq 1$$

$$\mathbb{C}H^n_\lambda \to \mathbb{C}P^\infty : z \mapsto (\ldots, \sqrt{\frac{\lambda(\lambda+1)\cdots(\lambda-1+|j|)}{j!}}\ z^j, \ldots),\ \ |j| \geq 0$$

$z^j = z_1^{j_1} \cdots z_n^{j_n},\ |j| = j_1 + \cdots + j_n,\ j! = j_1! \cdots j_n!$

Multiplying g_{FS} by some constant

$\mathbb{C}P^n_\lambda = (\mathbb{C}P^n, \lambda g_{FS}),\ \lambda > 0,\ \mathbb{C}P^n := \mathbb{C}P^n_1 = (\mathbb{C}P^n, g_{FS})$

$$\mathbb{C}P^n_\lambda \nrightarrow \mathbb{C}^{N\le\infty}, \mathbb{C}H^{N\le\infty}$$

$$\mathbb{C}P^n_\lambda \xrightarrow{f} \mathbb{C}P^{N\le\infty} \ \Leftrightarrow\ \lambda = k \in \mathbb{Z},\ N_k := \frac{(n+k)!}{n!k!} - 1 \le N$$

$$f : \mathbb{C}P^n_k \xrightarrow{V_k} \mathbb{C}P^{N_k} \xrightarrow{tot.geod.} \mathbb{C}P^N, \quad V_k^* g_{FS} = k g_{FS}$$

$$\mathbb{C}P^n_k \xrightarrow{V_k} \mathbb{C}P^{N_k} : [Z] \longmapsto [\ldots, \sqrt{\frac{|j|!}{j!}}\, Z^j, \ldots],\ |j| \ge 0$$

$Z^j = Z_0^{j_0} \cdots Z_n^{j_n},\ |j| = j_0 + \cdots + j_n,\ j! = j_0! \cdots j_n!$

Products into complex space forms

Products into the complex hyperbolic space

$$\mathbb{C}H^{n_1} \times \mathbb{C}H^{n_2} \to \mathbb{C}H^{N \leq \infty} \Leftrightarrow \ n_1 = 0 \text{ or } n_2 = 0$$

Products into the complex projective space

$$\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \stackrel{Segre}{\to} \mathbb{C}P^{(n_1 n_2 + n_1 + n_2)}, \ ([Z],[W]) \mapsto [\ldots, Z_j W_k, \ldots]$$

$$M_1 \to \mathbb{C}P^{N_1 \leq \infty}, M_2 \to \mathbb{C}P^{N_2 \leq \infty} \ \Rightarrow \ M_1 \times M_2 \to \mathbb{C}P^{N_1 N_2 + N_1 + N_2}$$

Products into the complex Euclidean space

$$f_1 : M_1 \to \mathbb{C}^{N_1 \leq \infty}, f_2 : M_2 \to \mathbb{C}^{N_2 \leq \infty} \ \Rightarrow \ f_1 \times f_2 : M_1 \times M_2 \to \mathbb{C}^{N_1 + N_2}$$

Remark (reducibility of a Kähler product into $\mathbb{C}^{N \leq \infty}$)

$$f : M_1 \times M_2 \to \mathbb{C}^N \ \Rightarrow \ f = f_1 \times f_2, f_1 : M_1 \to \mathbb{C}^{N_1}, f_2 : M_2 \to \mathbb{C}^{N_2}$$

Bounded symmetric domains into complex space forms

Theorem (L. , A.J. Di Scala, Geom. Dedicata (2007)) *Let* $\Omega \subset \mathbb{C}^n$ *be a irreducible bounded symmetric domain equipped with the Bergman metric* g_B. *Then:*

$$(\Omega, g_B) \nrightarrow \mathbb{C}^{N<\infty}$$

$$(\Omega, g_B) \nrightarrow \mathbb{C}H^{N\leq\infty}$$

$$(\Omega, g_B) \nrightarrow \mathbb{C}P^{N<\infty}$$

$$(\Omega, g_B) \rightarrow \mathbb{C}P^{\infty}$$

$$(\Omega, g_B) \rightarrow \ell^2(\mathbb{C}) \;\Leftrightarrow\; (\Omega = \mathbb{C}H^n,\ g_B = (n+1)g_{hyp})$$

Homogeneous Kähler manifolds: definitions

A homogeneous Kähler manifold (h.K.m.) is a Kähler manifold (M, g) such that the Lie group $G = \mathrm{Aut}(M) \cap \mathrm{Isom}(M, g)$ acts transitively on M.

Remark. *The metric g is not uniquely determined by G. There exist different (neither homothetic or isometric) G-invariant homogeneous metrics.*

Homogeneous bounded domains

Let $\Omega \subset \mathbb{C}^n$*,* Ω *bounded domain endowed with a homogeneous Kähler metric* g_Ω*. Then* (Ω, g_Ω) *is called a* homogeneous bounded domain *(h.b.d.).*

If $Aut(\Omega)$ *acts transitively on* $\Omega \subset \mathbb{C}^n$ *then* $(\Omega, g_\Omega = g_B)$ *is a h.b.d..*

Remark. *Every bounded symmetric domain* (Ω, g_B) *is a h.b.d. but there exist (Pyatetskii-Shapiro, 1969) h.b.d.* (Ω, g_B) *which are not bounded symmetric domains.*

Other examples of h.K.m.

Flat h.K.m. $\mathcal{E} = \mathbb{C}^k \times T_1 \times \cdots T_l$ *where* $T_j = \mathbb{C}^{n_j}/\Lambda_j$ *is a complex torus with the flat metric.*

Compact simply-connected h.K.m. These are also called Kähler C-spaces *or* Wang's spaces *or* rational homogeneous varieties.

Compact h.K.m. $(M, g) = \mathcal{C} \times T_1 \times \cdots T_l$, *$C$-space, T_j flat torus.*

Products of homogeneous Kähler manifolds The products of h.K.m. is a h.K.m.

Solution of the fundamental conjecture (FC) for h.K.m.

Theorem FC *(J. Dorfmeister, K. Nakajima, Acta Math. 1988)* A h.K.m. (M, g) is the total space of a holomorphic fiber bundle over a h.b.d. (Ω, g_Ω). Moreover the fiber $\mathcal{F} = \mathcal{E} \times \mathcal{C}$ is (with the induced Kähler metric) the Kähler product of a flat homogeneous Kähler manifold $\mathcal{E} = \mathbb{C}^k \times T_1 \times \cdots T_l$ and a C-space $\mathcal{C}$.

$$\begin{array}{ccc} \mathcal{F} = \mathcal{E} \times \mathcal{C} & \overset{\text{Kähler}}{\hookrightarrow} & (M, g) \\ & & \pi \downarrow \\ & & (\Omega, g_\Omega) \end{array}$$

Remark. $M \overset{top}{=} \Omega \times \mathcal{F}$ *as a complex manifold.*

Statements of the main results:

to appear in Asian J. Math.

joint work with A. J. Di Scala – H. Hishi

Homogeneous Kähler manifolds into $\mathbb{C}^{N\leq\infty}$

Theorem 1 Let (M,g) be a n-dimensonal h.K.m. which can be Kähler immersed into $\mathbb{C}^{N\leq\infty}$. Then

$$(M,g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l},$$

where $\mathbb{C}H^{n_r}_{\lambda_r} = (\mathbb{C}H^{n_r}, \lambda_r g_{hyp})$, $r = 1,\ldots,l$. Moreover, the immersion is given, up to a unitary transformation of $\mathbb{C}^N$ by $f_0 \times f_1 \times \cdots \times f_l$, where f_0 is the linear inclusion $\mathbb{C}^k \stackrel{tot.geod.}{\longrightarrow} \mathbb{C}^N$ and each $f_k : \mathbb{C}H^{n_k} \longrightarrow \ell^2(\mathbb{C})$ is $\sqrt{\lambda_k}$ times the map

$$z = (z_1,\ldots,z_{n_r}) \mapsto \left(\ldots, \sqrt{\frac{(j_1+\cdots+j_{n_r}-1)!}{j_1!\cdots j_{n_r}!}} z_1^{j_1}\cdots z_{n_r}^{j_{n_r}}, \ldots\right).$$

Homogeneous Kähler manifolds into $\mathbb{C}H^{N\leq\infty}$

Theorem 2 Let (M, g) be a n-dimensional h.K.m. which can be Kähler immersed into $\mathbb{C}H^{N\leq\infty}$. Then, up to a unitary transformation of $\mathbb{C}H^N$,

$$(M, g) = \mathbb{C}H^n \stackrel{tot.geod.}{\longrightarrow} \mathbb{C}H^N.$$

Two theorems on h.K.m. into $\mathbb{C}P^{N\leq\infty}$

Theorem 3 Let (M, g, ω) be a n-dimensional h.K.m. which can be Kähler immersed into (M, g) into $\mathbb{C}P^{N\leq\infty}$. *Then ω is integral, $\pi_1(M) = 1$ and the immersion is injective.*

Theorem 4 *Let (M, g, ω) be a simply-connected h.K.m. such that its associated Kähler form ω is integral. Then there exists $m_0 \in \mathbb{Z}$ such that*

$$(M, m_0 g) \to \mathbb{C}P^N.$$

Remark on the compact case

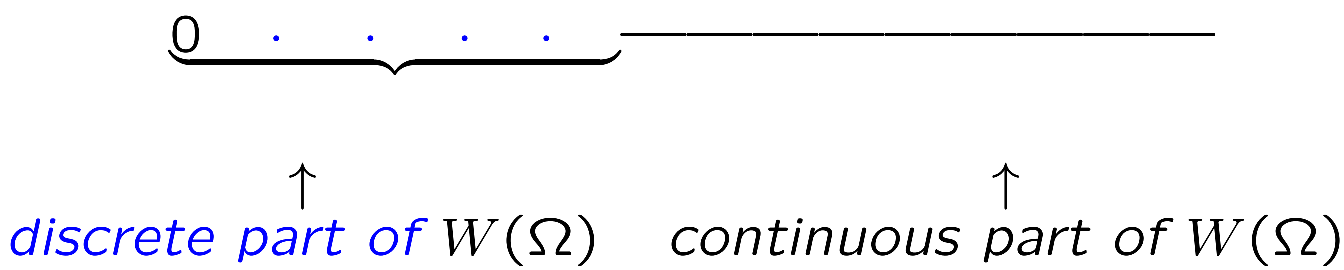
When M is compact Theorem 3 and Theorem 4 were proved by M. Takeuchi (Japan J. Math. 1978) using the theory of semisimple Lie groups and Dynkin diagrams (one can take $m_0 = 1$ in Theorem 4).

Notice that if a h.K.m. can be Kähler immersed into $\mathbb{C}P^{N<\infty}$ then M is a C-space, i.e. is a compact, simply-connected and homogeneous Kähler manifold.

Viceversa if M ia any compact (not necessarily homogeneous) Kähler manifold which can be Kähler immersed into $\mathbb{C}P^{N\leq\infty}$ one can assume $N < \infty$.

The case of a bounded symmetric domain $(\Omega, \lambda g_B)$

The Wallach set $W(\Omega) \subset \mathbb{R}$ of a irreducible bounded symmetric domain $\Omega \subset \mathbb{C}^n$ is a subset of $\mathbb{R}$ which "looks like":



Important property of the Wallach set: $W(\Omega) = \mathbb{R}$ (and hence the discrete part of $W(\Omega)$ is empty) if and only if $\Omega = \mathbb{C}H^n$.

The Wallach set and Kähler immersions into $\mathbb{C}P^\infty$

Theorem W (L. – M. Zedda, Math.Ann. 2010) *Let* (Ω, g_B) *be a irreducible bounded symmetric domain. Then* $(\Omega, \lambda g_B)$ *can be Kähler immersed into* $\mathbb{C}P^\infty$ *if and only if* $\lambda\gamma \in W(\Omega) \setminus \{0\}$*, where* $\gamma > 0$ *denotes the genus of* Ω*.*

Two consequences of Theorem W

First consequence: Let $(\Omega, g_B) \neq \mathbb{C}H^n$ be a irreducible bounded symmetric domain. One can find $\lambda > 0$ such that $\lambda\gamma \notin W(\Omega)$:

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$\uparrow$

$\lambda\gamma \notin W(\Omega)$

By Theorem W, λg_B is not projectively induced and $\lambda\omega_B$ is integral.

Second consequence: The complex hyperbolic space is the only irreducible bounded symmetric domain (Ω, g_B) where λg_B is projectively induced, for all $\lambda > 0$.

A conjecture and a theorem for hom. bounded domains

Conjecture : *Let (M, g) be a simply-connected h.K.m. such that its associated Kähler form ω is integral. If λg is projectively induced for all $\lambda \in \mathbb{R}^+$ then $(M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$.*

Theorem 5 *Let (Ω, g_Ω) be a h.b.d. If $(\Omega, \lambda g_\Omega)$ can be Kähler immersed into $\mathbb{C}P^\infty$ for all $\lambda > 0$, then $(\Omega, g_\Omega) = \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$.*

Ingredients for the proof. Unitary representation of semisimple Lie groups; reproducing kernels of weighted Bergman spaces.

Sketch of the proofs of Theorem 1, 2, 3, 4

Sketch of the proof of Theorem 1 (based on Theorem 5)

$(M,g) \xrightarrow{f} \mathbb{C}^{N\leq\infty}$ we want to prove that: $(M,g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$ and $f = f_0 \times f_1 \times \cdots \times f_l$.

1. Theorem FC + maximum principle

$$\mathcal{F} = \mathbb{C}^k \times \not{T}_1 \times \cdots \not{T}_l \times \not{\mathbb{C}} \quad \overset{\text{Kähler}}{\hookrightarrow} \quad \begin{array}{c} (M,g) \\ \pi\downarrow \\ (\Omega, g_\Omega) \end{array} \begin{array}{c} \to \mathbb{C}^{N\leq\infty} \\ \\ \\ \end{array}$$

2. Riemannian geometry + homogeneity $\Rightarrow$

$$(M,g) \overset{\text{Kähler}}{=} \mathbb{C}^k \times (\Omega, g_\Omega) \Rightarrow (\Omega, \lambda g_\Omega) \to \mathbb{C}^{N\leq\infty},\ \forall\lambda > 0.$$

3. S. Bochner (Bull.Amer.Math.Soc., 1947) $\Rightarrow (\Omega, \lambda g_\Omega) \to \mathbb{C}P^\infty, \ \forall \lambda$.

4. Theorem 5 $\Rightarrow (\Omega, g_\Omega) = \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l} \Rightarrow (M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$.

5. The fact that the immersion f is, up to a unitary transformation of $\mathbb{C}^N$, of the form $f = f_0 \times f_1 \times \cdots \times f_l$ follows by the reducibility of a Kähler product into $\mathbb{C}^{N \leq \infty}$ and by Calabi's rigidity theorem.

□

Sketch of the proof of Theorem 2 (based on Theorem 1)

If $(M,g) \to \mathbb{C}H^{N\leq\infty}$ we want to prove that

$$(M,g) = \mathbb{C}H^n \overset{tot.geod.}{\longrightarrow} \mathbb{C}H^N.$$

1. $(M,g) \to \mathbb{C}H^{N\leq\infty} \Rightarrow (M,g) \to \ell^2(\mathbb{C})$.

2. Theorem 1 $\Rightarrow (M,g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l} \Rightarrow M = \mathbb{C}H^n$. $\Box$

Sketch of the proof of Theorem 3

Let $f : (M, g, \omega) \to \mathbb{C}P^{N \leq \infty}$ be a Kähler immersion.

The integrality of $\omega = f^* \omega_{FS}$ is immediate since ω_{FS} is integral.

$$\text{Th. FC} \Rightarrow \begin{array}{ccc} \mathcal{F} = \mathbb{C}^k \times \not{T}_1 \times \cdots \not{T}_l \times C \overset{\text{Kähler}}{\hookrightarrow} & (M, g) & \to \mathbb{C}P^{N \leq \infty} \\ & \pi \downarrow & \\ & (\Omega, g_\Omega) & \end{array}$$

$\Rightarrow M \overset{top}{=} \Omega \times \mathbb{C}^n \times C$ is simply-connected.

Calabi's rigidity $\Rightarrow f \circ g = \mathcal{U}_g \circ f$, $\forall g \in G = \mathrm{Aut}(M) \cap \mathrm{Isom}(M, g)$
$\Rightarrow f(M)$ is a h.K.m. $\Rightarrow f(M) \subset \mathbb{C}P^N$ is simply-connected.

$f : M \to f(M)$ is a local isometry $\Rightarrow f$ is a covering map $\Rightarrow f$ is injective. $\square$

Sketch of the proof of Theorem 4

Let (M, g, ω) be a simply-connected h.K.m. with ω integral we want to show that $(M, m_0 g) \to \mathbb{C}P^{N \leq \infty}$, for some $m_0 \in \mathbb{Z}$.

1. Let L be a holomorphic line bundle with $c_1(L) = [\omega]$ and consider the Hilbert space

$$\mathcal{H}_m = \{s \in H^0(L) \mid \int_M h_m(s,s)\frac{\omega^n}{n!} < \infty\}$$

where h_m is an Hermitian metric on L^m such that $\mathrm{Ric}(h_m) = m\omega$.

2. There exists $m_0 \in \mathbb{Z}$ such that $\mathcal{H}_{m_0} \neq \{0\}$ (J. Rosenberg, M. Vergne, J. of Functional Analysis (1984));

3. the <u>*base point free condition*</u> is satisfied:

for all $x \in M$ there exists $s \in \mathcal{H}_{m_0}$ such that $s(x) \neq 0$ (homogeneity and $\pi_1(M) = 1$);

4. the "Kodaira map"

$$\varphi_{m_0} : M \to \mathbb{C}P^{d_{m_0}}, x \mapsto [s_0(x), \ldots, s_{d_{m_0}}(x)]$$

with respect to an orthonormal basis $\{s_0, \ldots, s_{d_{m_0}}\}$ of $\mathcal{H}_{m_0}$ is a Kähler immersion (homogeneity and $\pi_1(M) = 1$).

□