Firenze (7.04.2022)

Kähler-Einstein metrics and Kähler-Ricci solitons induced by complex space forms

Andrea Loi, University of Cagliari

Aim of the talk: *provide an overview of the main results on Kähler immersions of Kähler manifolds into complex space forms in the Kähler-Einstein and Kähler-Ricci solitons case.*

Advertising for the book: L., M. Zedda, *Kähler immersions of Kähler manifolds into complex space forms.*, Lectures Notes of the Unione Matematica Italiana, Springer 2018.

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0. Kähler manifolds and complex space forms

A *Kähler manifold* is a pair (M, ω) where M is a complex manifold and $\omega \in \Omega^2(M)$ (the *Kähler form*) such that for any $p \in M$ there exists a neighborhood $U \subset M$ of p equipped with complex coordinates $z_1, \ldots, z_n$, $n = \dim_{\mathbb{C}} M$, and $\Phi : U \to \mathbb{R}$ (a *Kähler potential) such that*

$$\omega_{|U} = \frac{i}{2}\partial\bar{\partial}\Phi = \frac{i}{2}\sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$$

$$g_{\alpha\bar{\beta}} := \{\frac{\partial^2\Phi}{\partial z_\alpha \partial \bar{z}_\beta}\}, \ \overline{g_{\alpha\bar{\beta}}} = g_{\beta\bar{\alpha}}, \ g_{\alpha\bar{\beta}} >> 0$$

If $\Phi : U \to \mathbb{R}$ *and* $\tilde{\Phi} : U \to \mathbb{R}$ *are two Kähler potential for a Kähler metric* g *then*

$$\tilde{\Phi} = \Phi + h + \bar{h}, \;\; h \in Hol(U).$$

Given a Kähler manifold (M, ω) *we can associate a Riemannian metric* g *(the* Kähler metric*) on* M *by*

$$g_{|U} = -\mathit{Real}\{\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \Phi}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \otimes d\bar{z}_\beta\}.$$

$$(M, \omega) \longleftrightarrow (M, g)$$

A complex space form (S, g_c) is a Kähler manifold with constant holomorphic sectional curvature $2c$. If S is assumed to be complete and simply-connected one has:

Complex Euclidean space $(\mathbb{C}^N, g_0)$

$$\omega_0 = \frac{i}{2}\partial\bar{\partial}|z|^2 = \frac{i}{2}\sum_{j=1}^{N} dz_j \wedge d\bar{z}_j,\ |z|^2 = |z_1|^2 + \cdots + |z_N|^2.$$

Complex hyperbolic space $(\mathbb{C}H^N := \{z \in \mathbb{C}^N \mid |z|^2 < 1\}, g_c)$, $c < 0$

$$\omega_c = \frac{i}{c}\partial\bar{\partial}\log(1 - |z|^2).$$

Complex projective space $(\mathbb{C}P^N = \mathbb{C}^{N+1} \setminus \{0\}/z \sim \lambda z, g_c)$, $c > 0$

$$\omega_c|_{U_0} = \frac{i}{c}\partial\bar{\partial}\log(1 + |z|^2),\ z_j = \frac{Z_j}{Z_0},\ j = 1, \ldots, N,\ \ U_0 = \{Z_0 \neq 0\}.$$

1. The work of E. Calabi, Ann. Math. 1953

Let (M, g) *be a real analytic Kähler manifold. Then in a neighborhood* U *of* $p \in M$ *one can find a Kähler potential (Calabi's diastasis function)* $D_p^g : U \to \mathbb{R}$ *characterized by the fact that in every complex coordinates* $\{z_1, \ldots, z_n\}$ *centered at* p

$$D_p^g(z, \bar{z}) = \sum_{|j|,|k| \geq 0} a_{jk} z^j \bar{z}^k,$$

with $a_{j0} = a_{0j} = 0$ *for all multi-indices* j*.*

Definition *A Kähler immersion of a Kähler manifold* (M, g) *into a complex space form* (S, g_c) *is a holomorphic map* $f : M \to S$ *which is isometric, i.e.* $f^* g_c = g$*.*

Theorem *(hereditary property of the diastasis)* if $f : M \to S$ is a Kähler immersion from a Kähler manifold (M, g) into a complex space form (S, g_c) then $D_p^g = \varphi^*(D_{\varphi(p)}^{g_c})$, for all $p \in M$.

Theorem 1 *(rigidity)* Given two Kähler immersions f_1 and f_2 from a Kähler manifold (M, g) into a complex space form (S, g_c) there exists $\mathcal{U} \in \mathrm{Aut}(S) \cap \mathrm{Isom}(S, g_S)$ such that $f_2 = \mathcal{U} \circ f_1$.

Theorem 2 *(extension)* A simply-connected Kähler manifold (M, g) admits a Kähler immersion into a complex space form (S, g_c) iff there exists an open set $U \subset M$ such that $(U, g_{|U})$ can be Kähler immersed into (S, g_c).

Theorem 3 If there exists a Kähler immersion of an open subset of a complex space form (S, g_c) into a complex space form $(\tilde{S}, g_{\tilde{c}})$ then $sign(c) = sign(\tilde{c})$.

2.The Kähler-Einstein (KE) case

Theorem (B. Smyth, Ann. Math. 1967) *A compact KE manifold of complex dimension n which admits a Kähler embedding into $(\mathbb{C}P^{n+1}, g_c)$ is totally geodesic or the complex quadric $Q = \{Z_0^2 + \cdots + Z_{n+1}^2 = 0\}$.*

Theorem (S. S. Chern, J. Diff. Geom. 1967) *A KE manifold of complex dimension n which admits a Kähler immersion into $(\mathbb{C}P^{n+1}, g_c)$ is totally geodesic or an open subset of the complex quadric.*

Theorem (K. Tsukada, Math. Ann. 1986) *A KE manifold of complex dimension* n *which admits a Kähler immersion into* $(\mathbb{C}P^{n+2}, g_c)$ *is totally geodesic or an open subset of the complex quadric.*

Theorem (J. Hano, Math. Ann. 1975) *Let* $M \subset \mathbb{C}P^N$ *be a compact complete intersection. If the restriction of* g_c *to* M *is Einstein then* M *is totally geodesic or the complex quadric.*

Theorem (D. Hulin, Journal of Geom. Anal. 2000) *A compact KE manifold which admits a Kähler embedding into* $(\mathbb{C}P^N, g_c)$ *has positive scalar curvature.*

Open problem: *Drop the compactness assumption and injectivity assumption.*

Theorem (G. Manno, F. Salis, NY Journal of Math. 2022) *Let g be a rotation invariant KE metric* * *on a complex surface M. If (M, g) admits a Kähler immersion into $(\mathbb{C}P^n, g_c)$, is an open subset of either $(\mathbb{C}P^2, qg_c)$ or $(\mathbb{C}P^1 \times \mathbb{C}P^1, q(g_c \oplus g_c))$, $q \in \mathbb{Z}^+$.*

Conjecture A: *A KE manifold which admits a Kähler immersion into $(\mathbb{C}P^N, g_c)$ is an open subset of a compact and simply-connected homogeneous Kähler manifold and the immersion is an embedding.*

Theorem (M. Umehara, Tohoku Math. J., 1987) *If a KE manifold (M, g) admits a Kähler immersion into a complex space form (S, g_c) with $c \leq 0$ then g is a complex space form (and the immersion is a totally geodesic).*

*A Kähler metric g is rotation invariant if it admits a Kähler potential $\Phi : U \to \mathbb{R}$ which depends only on $|z_1|^2, |z_2|^2, \ldots, |z_n|^2$.

3.The Kähler-Ricci solitons (KRS) case

Theorem KRS (-, Mossa, PAMS 2021) *Let* (g, X) *be a KRS*[†] *on complex manifold* M*. If* (M, g) *can be Kähler immersed into a complex space form* (S, g_c) *then* g *is KE. Moreover, its Einstein constant is a rational multiple of* c*.*

Remark *There are no topological assumptions and the immersion is not required to be injective.*

Remark *The theorem is valid also when the ambient space is an indefinite complex space form.*

[†] $Ric_g = \lambda g + L_X g$, where X is the real part of a holomorphic vector field.

Corollary *Let (g, X) be a KRS on a complex manifold M. A Kähler immersion of (M, g) into a complex space form of non-positive holomorphic sectional curvature is totally geodesic.*

Theorem KRS extends the following:

Theorem (Bedulli and Gori, PAMS 2014) A KRS on a compact Kähler submanifold $M \subset \mathbb{C}P^N$ which is a complete intersection is KE (and hence by Hano's theorem is the quadric or a complex projective space totally geodesically embedded in $\mathbb{C}P^N$).

4. Kähler immersions into (S^∞, g_c)

Theorem (Calabi, Ann. Math. 1953) *If there exist a Kähler immersion of an open subset of a complex space form* (S, g_c) *into an infinite dimensional complex space form* $(\tilde{S}^\infty, g_{\tilde{c}})$ *then* $c \leq \tilde{c}$.

Example (Calabi's embedding):

$$\mathbb{C}^n \to \mathbb{C}P^\infty : z \mapsto (\dots, \sqrt{\frac{1}{j!}}\ z^j, \dots),\ \ |j| \geq 0$$

$z^j = z_1^{j_1} \cdots z_n^{j_n}$ $|j| = j_1 + \cdots + j_n$, $j! = j_1! \cdots j_n!$

Theorem (-, M. Zedda, Math. Ann. 2010) *There exist continuous families of complete and nonhomogeneous KE manifolds which can be Kähler embedded into infinite dimensional complex projective space.*

Open question: *does there exist complete and nonhomogeneous KE manifolds Kähler immersed into non-elliptic infinite dimensional complex space forms?*

Theorem (-, F. Salis, F. Zuddas, Pacific J. Math. 2022) *There exist continuous families of non trivial (radial[‡]) KRS which can be Kähler embedded into any infinite dimensional complex space form.*

[‡]A Kähler metric g is radial if it admits a Kähler potential $\Phi : U \to \mathbb{R}$ which depends only on $|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$.

5.The Ricci flat case

Definition: A Kähler metric g on a complex manifold M is projectively induced if (M, g) can be Kähler immersed into a finite or infinite dimensional complex projective space.

Conjecture B: *A Ricci flat projectively induced Kähler metric is flat.*

The Taub-NUT metrics

The Taub-NUT metrics is the family of Kähler metrics g_m on $\mathbb{C}^2$ whose associated Kähler forms are given by

$$\omega_m = \frac{i}{2}\partial\bar{\partial}\left[u^2 + v^2 + m(u^4 + v^4)\right], \ m \geq 0,$$

where $|z_1| = e^{m(u^2-v^2)}u, \ |z_2| = e^{m(v^2-u^2)}v$. For $m = 0$, g_0 is flat metric and for $m \neq 0$, g_m is Ricci flat (not flat) and complete (C. LeBrun, Proceedings of Symposia in Pure Mathematics, 1991).

Theorem (-, M. Zedda, F. Zuddas, Ann. Global Anal. Geom. 2012) *For $m > \frac{1}{2}$ the Kähler metric g_m on $\mathbb{C}^2$ is not projectively induced.*

Open problem: *Show that g_m is projectively induced iff $m = 0$.*

The Eguchi–Hanson metric

Let g_{EH} be the Ricci flat and complete Kähler metric on $\widehat{\mathbb{C}}^2$ (the blow-up of $\mathbb{C}^2$ at the origin) given in $\widehat{\mathbb{C}}^2 \setminus E = \mathbb{C}^2 \setminus \{0\}$ (E exceptional divisor) by the potential

$$\Phi(x) = \sqrt{x^2+1} + \log x - \log(1+\sqrt{x^2+1}), \ x = |z_1|^2 + |z_2|^2$$

Theorem (-, F. Salis, F. Zuddas, MZ 2018) *The Eguchi-Hanson metric g_{EH} is not projectively induced.*

Theorem (-, M. Zedda, F. Zuddas, Tohoku Math. J. 2020) *The Eguchi-Hanson metric λg_{EH} is not projectively induced for all $\lambda > 0$.*

6. On the proof of Theorem KRS

Let M be a complex manifold and $p \in M$. Define

$\hat{\mathcal{O}}_p = \left\{\alpha = (\alpha_1, \ldots, \alpha_m) \mid \alpha_j \in \mathcal{O}_p, \alpha_j(p) = 0, \forall j = 1, \ldots, m, m \geq 1\right\}.$

Let $\alpha \in \hat{\mathcal{O}}_p$ and $\ell \in \mathbb{N}$ such that $\ell \leq |\alpha| := m$. Let

$$\langle \alpha, \alpha \rangle_\ell(z) = \sum_{j=1}^{\ell} |\alpha_j(z)|^2 - \sum_{k=\ell+1}^{|\alpha|} |\alpha_k(z)|^2.$$

The Umehara algebra is defined by

$$\Lambda_p = \left\{h + \bar{h} + \langle \alpha, \alpha \rangle_\ell \mid h \in \mathcal{O}_p,\ \alpha \in \hat{\mathcal{O}}_p,\ \ell \leq |\alpha|\right\}.$$

Consider the $\mathbb{R}$-algebra $\tilde{\Lambda}_p \subset \Lambda_p$ is defined by:

$$\tilde{\Lambda}_p = \left\{a + \langle \alpha, \alpha \rangle_\ell \mid a \in \mathbb{R},\ \alpha \in \hat{\mathcal{O}}_p,\ \ell \leq |\alpha|\right\}.$$

Remark *Notice that the germ of the real part of a nonconstant holomorphic function $h \in \mathcal{O}_p$ belongs to Λ_p but not to $\tilde{\Lambda}_p$.*

Let K_p (resp. $\tilde{K}_p$) be the field of fractions of Λ_p (resp. $\tilde{\Lambda}_p$).

Definition *A real analytic function defined on a neighborhood U of a point p of a complex manifold M is of diastasis-type if in one (and hence any) coordinate system $\{z_1, \ldots, z_n\}$ centered at p its expansion in z and $\bar{z}$ does not contains non constant purely holomorphic or anti-holomorphic terms (i.e. of the form z^j or $\bar{z}^j$ with $j > 0$).*

Remark *A function $f \in \Lambda_p$ (resp. K_p) belong to $\tilde{\Lambda}_p$ (resp. $\tilde{K}_p$) iff f is of diastasis-type.*

Fundamental Lemma (-, Mossa, PAMS 2021) *Let* μ *be a real number and* $g \in \tilde{K}_p$*. Then*

$$e^g \notin \tilde{\Lambda}_p^{\mu} \tilde{K}_p \setminus \mathbb{R}$$

where $\tilde{\Lambda}_p^{\mu} \tilde{K}_p = \left\{ f^{\mu} h \mid f \in \tilde{\Lambda}_p,\ h \in \tilde{K}_p \right\}$*.*

This lemma extends the following

Theorem (Cheng, Di Scala,Yuan, Int. J. Math. 2021) *Let* $f \in \tilde{\Lambda}_p$ *then*

$$e^f \notin \tilde{K}_p \setminus \mathbb{R}.$$

Moreover, if $f^{\alpha} \in \tilde{K}_p \setminus \mathbb{R}$ *then* $\alpha \in \mathbb{Q}$*.*

Lemma (Umehara, Tokyo J. Math. 1986) *Let* M *be an* n*-dimensional complex manifold,* $p \in M$ *and* $f \in \Lambda_p$*. Then*

$$f^{n+1} \det \left[\frac{\partial^2 \log f}{\partial z_\alpha \partial \bar{z}_\beta} \right] \in \Lambda_p.$$

Sketch of the Proof of Theorem KRS

Theorem KRS (-, Mossa, PAMS 2021) *Let* (g, X) *be a KRS on complex manifold* M*. If* (M, g) *can be Kähler immersed into a complex space form* (S, g_c) *then* g *is KE. Moreover, its Einstein constant is a rational multiple of* c*.*

Step 1. (Umehara's algebra and Kähler immersions) Let $p \in M$ and let D_p^g be the Calabi's diastasis. Then

$$D_p^g = \sum_{i=1}^{N} |\varphi_i|^2 \in \tilde{\Lambda}_p, \text{ if } c = 0 \tag{1}$$

$$e^{\frac{c}{2} D_p^g} = 1 + \frac{c}{|c|} \sum_{i=1}^{N} |\varphi_i|_s^2 \in \tilde{\Lambda}_p, \text{ if } c \neq 0, \tag{2}$$

$$\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta}\right] \in \tilde{K}_p, \forall c. \tag{3}$$

The proof of (3) for $c \neq 0$ follows by Umehara's lemma applied to

$$f = e^{\frac{c}{2} D_p^g} = 1 + \frac{c}{|c|} \sum_{i=1}^{N} |\varphi_i|_s^2 \in \tilde{\Lambda}_p.$$

Indeed

$$f^{n+1} \det\left[\frac{\partial^2 \log f}{\partial z_\alpha \partial \bar{z}_\beta}\right] = \left(\frac{c}{2}\right)^n e^{(n+1)\frac{c}{2} D_p^g} \det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta}\right] \in \tilde{\Lambda}_p,$$

Step 2. (KRS equation in terms of Calabi's diastasis function) In local complex coordinates $\{z_1, \ldots, z_n\}$ in a neighborhood U of a point $p \in M$ where the diastasis D_p^g for the metric g is defined one has

$$X = \sum_{j=1}^{n} \left(f_j \frac{\partial}{\partial z_j} + \bar{f}_j \frac{\partial}{\partial \bar{z}_j} \right)$$

for some holomorphic functions $f_j, j = 1, \ldots, n$, on U. Hence

$$L_X \omega = \frac{i}{2} \partial \bar{\partial} f_X. \tag{4}$$

where ω is the Kähler form associated to g and

$$f_X = \sum_{j=1}^{n} f_j \frac{\partial D_p^g}{\partial z_j} + \bar{f}_j \frac{\partial D_p^g}{\partial \bar{z}_j}. \tag{5}$$

The KRS equation can be written on U as

$$\rho_\omega = -i \partial \bar{\partial} \log \det \left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta} \right] = \lambda \omega + L_X \omega = \lambda \frac{i}{2} \partial \bar{\partial} D_p^g + \frac{i}{2} \partial \bar{\partial} f_X$$

where ρ_ω the Ricci form of ω.

Thus the local expression of the KRS equation is

$$\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta}\right] = e^{-\frac{\lambda}{2}D_p^g - \frac{f_X}{2} + h + \bar{h}}, \tag{6}$$

for a holomorphic function h on U.

Final step. We treat the two cases $c = 0$ and $c \neq 0$ separately.

If $c = 0$, we get by (1) and (5) that

$$\xi := -\frac{\lambda}{2}D_p^g - \frac{f_X}{2} + h + \bar{h} \in \Lambda_p.$$

Now (3) with $c = 0$ gives

$$e^\xi = \det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta}\right] \in \tilde{K}_p.$$

In particular e^{ξ} and so ξ is of diastasis-type. Then $\xi \in \tilde{\Lambda}_p$.

By the fundamental lemma (with $\mu = 0$)

$$e^{\xi} = \det \left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta} \right] = cost,$$

Hence

$$\rho_\omega = -i\partial\bar{\partial} \log \det \left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta} \right] = 0$$

and so g is Ricci flat.

If $c \neq 0$ we get by (2) and (5) that

$$\eta := -\frac{f_X}{2} + h + \bar{h} \in K_p.$$

By (2), (3) and (6) one deduces that

$$e^{\eta} = \left[e^{\frac{c}{2}D_p^g}\right]^{\frac{\lambda}{c}} \det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta}\right] \in \tilde{\Lambda}_p^{\mu}\tilde{K}_p,\ \ \mu = \frac{\lambda}{c}. \qquad (7)$$

In particular e^{η} and so η is of diastasis-type. Then $\eta \in \tilde{K}_p$.

By the fundamental lemma $\eta = cost$. So f_X is the real part of a holomorphic function and hence

$$\rho_\omega = \lambda\omega + L_X\omega = \lambda\omega + \frac{i}{2}\partial\bar{\partial} f_X = \lambda\omega$$

and so g is KE with Einstein constant λ.

Finally, $\frac{\lambda}{c}$ is forced to be rational by the last part of the theorem of Cheng-Di Scala-Yuan. □

Thank you for your attention!