The Gromov width of symmetric spaces

Andrea Loi

University of Cagliari-Italy

Evora, 3 September 2013

AL, R. Mossa, F. Zuddas,
Symplectic capacities of Hermitian symmetric spaces,
arXiv:1302.1984 (2013).

Darboux Theorem

Let (M^{2n}, ω) be a symplectic manifold and let $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$ be the standard symplectic form on $\mathbb{R}^{2n}$. Given $p \in M$ there exist an open set $U_p \subset M$ and a diffeomorphism

$$\psi : U_p \to \psi(U_p) \subset \mathbb{R}^{2n}$$

such that

$$\psi^* \omega_0 = \omega_{|U_p}$$

Darboux Theorem

Let (M^{2n}, ω) be a symplectic manifold and let $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$ be the standard symplectic form on $\mathbb{R}^{2n}$. Given $p \in M$ there exist an open set $U_p \subset M$ and a diffeomorphism

$$\psi : U_p \to \psi(U_p) \subset \mathbb{R}^{2n}$$

such that

$$\psi^* \omega_0 = \omega_{|U_p}$$

Question

How large U_p can be taken?

Darboux Theorem

Let (M^{2n}, ω) be a symplectic manifold and let $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$ be the standard symplectic form on $\mathbb{R}^{2n}$. Given $p \in M$ there exist an open set $U_p \subset M$ and a diffeomorphism

$$\psi : U_p \to \psi(U_p) \subset \mathbb{R}^{2n}$$

such that

$$\psi^* \omega_0 = \omega_{|U_p}$$

Question

How large U_p can be taken?

Gromov's exotic symplectic structures

There exists a symplectic form ω on $\mathbb{R}^{2n}$, $n \geq 2$, such that $(\mathbb{R}^{2n}, \omega)$ cannot be symplectically embedded into $(\mathbb{R}^{2n}, \omega_0)$. [Gromov, Inv. Math., 1985]

Gromov width

The Gromov width [Gromov, Inv. Math., 1985] of a $2n$-dimensional symplectic manifold (M,ω) is defined as

$$c_G(M,\omega) = \sup\{\pi r^2 \mid B^{2n}(r) \text{ symplectically embeds into } (M,\omega)\},$$

where

$$B^{2n}(r) = \{(x,y) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n} |x_j|^2 + |y_j|^2 < r^2\}$$

is the open ball of radius r in $(\mathbb{R}^{2n}, \omega_0)$.

Gromov width

The Gromov width [Gromov, Inv. Math., 1985] of a $2n$-dimensional symplectic manifold (M, ω) is defined as

$$c_G(M, \omega) = \sup\{\pi r^2 \mid B^{2n}(r) \text{ symplectically embeds into } (M, \omega)\},$$

where

$$B^{2n}(r) = \{(x, y) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n} |x_j|^2 + |y_j|^2 < r^2\}$$

is the open ball of radius r in $(\mathbb{R}^{2n}, \omega_0)$.

Remarks

Gromov width

The Gromov width [Gromov, Inv. Math., 1985] of a $2n$-dimensional symplectic manifold (M, ω) is defined as

$$c_G(M,\omega) = \sup\{\pi r^2 \mid B^{2n}(r) \text{ symplectically embeds into } (M,\omega)\},$$

where

$$B^{2n}(r) = \{(x,y) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n} |x_j|^2 + |y_j|^2 < r^2\}$$

is the open ball of radius r in $(\mathbb{R}^{2n}, \omega_0)$.

Remarks

- $c_G > 0$ by Darboux theorem.

Gromov width

The Gromov width [Gromov, Inv. Math., 1985] of a $2n$-dimensional symplectic manifold (M,ω) is defined as

$$c_G(M,\omega) = \sup\{\pi r^2 \mid B^{2n}(r) \text{ symplectically embeds into } (M,\omega)\},$$

where

$$B^{2n}(r) = \{(x,y) \in \mathbb{R}^{2n} \mid \sum_{j=1}^{n} |x_j|^2 + |y_j|^2 < r^2\}$$

is the open ball of radius r in $(\mathbb{R}^{2n}, \omega_0)$.

Remarks

- $c_G > 0$ by Darboux theorem.
- M compact $\Rightarrow$ $c_G(M,\omega) < \infty$.

Symplectic capacities

A map c from the class $\mathcal{C}(2n)$ of all symplectic manifolds of dimension $2n$ to $[0, +\infty]$ is called a *symplectic capacity* [H. Hofer, E. Zehnder, 1990] if it satisfies the following conditions:

Symplectic capacities

A map c from the class $\mathcal{C}(2n)$ of all symplectic manifolds of dimension $2n$ to $[0, +\infty]$ is called a *symplectic capacity* [H. Hofer, E. Zehnder, 1990] if it satisfies the following conditions:

- (**monotonicity**) if there exists a symplectic embedding $(M_1, \omega_1) \to (M_2, \omega_2)$ then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$;

Symplectic capacities

A map c from the class $\mathcal{C}(2n)$ of all symplectic manifolds of dimension $2n$ to $[0, +\infty]$ is called a *symplectic capacity* [H. Hofer, E. Zehnder, 1990] if it satisfies the following conditions:

- (**monotonicity**) if there exists a symplectic embedding $(M_1, \omega_1) \to (M_2, \omega_2)$ then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$;
- (**conformality**) $c(M, \lambda\omega) = |\lambda| c(M, \omega)$, for every $\lambda \in \mathbb{R} \setminus \{0\}$;

Symplectic capacities

A map c from the class $\mathcal{C}(2n)$ of all symplectic manifolds of dimension $2n$ to $[0, +\infty]$ is called a *symplectic capacity* [H. Hofer, E. Zehnder, 1990] if it satisfies the following conditions:

- (**monotonicity**) if there exists a symplectic embedding $(M_1, \omega_1) \to (M_2, \omega_2)$ then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$;
- (**conformality**) $c(M, \lambda\omega) = |\lambda| c(M, \omega)$, for every $\lambda \in \mathbb{R} \setminus \{0\}$;
- (**nontriviality**) $c(B^{2n}(r), \omega_0) = \pi r^2 = c(Z^{2n}(r), \omega_0)$,

Symplectic capacities

A map c from the class $\mathcal{C}(2n)$ of all symplectic manifolds of dimension $2n$ to $[0,+\infty]$ is called a *symplectic capacity* [H. Hofer, E. Zehnder, 1990] if it satisfies the following conditions:

- (**monotonicity**) if there exists a symplectic embedding $(M_1,\omega_1) \to (M_2,\omega_2)$ then $c(M_1,\omega_1) \leq c(M_2,\omega_2)$;
- (**conformality**) $c(M,\lambda\omega) = |\lambda| c(M,\omega)$, for every $\lambda \in \mathbb{R} \setminus \{0\}$;
- (**nontriviality**) $c(B^{2n}(r),\omega_0) = \pi r^2 = c(Z^{2n}(r),\omega_0)$,

where

$$Z^{2n}(r) = B^2(r) \times \mathbb{R}^{2n-2} = \{(x,y) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < r^2\}.$$

Some remarks on symplectic capacities 1

Some remarks on symplectic capacities 1

When $n = 1$ (2-dimensional symplectic manifolds)

$$c(M, \omega) := \left| \int_M \omega \right|$$

defines a symplecitc capacity which agrees with the Lebesgue measure in $(\mathbb{R}^2, \omega_0)$.

Some remarks on symplectic capacities 1

When $n = 1$ (2-dimensional symplectic manifolds)

$$c(M, \omega) := \left| \int_M \omega \right|$$

defines a symplecitc capacity which agrees with the Lebesgue measure in $(\mathbb{R}^2, \omega_0)$.

In contrast, when $n > 1$,

$$c(M, \omega) := \left(\int_M \frac{\omega^n}{n!} \right)^{\frac{1}{n}}$$

does not define a symplectic capacity since $Z^{2n}(r)$ has infinite volume.

Some remarks on symplectic capacities 2

Some remarks on symplectic capacities 2

(monotonicity) $\Rightarrow$ symplectic capacities are symplectic invariants.

Some remarks on symplectic capacities 2

(monotonicity) $\Rightarrow$ symplectic capacities are symplectic invariants.

for every open set $U \subset \mathbb{R}^n$ such that $B^{2n}(r) \subset U \subset Z^{2n}(r)$ $\Rightarrow$ $c(U) = \pi r^2$.

Some remarks on symplectic capacities 2

(monotonicity) $\Rightarrow$ symplectic capacities are symplectic invariants.

for every open set $U \subset \mathbb{R}^n$ such that $B^{2n}(r) \subset U \subset Z^{2n}(r)$ $\Rightarrow$ $c(U) = \pi r^2$.

It is hard to prove the existence of a symplectic capacity.

Theorem

The Gromov width c_G is a symplectic capacity. Moreover

$$c_G(M,\omega) \leq c(M,\omega)$$

for every capacity c.

Theorem

The Gromov width c_G is a symplectic capacity. Moreover

$$c_G(M, \omega) \leq c(M, \omega)$$

for every capacity c.

Proof

(monotonicity) and (conformality) for c_G are easy consequences of the definition of Gromov width. details

Theorem

The Gromov width c_G is a symplectic capacity. Moreover

$$c_G(M,\omega) \leq c(M,\omega)$$

for every capacity c.

Proof

(monotonicity) and (conformality) for c_G are easy consequences of the definition of Gromov width. ▸ details

Let $\varphi : B^{2n}(r) \to M$ be a symplectic embedding. Then

$$\pi r^2 = c(B^{2n}(r),\omega_0) \leq c(M,\omega) \Rightarrow c_G(M,\omega) \leq c(M,\omega).$$

Theorem

The Gromov width c_G is a symplectic capacity. Moreover

$$c_G(M,\omega) \leq c(M,\omega)$$

for every capacity c.

Proof

(monotonicity) and (conformality) for c_G are easy consequences of the definition of Gromov width. ▸ details

Let $\varphi : B^{2n}(r) \to M$ be a symplectic embedding. Then

$$\pi r^2 = c(B^{2n}(r),\omega_0) \leq c(M,\omega) \Rightarrow c_G(M,\omega) \leq c(M,\omega).$$

The (nontriviality) for c_G, i.e. $c_G(B^{2n}(r),\omega_0) = \pi = c_G(Z^{2n}(r),\omega_0)$, follows by the celebrated Gromov's nonsqueezing theorem:

Gromov nonsqueezing theorem

There exists a symplectic embedding $B^{2n}(r) \hookrightarrow Z^{2n}(R)$ iff $r \leq R$.

[Gromov, Inv. Math., 1985]

Gromov nonsqueezing theorem

There exists a symplectic embedding $B^{2n}(r) \hookrightarrow Z^{2n}(R)$ iff $r \leq R$.

[Gromov, Inv. Math., 1985]

Remark

Assuming the existence of any symplectic capacity c one easily deduces Gromov's nonsqueezing theorem. Indeed, let $\varphi : B^{2n}(r) \to Z^{2n}(R)$ be a symplectic embedding. Then (monotonicity)+(nontriviality) $\Rightarrow$

$$\pi r^2 = c(B^{2n}(r), \omega_0) \leq c(Z^{2n}(R), \omega_0) = \pi R^2.$$

Some known results for homogeneous Kähler manifolds

- Upper and lower bounds of the Gromov width of some coadjoint orbits

  [A. C. Castro, Upper bound for the Gromov width of coadjoint orbits of type A, arXiv:1301.0158v1]

  [Taekgyu Hwang, Dong Youp Suh, The Gromov width from Hamiltonian circle actions, arXiv:1305.2989v2]

Some known results for homogeneous Kähler manifolds

- Upper and lower bounds of the Gromov width of some coadjoint orbits

  [A. C. Castro, Upper bound for the Gromov width of coadjoint orbits of type A, arXiv:1301.0158v1]

  [Taekgyu Hwang, Dong Youp Suh, The Gromov width from Hamiltonian circle actions, arXiv:1305.2989v2]

- Computation of the Gromov width of the complex Grassmannian [Y. Karshon, S. Tolman, Algebr. Geom. Topol., 2005] and product of Grassmannians [G. Lu, Israel J. Math., 2006]

Some known results for homogeneous Kähler manifolds

- Upper and lower bounds of the Gromov width of some coadjoint orbits

  [A. C. Castro, Upper bound for the Gromov width of coadjoint orbits of type A, arXiv:1301.0158v1]

  [Taekgyu Hwang, Dong Youp Suh, The Gromov width from Hamiltonian circle actions, arXiv:1305.2989v2]

- Computation of the Gromov width of the complex Grassmannian [Y. Karshon, S. Tolman, Algebr. Geom. Topol., 2005] and product of Grassmannians

  [G. Lu, Israel J. Math., 2006]

- Computation of the Gromov width of the 4-dimensional torus.

  [J. Latschev, D. McDuff and F. Schlenk, Te Gromov width of the 4-dimensional tori, arXiv:1111.6566]

Some known results for homogeneous Kähler manifolds

- Upper and lower bounds of the Gromov width of some coadjoint orbits

  [A. C. Castro, Upper bound for the Gromov width of coadjoint orbits of type A, arXiv:1301.0158v1]

  [Taekgyu Hwang, Dong Youp Suh, The Gromov width from Hamiltonian circle actions, arXiv:1305.2989v2]

- Computation of the Gromov width of the complex Grassmannian [Y. Karshon, S. Tolman, Algebr. Geom. Topol., 2005] and product of Grassmannians [G. Lu, Israel J. Math., 2006]

- Computation of the Gromov width of the 4-dimensional torus.

  [J. Latschev, D. McDuff and F. Schlenk, Te Gromov width of the 4-dimensional tori, arXiv:1111.6566]

- Computation of the Gromov width of the first Cartan domain and upper and lower bounds for the classical ones (endowed with ω_0)

  [G. Lu, H. Ding, Q. Zhang, Int. Math. Forum 2, 2007].

Some known results for homogeneous Kähler manifolds

- Upper and lower bounds of the Gromov width of some coadjoint orbits

  [A. C. Castro, Upper bound for the Gromov width of coadjoint orbits of type A, arXiv:1301.0158v1]

  [Taekgyu Hwang, Dong Youp Suh, The Gromov width from Hamiltonian circle actions, arXiv:1305.2989v2]

- Computation of the Gromov width of the complex Grassmannian [Y. Karshon, S. Tolman, Algebr. Geom. Topol., 2005] and product of Grassmannians [G. Lu, Israel J. Math., 2006]

- Computation of the Gromov width of the 4-dimensional torus.

  [J. Latschev, D. McDuff and F. Schlenk, Te Gromov width of the 4-dimensional tori, arXiv:1111.6566]

- Computation of the Gromov width of the first Cartan domain and upper and lower bounds for the classical ones (endowed with ω_0) [G. Lu, H. Ding, Q. Zhang, Int. Math. Forum 2, 2007].

Aim of this talk: to compute the Gromov width for all Hermitian symmetric spaces of compact and noncompact type (bounded symmetric domains) and their products.

Basic facts on Hermitian spaces

- A Hermitian symmetric space is a connected Kähler manifold (M, ω) such that each point $p \in M$ is an isolated fixed point of some holomorphic involutory isometry s_p of M.

Basic facts on Hermitian spaces

- A Hermitian symmetric space is a connected Kähler manifold (M, ω) such that each point $p \in M$ is an isolated fixed point of some holomorphic involutory isometry s_p of M.
- The component of the identity of the group of holomorphic isometries of M acts transitively on M and hence every Hermitian symmetric space is a homogeneous space.

Basic facts on Hermitian spaces

- A Hermitian symmetric space is a connected Kähler manifold (M, ω) such that each point $p \in M$ is an isolated fixed point of some holomorphic involutory isometry s_p of M.
- The component of the identity of the group of holomorphic isometries of M acts transitively on M and hence every Hermitian symmetric space is a homogeneous space.
- A Hermitian symmetric space M is said to be of *compact* or *noncompact type* if M is compact or noncompact (and non flat).

Basic facts on Hermitian spaces

- A Hermitian symmetric space is a connected Kähler manifold (M, ω) such that each point $p \in M$ is an isolated fixed point of some holomorphic involutory isometry s_p of M.
- The component of the identity of the group of holomorphic isometries of M acts transitively on M and hence every Hermitian symmetric space is a homogeneous space.
- A Hermitian symmetric space M is said to be of *compact* or *noncompact type* if M is compact or noncompact (and non flat).
- Every Hermitian symmetric space is a direct product

$$M_0 \times M_- \times M_+$$

where all the factors are simply-connected Hermitian symmetric spaces, $M_0 = \mathbb{C}^n$ and M_- and M_+ are spaces of compact and noncompact type, respectively.

Basic facts on Hermitian spaces

- A Hermitian symmetric space is a connected Kähler manifold (M, ω) such that each point $p \in M$ is an isolated fixed point of some holomorphic involutory isometry s_p of M.
- The component of the identity of the group of holomorphic isometries of M acts transitively on M and hence every Hermitian symmetric space is a homogeneous space.
- A Hermitian symmetric space M is said to be of *compact* or *noncompact type* if M is compact or noncompact (and non flat).
- Every Hermitian symmetric space is a direct product

  $$M_0 \times M_- \times M_+$$

  where all the factors are simply-connected Hermitian symmetric spaces, $M_0 = \mathbb{C}^n$ and M_- and M_+ are spaces of compact and noncompact type, respectively.
- Any Hermitian symmetric space of compact or non-compact type is simply connected and is a direct product of irreducible Hermitian symmetric spaces.

Hermitian symmetric spaces of noncompact type (HSSNT)

An irreducible HSSNT is holomorphically isometric to a bounded symmetric domain $\Omega \subset \mathbb{C}^n$ centered at the origin $0 \in \mathbb{C}^n$ equipped with (a multiple of) the Bergman metric ω_{Berg}.

Hermitian symmetric spaces of noncompact type (HSSNT)

An irreducible HSSNT is holomorphically isometric to a bounded symmetric domain $\Omega \subset \mathbb{C}^n$ centered at the origin $0 \in \mathbb{C}^n$ equipped with (a multiple of) the Bergman metric ω_{Berg}.

There exists homogeneous bounded domains (equipped with the Bergman metric) which are not HSSNT (first examples due to Pyateskii–Shapiro).

Hermitian symmetric spaces of noncompact type (HSSNT)

An irreducible HSSNT is holomorphically isometric to a bounded symmetric domain $\Omega \subset \mathbb{C}^n$ centered at the origin $0 \in \mathbb{C}^n$ equipped with (a multiple of) the Bergman metric ω_{Berg}.

There exists homogeneous bounded domains (equipped with the Bergman metric) which are not HSSNT (first examples due to Pyateskii–Shapiro).

There is a complete classification of irreducible HSSNT, with four classical series, studied by Cartan, and two exceptional cases.

Hermitian symmetric spaces of noncompact type (HSSNT)

An irreducible HSSNT is holomorphically isometric to a bounded symmetric domain $\Omega \subset \mathbb{C}^n$ centered at the origin $0 \in \mathbb{C}^n$ equipped with (a multiple of) the Bergman metric ω_{Berg}.

There exists homogeneous bounded domains (equipped with the Bergman metric) which are not HSSNT (first examples due to Pyateskii–Shapiro).

There is a complete classification of irreducible HSSNT, with four classical series, studied by Cartan, and two exceptional cases.

Hermitian symmetric spaces of compact type (HSSCT)

Let (M, ω_{FS}) be an irreducible HSSCT, where ω_{FS} is the *canonical Kähler form*, i.e. the Kähler-Einstein form such that

$$\omega_{FS}(A) = \int_A \omega_{FS} = \pi$$

for the generator $A = [\mathbb{C}P^1] \in H_2(M, \mathbb{Z})$.

Hermitian symmetric spaces of compact type (HSSCT)

Let (M, ω_{FS}) be an irreducible HSSCT, where ω_{FS} is the *canonical Kähler form*, i.e. the Kähler-Einstein form such that

$$\omega_{FS}(A) = \int_A \omega_{FS} = \pi$$

for the generator $A = [\mathbb{C}P^1] \in H_2(M, \mathbb{Z})$.

There exists a natural number N and a holomorphic embedding

$$BW : M \to \mathbb{C}P^N$$

called the *Borel–Weil embedding*, such that

$$\omega_{FS} = BW^* \Omega_{FS}.$$

Duality

To every bounded symmetric domain $\Omega \subset \mathbb{C}^n$ one can associate an irreducible HSSCT (M, ω) called the *compact dual* of Ω (and viceversa) such that Ω is holomorphically embedded into M.

Duality

To every bounded symmetric domain $\Omega \subset \mathbb{C}^n$ one can associate an irreducible HSSCT (M, ω) called the *compact dual* of Ω (and viceversa) such that Ω is holomorphically embedded into M.

More precisely we have the following holomorphic embeddings:

$$\Omega \overset{Harish-Chandra}{\subset} \mathbb{C}^n \overset{Borel}{\subset} M \overset{BW}{\hookrightarrow} \mathbb{C}P^N$$

The first Cartan domain and the complex Grassmannian

The first Cartan domain and the complex Grassmannian

Let

$$D_I[k,n] = \{Z \in M_{k,n}(\mathbb{C}) \mid I_k - ZZ^* > 0\}$$

be the first Cartan domain.

The first Cartan domain and the complex Grassmannian

Let

$$D_I[k,n] = \{Z \in M_{k,n}(\mathbb{C}) \mid I_k - ZZ^* > 0\}$$

be the first Cartan domain.
The compact dual of $D_I[k,n]$ is the complex Grassmannian $\mathrm{Grass}_k(\mathbb{C}^{n+k})$ endowed with the Fubini-Study form ω_{FS}. More precisely,

$$D_I[k,n] \overset{Harish-Chandra}{\subset} M_{k,n}(\mathbb{C}) = \mathbb{C}^{kn} \overset{Borel}{\subset} \mathrm{Grass}_k(\mathbb{C}^{n+k}) \overset{P=Plucker}{\hookrightarrow} \mathbb{C}P^N,$$

$N = \left({n+k \atop k} \right) - 1,$

The first Cartan domain and the complex Grassmannian

Let

$$D_I[k,n] = \{Z \in M_{k,n}(\mathbb{C}) \mid I_k - ZZ^* > 0\}$$

be the first Cartan domain.
The compact dual of $D_I[k,n]$ is the complex Grassmannian $\mathrm{Grass}_k(\mathbb{C}^{n+k})$ endowed with the Fubini-Study form ω_{FS}. More precisely,

$$D_I[k,n] \overset{Harish-Chandra}{\subset} M_{k,n}(\mathbb{C}) = \mathbb{C}^{kn} \overset{Borel}{\subset} \mathrm{Grass}_k(\mathbb{C}^{n+k}) \overset{P=Plucker}{\hookrightarrow} \mathbb{C}P^N,$$

$N = \binom{n+k}{k} - 1,$

$$\omega_{FS} = P^*\Omega_{FS}, \ \omega_{FS}|_{\mathbb{C}^{kn}} = \frac{i}{2\pi}\partial\bar{\partial}\log\det(I_k + ZZ^*)$$

$$\omega_{Berg} = -2n\frac{i}{2\pi}\partial\bar{\partial}\log\det(I_k - ZZ^*).$$

Theorem 1

Let (M, ω_{FS}) be an irreducible HSSCT endowed with the canonical symplectic (Kähler) form ω_{FS}.

Theorem 1

Let (M, ω_{FS}) be an irreducible HSSCT endowed with the canonical symplectic (Kähler) form ω_{FS}. Then

$$c_G(M, \omega_{FS}) = \pi.$$

Theorem 1

Let (M, ω_{FS}) be an irreducible HSSCT endowed with the canonical symplectic (Kähler) form ω_{FS}. Then

$$c_G(M, \omega_{FS}) = \pi.$$

Theorem 2

Let (M_i, ω^i_{FS}), $i = 1, \ldots, r$, be irreducible HSSCT of complex dimension n_i endowed with the canonical symplectic (Kähler) forms ω^i_{FS}.

Theorem 1

Let (M, ω_{FS}) be an irreducible HSSCT endowed with the canonical symplectic (Kähler) form ω_{FS}. Then

$$c_G(M, \omega_{FS}) = \pi.$$

Theorem 2

Let (M_i, ω^i_{FS}), $i = 1, \ldots, r$, be irreducible HSSCT of complex dimension n_i endowed with the canonical symplectic (Kähler) forms ω^i_{FS}. Then

$$c_G\left(M_1 \times \cdots \times M_r, \omega^1_{FS} \oplus \cdots \oplus \omega^r_{FS}\right) = \pi.$$

Theorem 1

Let (M, ω_{FS}) be an irreducible HSSCT endowed with the canonical symplectic (Kähler) form ω_{FS}. Then

$$c_G(M, \omega_{FS}) = \pi.$$

Theorem 2

Let (M_i, ω^i_{FS}), $i = 1, \dots, r$, be irreducible HSSCT of complex dimension n_i endowed with the canonical symplectic (Kähler) forms ω^i_{FS}. Then

$$c_G\left(M_1 \times \cdots \times M_r, \omega^1_{FS} \oplus \cdots \oplus \omega^r_{FS}\right) = \pi.$$

Moreover, if $a_1, \dots, a_r$ are nonzero constants,

Theorem 1

Let (M, ω_{FS}) be an irreducible HSSCT endowed with the canonical symplectic (Kähler) form ω_{FS}. Then

$$c_G(M, \omega_{FS}) = \pi.$$

Theorem 2

Let (M_i, ω^i_{FS}), $i = 1, \ldots, r$, be irreducible HSSCT of complex dimension n_i endowed with the canonical symplectic (Kähler) forms ω^i_{FS}. Then

$$c_G\left(M_1 \times \cdots \times M_r, \omega^1_{FS} \oplus \cdots \oplus \omega^r_{FS}\right) = \pi.$$

Moreover, if $a_1, \ldots, a_r$ are nonzero constants, then

$$c_G\left(M_1 \times \cdots \times M_r, a_1\omega^1_{FS} \oplus \cdots \oplus a_r\omega^r_{FS}\right) \leq \min\{|a_1|, \ldots, |a_r|\}\pi.$$

Remarks on Theorem 1 and Theorem 2

Remarks on Theorem 1 and Theorem 2

Theorem 1 extends (to the case of HSSCT) the results of Y. Karshon, S. Tolman [Y. Karshon, S. Tolman, Algebr. Geom. Topol., 2005] when M is the complex Grassmannian.

Remarks on Theorem 1 and Theorem 2

Theorem 1 extends (to the case of HSSCT) the results of Y. Karshon, S. Tolman [Y. Karshon, S. Tolman, Algebr. Geom. Topol., 2005] when M is the complex Grassmannian.

Theorem 2 extends (to the case of HSSCT) G. Lu's results [G. Lu, Israel J. Math., 2006] when M_j are complex Grassmannians.

Remarks on Theorem 1 and Theorem 2

Theorem 1 extends (to the case of HSSCT) the results of Y. Karshon, S. Tolman [Y. Karshon, S. Tolman, Algebr. Geom. Topol., 2005] when M is the complex Grassmannian.

Theorem 2 extends (to the case of HSSCT) G. Lu's results [G. Lu, Israel J. Math., 2006] when M_j are complex Grassmannians.

When $M_j = \mathbb{C}P^1$ for all $j = 1, \ldots, r$,

$$c_G(\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1, a_1\omega_{FS} \oplus \cdots \oplus a_r\omega_{FS}) = \min\{|a_1|, \ldots, |a_r|\}\pi.$$

[G. Lu, Israel J. Math., 2006]

Remarks on Theorem 1 and Theorem 2

Theorem 1 extends (to the case of HSSCT) the results of Y. Karshon, S. Tolman [Y. Karshon, S. Tolman, Algebr. Geom. Topol., 2005] when M is the complex Grassmannian.

Theorem 2 extends (to the case of HSSCT) G. Lu's results [G. Lu, Israel J. Math., 2006] when M_j are complex Grassmannians.

When $M_j = \mathbb{C}P^1$ for all $j = 1, \ldots, r$,

$$c_G(\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1, a_1\omega_{FS} \oplus \cdots \oplus a_r\omega_{FS}) = \min\{|a_1|, \ldots, |a_r|\}\pi.$$

[G. Lu, Israel J. Math., 2006]

$$c_G(\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_r}, a_1\omega^1_{FS} \oplus \cdots \oplus a_r\omega^r_{FS}) = ?$$

if some $n_j > 1$ or $|a_j| \neq 1$.

Idea of the proof of Theorem 1: the upper bound

Step 1

Idea of the proof of Theorem 1: the upper bound

Step 1

- The proof of the upper bound $c_G(M, \omega_{FS}) \leq \pi$ is obtained by the computations of some genus-zero three-points Gromov-Witten invariants for irreducible HSSCT [A. Beauville, Mat. Fiz. Anal. Geom., 1995], [P. E. Chaput, L. Manivel, N. Perrin, Transform. Groups, 2008], [B. Siebert, G. Tian, Asian J. Math., 1997]

Idea of the proof of Theorem 1: the upper bound

Step 1

- The proof of the upper bound $c_G(M, \omega_{FS}) \leq \pi$ is obtained by the computations of some genus-zero three-points Gromov-Witten invariants for irreducible HSSCT [A. Beauville, Mat. Fiz. Anal. Geom., 1995], [P. E. Chaput, L. Manivel, N. Perrin, Transform. Groups, 2008], [B. Siebert, G. Tian, Asian J. Math., 1997] and through nonsqueezing theorem techinques using and extending the ideas in [Y. Karshon, S. Tolman, Algebr. Geom. Topol., 2005] for the complex Grassmannian.

  ▸ details

Idea of the proof of Theorem 1: the lower bound

Step 2 The lower bound $c_G(M, \omega_{FS}) \geq \pi$ is obtained as follows.

Idea of the proof of Theorem 1: the lower bound

Step 2 The lower bound $c_G(M, \omega_{FS}) \geq \pi$ is obtained as follows.

- Let (Ω, ω_0), $\Omega \subset \mathbb{C}^n$, be the bounded symmetric domain noncompact dual of (M, ω_{FS}) equipped with the canonical symplectic form ω_0 of $\mathbb{R}^{2n}$.

Idea of the proof of Theorem 1: the lower bound

Step 2 The lower bound $c_G(M, \omega_{FS}) \geq \pi$ is obtained as follows.

- Let (Ω, ω_0), $\Omega \subset \mathbb{C}^n$, be the bounded symmetric domain noncompact dual of (M, ω_{FS}) equipped with the canonical symplectic form ω_0 of $\mathbb{R}^{2n}$. Then there exists a symplectic embedding

$$\Phi_\Omega : (\Omega, \omega_0) \to (M, \omega_{FS})$$

go to Example [A. J. Di Scala, AL, Symplectic duality of Symmetric Spaces, Adv.Math., 2008]

Idea of the proof of Theorem 1: the lower bound

Step 2 The lower bound $c_G(M, \omega_{FS}) \geq \pi$ is obtained as follows.

- Let (Ω, ω_0), $\Omega \subset \mathbb{C}^n$, be the bounded symmetric domain noncompact dual of (M, ω_{FS}) equipped with the canonical symplectic form ω_0 of $\mathbb{R}^{2n}$. Then there exists a symplectic embedding

$$\Phi_\Omega : (\Omega, \omega_0) \to (M, \omega_{FS})$$

▸ go to Example [A. J. Di Scala, AL, Symplectic duality of Symmetric Spaces, Adv.Math., 2008]

- Using Jordan triple systems tools one can prove that there exists a symplectic embedding

$$(B^{2n}(1), \omega_0) \hookrightarrow (\Omega, \omega_0)$$

Idea of the proof of Theorem 1: the lower bound

Step 2 The lower bound $c_G(M, \omega_{FS}) \geq \pi$ is obtained as follows.

- Let (Ω, ω_0), $\Omega \subset \mathbb{C}^n$, be the bounded symmetric domain noncompact dual of (M, ω_{FS}) equipped with the canonical symplectic form ω_0 of $\mathbb{R}^{2n}$. Then there exists a symplectic embedding

$$\Phi_\Omega : (\Omega, \omega_0) \to (M, \omega_{FS})$$

go to Example [A. J. Di Scala, AL, Symplectic duality of Symmetric Spaces, Adv.Math., 2008]

- Using Jordan triple systems tools one can prove that there exists a symplectic embedding

$$(B^{2n}(1), \omega_0) \hookrightarrow (\Omega, \omega_0) \stackrel{\Phi_\Omega}{\to} (M, \omega_{FS})$$

Idea of the proof of Theorem 1: the lower bound

Step 2 The lower bound $c_G(M, \omega_{FS}) \geq \pi$ is obtained as follows.

- Let (Ω, ω_0), $\Omega \subset \mathbb{C}^n$, be the bounded symmetric domain noncompact dual of (M, ω_{FS}) equipped with the canonical symplectic form ω_0 of $\mathbb{R}^{2n}$. Then there exists a symplectic embedding

$$\Phi_\Omega : (\Omega, \omega_0) \to (M, \omega_{FS})$$

go to Example [A. J. Di Scala, AL, Symplectic duality of Symmetric Spaces, Adv.Math., 2008]

- Using Jordan triple systems tools one can prove that there exists a symplectic embedding

$$(B^{2n}(1), \omega_0) \hookrightarrow (\Omega, \omega_0) \stackrel{\Phi_\Omega}{\to} (M, \omega_{FS}) \Rightarrow c_G(M, \omega_{FS}) \geq \pi.$$

Idea of the proof of Theorem 1: the lower bound

Step 2 The lower bound $c_G(M, \omega_{FS}) \geq \pi$ is obtained as follows.

- Let (Ω, ω_0), $\Omega \subset \mathbb{C}^n$, be the bounded symmetric domain noncompact dual of (M, ω_{FS}) equipped with the canonical symplectic form ω_0 of $\mathbb{R}^{2n}$. Then there exists a symplectic embedding

$$\Phi_\Omega : (\Omega, \omega_0) \to (M, \omega_{FS})$$

go to Example [A. J. Di Scala, AL, Symplectic duality of Symmetric Spaces, Adv.Math., 2008]

- Using Jordan triple systems tools one can prove that there exists a symplectic embedding

$$(B^{2n}(1), \omega_0) \hookrightarrow (\Omega, \omega_0) \stackrel{\Phi_\Omega}{\to} (M, \omega_{FS}) \Rightarrow c_G(M, \omega_{FS}) \geq \pi.$$

The embedding Φ_Ω induces a global symplectomorphism

$$(\Omega, \omega_0) \stackrel{\Phi_\Omega}{\to} (M \setminus \mathrm{Cut}_0(M) \cong \mathbb{C}^n, \omega_{FS}) \stackrel{Borel}{\subset} (M, \omega_{FS}) \stackrel{BW}{\hookrightarrow} (\mathbb{C}P^N, \omega_{FS})$$

Idea of the proof of Theorem 2

Steps

Idea of the proof of Theorem 2

Steps

- The lower bound $c_G\left(M_1 \times \cdots \times M_r, \omega_{FS}^1 \oplus \cdots \oplus \omega_{FS}^r\right) \geq \pi$ is obtained by (motonicity)+(nontriviality) of c_G and the embeddings

$$B^{2n_1+\cdots+2n_r}(1) \subset \times_{j=1}^r B^{2n_j}(1) \subset \times_{j=1}^r \Omega_j \overset{\Phi_{\Omega_1}\times\cdots\times\Phi_{\Omega_r}}{\longrightarrow} \times_{j=1}^r M_j.$$

Idea of the proof of Theorem 2

Steps

- The lower bound $c_G\left(M_1\times\cdots\times M_r, \omega_{FS}^1\oplus\cdots\oplus\omega_{FS}^r\right)\geq\pi$ is obtained by (motonicity)+(nontriviality) of c_G and the embeddings

$$B^{2n_1+\cdots+2n_r}(1)\subset\times_{j=1}^r B^{2n_j}(1)\subset\times_{j=1}^r\Omega_j\stackrel{\Phi_{\Omega_1}\times\cdots\times\Phi_{\Omega_r}}{\longrightarrow}\times_{j=1}^r M_j.$$

- The upper bound

$$c_G\left(M_1\times\cdots\times M_r, a_1\omega_{FS}^1\oplus\cdots\oplus a_r\omega_{FS}^r\right)\leq\min\{|a_1|,\ldots,|a_r|\}\pi$$

and hence $c_G\left(M_1\times\cdots\times M_r, \omega_{FS}^1\oplus\cdots\oplus\omega_{FS}^r\right)\leq\pi$ is obtained by combining $c_G(M_j,\omega_{FS}^j)\leq\pi$ with the following theorem.

Theorem

Let (M, ω_{FS}) be an irreducible HSSCT and (N, ω) be any closed symplectic manifold. Then, for any nonzero real number a,

$$c_G(N \times M, \omega \oplus a\omega_{FS}) \leq |a|\pi.$$

Theorem

Let (M, ω_{FS}) be an irreducible HSSCT and (N, ω) be any closed symplectic manifold. Then, for any nonzero real number a,

$$c_G(N \times M, \omega \oplus a\omega_{FS}) \leq |a|\pi.$$

Remark

The proof of the theorem uses Lu's pseudo symplectic capacities and their estimation in terms of Gromov-Witten invariants.

Theorem 3

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain.

Theorem 3

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain. Then

$$c_G(\Omega, \omega_0) = \pi.$$

Theorem 3

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain. Then

$$c_G(\Omega, \omega_0) = \pi.$$

Theorem 4

Let $\Omega_i \subset \mathbb{C}^{n_i}$, $i = 1, \ldots, r$, be bounded symmetric domains of complex dimension n_i equipped with the standard symplectic form ω_0^i of $\mathbb{R}^{2n_i} = \mathbb{C}^{n_i}$.

Theorem 3

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain. Then

$$c_G(\Omega, \omega_0) = \pi.$$

Theorem 4

Let $\Omega_i \subset \mathbb{C}^{n_i}$, $i = 1, \dots, r$, be bounded symmetric domains of complex dimension n_i equipped with the standard symplectic form ω_0^i of $\mathbb{R}^{2n_i} = \mathbb{C}^{n_i}$. Then

$$c_G\left(\Omega_1 \times \cdots \times \Omega_r, \omega_0^1 \oplus \cdots \oplus \omega_0^r\right) = \pi.$$

Theorem 3

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain. Then

$$c_G(\Omega, \omega_0) = \pi.$$

Theorem 4

Let $\Omega_i \subset \mathbb{C}^{n_i}$, $i = 1, \ldots, r$, be bounded symmetric domains of complex dimension n_i equipped with the standard symplectic form ω_0^i of $\mathbb{R}^{2n_i} = \mathbb{C}^{n_i}$. Then

$$c_G\left(\Omega_1 \times \cdots \times \Omega_r, \omega_0^1 \oplus \cdots \oplus \omega_0^r\right) = \pi.$$

Moreover, If $a_1, \ldots, a_r$ are nonzero constants,

Theorem 3

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain. Then

$$c_G(\Omega, \omega_0) = \pi.$$

Theorem 4

Let $\Omega_i \subset \mathbb{C}^{n_i}$, $i = 1, \ldots, r$, be bounded symmetric domains of complex dimension n_i equipped with the standard symplectic form ω_0^i of $\mathbb{R}^{2n_i} = \mathbb{C}^{n_i}$. Then

$$c_G\left(\Omega_1 \times \cdots \times \Omega_r, \omega_0^1 \oplus \cdots \oplus \omega_0^r\right) = \pi.$$

Moreover, If $a_1, \ldots, a_r$ are nonzero constants, then

$$c_G\left(\Omega_1 \times \cdots \times \Omega_r, a_1\omega_0^1 \oplus \cdots \oplus a_r\omega_0^r\right) \leq \min\{|a_1|, \ldots, |a_r|\}\pi.$$

Remarks on Theorem 3 and Theorem 4

Remarks on Theorem 3 and Theorem 4

Theorem 3 extends to the product of HSSNT (including the exceptional ones) the results in [G. Lu, H. Ding, Q. Zhang, Int. Math. Forum 2, 2007] valid for classical Cartan domains.

Remarks on Theorem 3 and Theorem 4

Theorem 3 extends to the product of HSSNT (including the exceptional ones) the results in [G. Lu, H. Ding, Q. Zhang, Int. Math. Forum 2, 2007] valid for classical Cartan domains.

Notice that

$$c_G(\Omega, \omega_{Berg}) = +\infty.$$

Remarks on Theorem 3 and Theorem 4

Theorem 3 extends to the product of HSSNT (including the exceptional ones) the results in [G. Lu, H. Ding, Q. Zhang, Int. Math. Forum 2, 2007] valid for classical Cartan domains.

Notice that

$$c_G(\Omega, \omega_{Berg}) = +\infty.$$

Indeed, by the following result of D. McDuff [D. McDuff, J. Diff. Geometry, 1988] (Ω, ω_{Berg}) is globally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Remarks on Theorem 3 and Theorem 4

Theorem 3 extends to the product of HSSNT (including the exceptional ones) the results in [G. Lu, H. Ding, Q. Zhang, Int. Math. Forum 2, 2007] valid for classical Cartan domains.

Notice that

$$c_G(\Omega, \omega_{Berg}) = +\infty.$$

Indeed, by the following result of D. McDuff [D. McDuff, J. Diff. Geometry, 1988] (Ω, ω_{Berg}) is globally symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

Theorem (McDuff)

Let (M, ω) be a Kähler manifold. Assume that $\pi_1(M) = \{1\}$, M is complete and $K \leq 0$. Then there exists a symplectomorphism

$$\psi : (M, \omega) \to (\mathbb{R}^{2n}, \omega_0).$$

The proof of Theorem 3

The proof of Theorem 3

$(B^{2n}(1), \omega_0) \hookrightarrow (\Omega, \omega_0) \stackrel{\Phi_\Omega}{\to} (M, \omega_{FS})$,

The proof of Theorem 3

$$(B^{2n}(1), \omega_0) \hookrightarrow (\Omega, \omega_0) \stackrel{\Phi_\Omega}{\to} (M, \omega_{FS}),\ c_G(B^{2n}(1), \omega_0) = c_G(M, \omega_{FS}) \stackrel{Th1}{=} \pi$$

The proof of Theorem 3

$(B^{2n}(1),\omega_0)\hookrightarrow(\Omega,\omega_0)\overset{\Phi_\Omega}{\to}(M,\omega_{FS}),\ c_G(B^{2n}(1),\omega_0)=c_G(M,\omega_{FS})\overset{Th1}{=}\pi$

$\Rightarrow$

The proof of Theorem 3

$(B^{2n}(1), \omega_0) \hookrightarrow (\Omega, \omega_0) \stackrel{\Phi_\Omega}{\to} (M, \omega_{FS})$, $c_G(B^{2n}(1), \omega_0) = c_G(M, \omega_{FS}) \stackrel{Th1}{=} \pi$

$\Rightarrow$ $c_G(\Omega, \omega_0) = \pi$

The proof of Theorem 3

$(B^{2n}(1), \omega_0) \hookrightarrow (\Omega, \omega_0) \overset{\Phi_\Omega}{\to} (M, \omega_{FS})$, $c_G(B^{2n}(1), \omega_0) = c_G(M, \omega_{FS}) \overset{Th1}{=} \pi$

$\Rightarrow$ $c_G(\Omega, \omega_0) = \pi$

The proof of Theorem 4

The proof of Theorem 3

$(B^{2n}(1), \omega_0) \hookrightarrow (\Omega, \omega_0) \overset{\Phi_\Omega}{\to} (M, \omega_{FS})$, $c_G(B^{2n}(1), \omega_0) = c_G(M, \omega_{FS}) \overset{Th1}{=} \pi$

$\Rightarrow$ $c_G(\Omega, \omega_0) = \pi$

The proof of Theorem 4

$$B^{2n_1+\cdots+2n_r}(1) \subset \times_{j=1}^{r} B^{2n_j}(1) \subset \times_{j=1}^{r} \Omega_j \xrightarrow{\Phi_{\Omega_1}\times\cdots\times\Phi_{\Omega_r}} \times_{j=1}^{r} M_j$$

The proof of Theorem 3

$(B^{2n}(1), \omega_0) \hookrightarrow (\Omega, \omega_0) \overset{\Phi_\Omega}{\to} (M, \omega_{FS})$, $c_G(B^{2n}(1), \omega_0) = c_G(M, \omega_{FS}) \overset{Th1}{=} \pi$

$\Rightarrow c_G(\Omega, \omega_0) = \pi$

The proof of Theorem 4

$$B^{2n_1+\cdots+2n_r}(1) \subset \times_{j=1}^{r} B^{2n_j}(1) \subset \times_{j=1}^{r} \Omega_j \xrightarrow{\Phi_{\Omega_1} \times \cdots \times \Phi_{\Omega_r}} \times_{j=1}^{r} M_j$$

$$c_G(B^{2n_1+\cdots+2n_r}(1), \omega_0) = c_G\left(M_1 \times \cdots \times M_r, \omega_{FS}^1 \oplus \cdots \oplus \omega_{FS}^r\right) \overset{Th2}{=} \pi$$

The proof of Theorem 3

$$(B^{2n}(1),\omega_0) \hookrightarrow (\Omega,\omega_0) \stackrel{\Phi_\Omega}{\to} (M,\omega_{FS}),\ c_G(B^{2n}(1),\omega_0) = c_G(M,\omega_{FS}) \stackrel{Th1}{=} \pi$$

$\Rightarrow$ $c_G(\Omega,\omega_0)=\pi$

The proof of Theorem 4

$$B^{2n_1+\cdots+2n_r}(1) \subset \times_{j=1}^{r} B^{2n_j}(1) \subset \times_{j=1}^{r}\Omega_j \stackrel{\Phi_{\Omega_1}\times\cdots\times\Phi_{\Omega_r}}{\longrightarrow} \times_{j=1}^{r} M_j$$

$$c_G(B^{2n_1+\cdots+2n_r}(1),\omega_0) = c_G\left(M_1\times\cdots\times M_r, \omega_{FS}^1\oplus\cdots\oplus\omega_{FS}^r\right) \stackrel{Th2}{=} \pi$$

$\Rightarrow$

The proof of Theorem 3

$(B^{2n}(1),\omega_0)\hookrightarrow(\Omega,\omega_0)\overset{\Phi_\Omega}{\to}(M,\omega_{FS}),\ c_G(B^{2n}(1),\omega_0)=c_G(M,\omega_{FS})\overset{Th1}{=}\pi$

$\Rightarrow c_G(\Omega,\omega_0)=\pi$

The proof of Theorem 4

$$B^{2n_1+\cdots+2n_r}(1)\subset\times_{j=1}^r B^{2n_j}(1)\subset\times_{j=1}^r\Omega_j\overset{\Phi_{\Omega_1}\times\cdots\times\Phi_{\Omega_r}}{\longrightarrow}\times_{j=1}^r M_j$$

$$c_G(B^{2n_1+\cdots+2n_r}(1),\omega_0)=c_G\left(M_1\times\cdots\times M_r,\omega_{FS}^1\oplus\cdots\oplus\omega_{FS}^r\right)\overset{Th2}{=}\pi$$

$\Rightarrow c_G\left(\Omega_1\times\cdots\times\Omega_r,\omega_0^1\oplus\cdots\oplus\omega_0^r\right)=\pi$

The proof of Theorem 3

$(B^{2n}(1), \omega_0) \hookrightarrow (\Omega, \omega_0) \overset{\Phi_\Omega}{\to} (M, \omega_{FS})$, $c_G(B^{2n}(1), \omega_0) = c_G(M, \omega_{FS}) \overset{Th1}{=} \pi$

$\Rightarrow$ $c_G(\Omega, \omega_0) = \pi$

The proof of Theorem 4

$$B^{2n_1+\cdots+2n_r}(1) \subset \times_{j=1}^r B^{2n_j}(1) \subset \times_{j=1}^r \Omega_j \overset{\Phi_{\Omega_1}\times\cdots\times\Phi_{\Omega_r}}{\longrightarrow} \times_{j=1}^r M_j$$

$$c_G(B^{2n_1+\cdots+2n_r}(1), \omega_0) = c_G\left(M_1 \times \cdots \times M_r, \omega_{FS}^1 \oplus \cdots \oplus \omega_{FS}^r\right) \overset{Th2}{=} \pi$$

$\Rightarrow$ $c_G\left(\Omega_1 \times \cdots \times \Omega_r, \omega_0^1 \oplus \cdots \oplus \omega_0^r\right) = \pi$

$$c_G\left(M_1 \times \cdots \times M_r, a_1\omega_{FS}^1 \oplus \cdots \oplus a_r\omega_{FS}^r\right) \overset{Th2}{\leq} \min\{|a_1|, \ldots, |a_r|\}\pi$$

The proof of Theorem 3

$$(B^{2n}(1),\omega_0)\hookrightarrow(\Omega,\omega_0)\stackrel{\Phi_\Omega}{\to}(M,\omega_{FS}),\ c_G(B^{2n}(1),\omega_0)=c_G(M,\omega_{FS})\stackrel{Th1}{=}\pi$$

$\Rightarrow$ $c_G(\Omega,\omega_0)=\pi$

The proof of Theorem 4

$$B^{2n_1+\cdots+2n_r}(1)\subset\times_{j=1}^r B^{2n_j}(1)\subset\times_{j=1}^r\Omega_j\stackrel{\Phi_{\Omega_1}\times\cdots\times\Phi_{\Omega_r}}{\longrightarrow}\times_{j=1}^r M_j$$

$$c_G(B^{2n_1+\cdots+2n_r}(1),\omega_0)=c_G\left(M_1\times\cdots\times M_r,\omega_{FS}^1\oplus\cdots\oplus\omega_{FS}^r\right)\stackrel{Th2}{=}\pi$$

$\Rightarrow$ $c_G\left(\Omega_1\times\cdots\times\Omega_r,\omega_0^1\oplus\cdots\oplus\omega_0^r\right)=\pi$

$$c_G\left(M_1\times\cdots\times M_r,a_1\omega_{FS}^1\oplus\cdots\oplus a_r\omega_{FS}^r\right)\stackrel{Th2}{\leq}\min\{|a_1|,\ldots,|a_r|\}\pi$$

$\Rightarrow$

The proof of Theorem 3

$$(B^{2n}(1), \omega_0) \hookrightarrow (\Omega, \omega_0) \overset{\Phi_\Omega}{\to} (M, \omega_{FS}),\ c_G(B^{2n}(1), \omega_0) = c_G(M, \omega_{FS}) \overset{Th1}{=} \pi$$

$\Rightarrow$ $c_G(\Omega, \omega_0) = \pi$

The proof of Theorem 4

$$B^{2n_1+\cdots+2n_r}(1) \subset \times_{j=1}^r B^{2n_j}(1) \subset \times_{j=1}^r \Omega_j \overset{\Phi_{\Omega_1}\times\cdots\times\Phi_{\Omega_r}}{\longrightarrow} \times_{j=1}^r M_j$$

$$c_G(B^{2n_1+\cdots+2n_r}(1), \omega_0) = c_G\left(M_1 \times \cdots \times M_r, \omega_{FS}^1 \oplus \cdots \oplus \omega_{FS}^r\right) \overset{Th2}{=} \pi$$

$\Rightarrow$ $c_G\left(\Omega_1 \times \cdots \times \Omega_r, \omega_0^1 \oplus \cdots \oplus \omega_0^r\right) = \pi$

$$c_G\left(M_1 \times \cdots \times M_r, a_1\omega_{FS}^1 \oplus \cdots \oplus a_r\omega_{FS}^r\right) \overset{Th2}{\leq} \min\{|a_1|, \ldots, |a_r|\}\pi$$

$\Rightarrow$ $c_G\left(\Omega_1 \times \cdots \times \Omega_r, a_1\omega_0^1 \oplus \cdots \oplus a_r\omega_0^r\right) \leq \min\{|a_1|, \ldots, |a_r|\}\pi.$

Theorem 5

Theorem 5

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain.

Theorem 5

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain. Then

$$c(\Omega, \omega_0) = \pi$$

for any symplectic capacity c.

Theorem 5

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain. Then

$$c(\Omega, \omega_0) = \pi$$

for any symplectic capacity c.

Idea of the proof

Theorem 5

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain. Then

$$c(\Omega, \omega_0) = \pi$$

for any symplectic capacity c.

Idea of the proof

Using Jordan triple systems tools one can prove that

$$B^{2n}(1) \subset \Omega \subset Z^{2n}(1).$$

Theorem 5

Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain. Then

$$c(\Omega, \omega_0) = \pi$$

for any symplectic capacity c.

Idea of the proof

Using Jordan triple systems tools one can prove that

$$B^{2n}(1) \subset \Omega \subset Z^{2n}(1).$$

Hence the conclusion follows by (monotonicity)+(nontriviality) of c.

Biran's conjecture

Biran's conjecture

Let (M, ω) be a closed symplectic manifold with $[\omega] \in H^2(M, \mathbb{Z})$.

Biran's conjecture

Let (M, ω) be a closed symplectic manifold with $[\omega] \in H^2(M, \mathbb{Z})$. Then $c_G(M, \omega) \geq \pi$.

Hofer–Zehnder capacity c_{HZ}

Hofer–Zehnder capacity c_{HZ}

In [H. Hofer, E. Zehnder, A new capacity for symplectic manifolds, Academic Press, New York 1990]
Hofer and Zehnder defines a symplectic capacity c_{HZ}, which satisfies

$$c_{HZ}(M,\omega) \geq c(M,\omega)$$

for all symplectic capacity c.

Known results on c_{HZ}

Known results on c_{HZ}

Theorem (Hofer–Viterbo)

$$c_{HZ}(\mathbb{C}P^n, \omega_{FS}) = \pi$$

[H. Hofer and C. Viterbo, The Weinstein conjecture...., Comm. Pure and Applied Math. 45, 1992]

Known results on c_{HZ}

Theorem (Hofer–Viterbo)

$$c_{HZ}(\mathbb{C}P^n, \omega_{FS}) = \pi$$

[H. Hofer and C. Viterbo, The Weinstein conjecture...., Comm. Pure and Applied Math. 45, 1992]

Theorem (Lu)

Let $a_j \neq 0, j = 1, \ldots r$. Then

$$c_{HZ}(\mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_r}, a_1\omega^1_{FS} \oplus \cdots \oplus a_r\omega^r_{FS}) = (|a_1| + \cdots + |a_r|)\pi.$$

[G. Lu, Israel J. Math., 2006].

Results on c_{HZ}

Results on c_{HZ}

Theorem 6

Let (M_i, ω_{FS}^i), $i = 1, \ldots, r$, be irreducible HSSCT of complex dimension n_i

Results on c_{HZ}

Theorem 6

Let (M_i, ω^i_{FS}), $i = 1, \ldots, r$, be irreducible HSSCT of complex dimension n_i
Then

$$c_{HZ}\left(M_1 \times \cdots \times M_r, a_1\omega^1_{FS} \oplus \cdots \oplus a_r\omega^r_{FS}\right) \geq \{|a_1| + \cdots + |a_r|\}\pi$$

Results on c_{HZ}

Theorem 6

Let (M_i, ω^i_{FS}), $i = 1, \ldots, r$, be irreducible HSSCT of complex dimension n_i Then

$$c_{HZ}\left(M_1 \times \cdots \times M_r, a_1\omega^1_{FS} \oplus \cdots \oplus a_r\omega^r_{FS}\right) \geq \{|a_1| + \cdots + |a_r|\}\pi$$

Remark

Theorem 6 extends a theorem of Lu when M_j are complex Grassmannians.

Results on c_{HZ}

Theorem 6

Let (M_i, ω^i_{FS}), $i = 1, \dots, r$, be irreducible HSSCT of complex dimension n_i Then

$$c_{HZ}\left(M_1 \times \cdots \times M_r, a_1\omega^1_{FS} \oplus \cdots \oplus a_r\omega^r_{FS}\right) \geq \{|a_1| + \cdots + |a_r|\}\pi$$

Remark

Theorem 6 extends a theorem of Lu when M_j are complex Grassmannians.

Open problem

$$c_{HZ}(M, \omega_{FS}) = ?$$

The symplectic Lusternik-Schnirelmann category

The symplectic Lusternik-Schnirelmann category

Let (N, ω) be a closed symplectc manifold. The *symplectic Lusternik-Schnirelmann category* $S(N, \omega)$ [Y. B. Rudyak, F. Schlenk, Commun. Contemp. Math., 2007] is defined as

$$S(N, \omega) = \min\{k \mid N = V_1 \cup \cdots \cup V_k\}$$

The symplectic Lusternik-Schnirelmann category

Let (N, ω) be a closed symplectc manifold. The *symplectic Lusternik-Schnirelmann category* $S(N, \omega)$ [Y. B. Rudyak, F. Schlenk, Commun. Contemp. Math., 2007] is defined as

$$S(N, \omega) = \min\{k \mid N = V_1 \cup \cdots \cup V_k\}$$

where each $V_i = \Phi_i(U_i)$ by a symplectic embedding $\Phi_i : U_i \to V_i \subset N$, U_i bounded subset of $(\mathbb{R}^{2n}, \omega_0)$ diffeomorphic to an open ball in $\mathbb{R}^{2n}$.

The symplectic Lusternik-Schnirelmann category

Let (N, ω) be a closed symplectc manifold. The *symplectic Lusternik-Schnirelmann category* $S(N, \omega)$ [Y. B. Rudyak, F. Schlenk, Commun. Contemp. Math., 2007] is defined as

$$S(N, \omega) = \min\{k \mid N = V_1 \cup \cdots \cup V_k\}$$

where each $V_i = \Phi_i(U_i)$ by a symplectic embedding $\Phi_i : U_i \to V_i \subset N$, U_i bounded subset of $(\mathbb{R}^{2n}, \omega_0)$ diffeomorphic to an open ball in $\mathbb{R}^{2n}$.

Theorem 7

Let (M, ω_{FS}) be an irreducible HSSCT and $BW : M \to \mathbb{C}P^N$ the Borel–Weil embedding.

The symplectic Lusternik-Schnirelmann category

Let (N, ω) be a closed symplectc manifold. The *symplectic Lusternik-Schnirelmann category* $S(N, \omega)$ [Y. B. Rudyak, F. Schlenk, Commun. Contemp. Math., 2007] is defined as

$$S(N, \omega) = \min\{k \mid N = V_1 \cup \cdots \cup V_k\}$$

where each $V_i = \Phi_i(U_i)$ by a symplectic embedding $\Phi_i : U_i \to V_i \subset N$, U_i bounded subset of $(\mathbb{R}^{2n}, \omega_0)$ diffeomorphic to an open ball in $\mathbb{R}^{2n}$.

Theorem 7

Let (M, ω_{FS}) be an irreducible HSSCT and $BW : M \to \mathbb{C}P^N$ the Borel–Weil embedding. Then

$$S(M, \omega_{FS}) \leq N + 1$$

Darboux charts

Darboux charts

Let (M, ω) be a closed symplectic manifold. By Darboux theorem for each point $p \in M$ there exists a symplectic embedding $\varphi : (B^{2n}(r), \omega_0) \to (M, \omega)$, for some $r > 0$. One calls $(B^{2n}(r), \varphi)$ a Darboux chart. [Y. B. Rudyak, F. Schlenk, Commun. Contemp. Math., 2007] .

Darboux charts

Let (M, ω) be a closed symplectic manifold. By Darboux theorem for each point $p \in M$ there exists a symplectic embedding $\varphi : (B^{2n}(r), \omega_0) \to (M, \omega)$, for some $r > 0$. One calls $(B^{2n}(r), \varphi)$ a Darboux chart. [Y. B. Rudyak, F. Schlenk, Commun. Contemp. Math., 2007] .

Let

$$S_B(M, \omega) = \min\{k \mid M = B_1 \cup \cdots \cup B_k\}$$

Darboux charts

Let (M, ω) be a closed symplectic manifold. By Darboux theorem for each point $p \in M$ there exists a symplectic embedding $\varphi : (B^{2n}(r), \omega_0) \to (M, \omega)$, for some $r > 0$. One calls $(B^{2n}(r), \varphi)$ a Darboux chart. [Y. B. Rudyak, F. Schlenk, Commun. Contemp. Math., 2007] .

Let

$$S_B(M, \omega) = \min\{k \mid M = B_1 \cup \cdots \cup B_k\}$$

where each B_i is the image $\varphi_i(B^{2n}(r_i))$ of a Darboux chart.

Darboux charts

Let (M, ω) be a closed symplectic manifold. By Darboux theorem for each point $p \in M$ there exists a symplectic embedding $\varphi : (B^{2n}(r), \omega_0) \to (M, \omega)$, for some $r > 0$. One calls $(B^{2n}(r), \varphi)$ a Darboux chart. [Y. B. Rudyak, F. Schlenk, Commun. Contemp. Math., 2007] .

Let

$$S_B(M, \omega) = \min\{k \mid M = B_1 \cup \cdots \cup B_k\}$$

where each B_i is the image $\varphi_i(B^{2n}(r_i))$ of a Darboux chart.

Problem

Let (M, ω_{FS}) be an irreducible HSSCT. Compute (or estimate) $S_B(M, \omega_{FS})$.

Symplectic packings

Symplectic packings

For an integer $k > 0$ and $r > 0$, a **symplectic k-packing by balls of radius** r of a $2n$-dimensional symplectic manifold (M, ω) is a set of symplectic embeddings

$$\varphi_i : (B^{2n}(r), \omega_0) \to (M, \omega), \;\; i = 1, \ldots, k$$

such that $\varphi_i(B^{2n}(r)) \cap \varphi_j(B^{2n}(r)) = \emptyset$, for $i \neq j$.

Symplectic packings

For an integer $k > 0$ and $r > 0$, a **symplectic k-packing by balls of radius r** of a $2n$-dimensional symplectic manifold (M, ω) is a set of symplectic embeddings

$$\varphi_i : (B^{2n}(r), \omega_0) \to (M, \omega), \; i = 1, \dots, k$$

such that $\varphi_i(B^{2n}(r)) \cap \varphi_j(B^{2n}(r)) = \emptyset$, for $i \neq j$. A **full symplectic k-packing** is a symplectic k-packing such that

$$M = \overline{\cup_{i=1}^{k} \varphi_i(B^{2n}(r))}$$

[Gromov, Inv. Math., 1985], [McDuff, Polterovich and Karshon, Inv. Math., 1994], [Traynor, J. Diff. Geom., 1995], [Biran, Inv. Math., 1997].

Symplectic packings

For an integer $k > 0$ and $r > 0$, a **symplectic k-packing by balls of radius r** of a $2n$-dimensional symplectic manifold (M, ω) is a set of symplectic embeddings

$$\varphi_i : (B^{2n}(r), \omega_0) \to (M, \omega), \ \ i = 1, \ldots, k$$

such that $\varphi_i(B^{2n}(r)) \cap \varphi_j(B^{2n}(r)) = \emptyset$, for $i \neq j$. A **full symplectic k-packing** is a symplectic k-packing such that

$$M = \overline{\cup_{i=1}^{k} \varphi_i(B^{2n}(r))}$$

[Gromov, Inv. Math., 1985], [McDuff, Polterovich and Karshon, Inv. Math., 1994], [Traynor, J. Diff. Geom., 1995], [Biran, Inv. Math., 1997].

Problem

Studying (full) symplectic k-packings of HSSCT.

Fefferman invariant

Fefferman invariant

Let (M, ω) be a closed symplectic manifold. Its *Fefferman invariant* $F(M, \omega)$ is the largest integer p for which there exists a symplectic p-packing (not necessarily full) of ball of radius 1.

Fefferman invariant

Let (M,ω) be a closed symplectic manifold. Its *Fefferman invariant* $F(M,\omega)$ is the largest integer p for which there exists a symplectic p-packing (not necessarily full) of ball of radius 1.

When $\mathrm{Grass}_k(\mathbb{C}^n)$ G. Lu [G. Lu, Israel J. Math., 2006] shows that

$$F(\mathrm{Grass}_k(\mathbb{C}^n),\omega_{FS}) \leq [n/k].$$

Fefferman invariant

Let (M, ω) be a closed symplectic manifold. Its *Fefferman invariant* $F(M, \omega)$ is the largest integer p for which there exists a symplectic p-packing (not necessarily full) of ball of radius 1.

When $\mathrm{Grass}_k(\mathbb{C}^n)$ G. Lu [G. Lu, Israel J. Math., 2006] shows that

$$F(\mathrm{Grass}_k(\mathbb{C}^n), \omega_{FS}) \leq [n/k].$$

Problem : find a similar upper bound for HSSCT.

THANK YOU FOR YOUR ATTENTION!

- (monotonicity) for c_G follows immediately by the definition of Gromov width.

- (monotonicity) for c_G follows immediately by the definition of Gromov width.
- Given a symplectic embedding

$$\varphi : (B^{2n}(r), \omega_0) \to (M, \lambda\omega)$$

it is not hard to construct a symplectic embedding

$$\hat{\varphi} : \left(B^{2n}(\frac{r}{\sqrt{|\lambda|}}), \omega_0\right) \to (M, \omega)$$

and viceversa. Thus (conformality) for c_G follows by the definition of Gromov width. go back

Lemma

Let (M, ω) be a monotone symplectic manifold (i.e. there exists $\lambda > 0$ such that

$$\omega(B) = \lambda c_1(M)(B)$$

for all spherical classes $B = [\mathbb{C}P^1] \in H^2(M, \mathbb{Z})$). Let $A \in H_2(M, \mathbb{Z})$ be an indecomposable spherical class. (it cannot be decomposed as a sum $A = A_1 + \cdots + A_k$, $k \geq 2$, of classes which are spherical and satisfy $\omega(A_i) > 0$ for $i = 1, \ldots, k$). Let pt denote the homology class of a point. Suppose that there exist submanifolds X and Y of M such that

$$\dim X + \dim Y = 4n - 2c_1(M)(A)$$

and so that

$$\Phi_A(pt, [X], [Y]) \neq 0.$$

If $\pi r^2 > \omega(A) = \int_A \omega$, there does **not** exist a symplectic embedding of $(B^{2n}(r), \omega_0)$ into (M, ω).

go back

Example: symplectic embedding of the first Cartan domain into its complex dual $\mathrm{Grass}_k(\mathbb{C}^{n+k})$

Example: symplectic embedding of the first Cartan domain into its complex dual $\mathrm{Grass}_k(\mathbb{C}^{n+k})$

Let

$$D_I[k,n] = \{Z \in M_{k,n}(\mathbb{C}) \mid I_k - ZZ^* > 0\}$$

be the first Cartan domain.

Example: symplectic embedding of the first Cartan domain into its complex dual $\mathrm{Grass}_k(\mathbb{C}^{n+k})$

Let

$$D_I[k,n] = \{Z \in M_{k,n}(\mathbb{C}) \mid I_k - ZZ^* > 0\}$$

be the first Cartan domain.

The compact dual of $D_I[k,n]$ is the complex Grassmannian

$$(\mathrm{Grass}_k(\mathbb{C}^{n+k}), \omega_{FS}).$$

Example: symplectic embedding of the first Cartan domain into its complex dual $\mathrm{Grass}_k(\mathbb{C}^{n+k})$

Let

$$D_I[k,n] = \{Z \in M_{k,n}(\mathbb{C}) \mid I_k - ZZ^* > 0\}$$

be the first Cartan domain.
The compact dual of $D_I[k,n]$ is the complex Grassmannian

$$(\mathrm{Grass}_k(\mathbb{C}^{n+k}), \omega_{FS}).$$

The map

$$\Phi : D_I[k,n] \to M_{k,n}(\mathbb{C}) = \mathbb{C}^{kn} \subset \mathrm{Grass}_k(\mathbb{C}^{n+k})$$

defined by

$$\Phi(Z) = (I_k - ZZ^*)^{-\frac{1}{2}} Z$$

Example: symplectic embedding of the first Cartan domain into its complex dual $\mathrm{Grass}_k(\mathbb{C}^{n+k})$

Let

$$D_I[k,n] = \{Z \in M_{k,n}(\mathbb{C}) \mid I_k - ZZ^* > 0\}$$

be the first Cartan domain.

The compact dual of $D_I[k,n]$ is the complex Grassmannian

$$(\mathrm{Grass}_k(\mathbb{C}^{n+k}), \omega_{FS}).$$

The map

$$\Phi : D_I[k,n] \to M_{k,n}(\mathbb{C}) = \mathbb{C}^{kn} \subset \mathrm{Grass}_k(\mathbb{C}^{n+k})$$

defined by

$$\Phi(Z) = (I_k - ZZ^*)^{-\frac{1}{2}} Z$$

is a diffeomorphism such that

$$\Phi^* \omega_{FS} = \omega_0.$$

go back

Pseudo symplectic capacities

A map c^k from the set $\mathcal{C}(2n, k)$ of all tuples $(M, \omega; \alpha_1, \ldots, \alpha_k)$ consisting of a $2n$-dimensional connected symplectic manifold (M, ω) and k nonzero homology classes $\alpha_i \in H_*(M; \mathbb{Q})$, $i = 1, \ldots, k$ to $[0, +\infty]$ is called a *k-pseudo symplectic capacity* [G. Lu, Israel J. Math., 2006] if it satisfies the following properties:

Pseudo symplectic capacities

A map c^k from the set $\mathcal{C}(2n, k)$ of all tuples $(M, \omega; \alpha_1, \ldots, \alpha_k)$ consisting of a $2n$-dimensional connected symplectic manifold (M, ω) and k nonzero homology classes $\alpha_i \in H_*(M; \mathbb{Q})$, $i = 1, \ldots, k$ to $[0, +\infty]$ is called a *k-pseudo symplectic capacity* [G. Lu, Israel J. Math., 2006] if it satisfies the following properties:

- (**pseudo monotonicity**) if there exists a symplectic embedding $\varphi : (M, \omega_1) \to (M, \omega_2)$ then, for any $\alpha_i \in H_*(M_1; \mathbb{Q})$, $i = 1, \ldots, k$,

$$c^{(k)}(M_1, \omega_1; \alpha_1, \ldots, \alpha_k) \leq c^{(k)}(M_2, \omega_2; \varphi_*(\alpha_1), \ldots, \varphi_*(\alpha_k));$$

Pseudo symplectic capacities

A map c^k from the set $\mathcal{C}(2n, k)$ of all tuples $(M, \omega; \alpha_1, \ldots, \alpha_k)$ consisting of a $2n$-dimensional connected symplectic manifold (M, ω) and k nonzero homology classes $\alpha_i \in H_*(M; \mathbb{Q})$, $i = 1, \ldots, k$ to $[0, +\infty]$ is called a *k-pseudo symplectic capacity* [G. Lu, Israel J. Math., 2006] if it satisfies the following properties:

- (**pseudo monotonicity**) if there exists a symplectic embedding $\varphi : (M, \omega_1) \to (M, \omega_2)$ then, for any $\alpha_i \in H_*(M_1; \mathbb{Q})$, $i = 1, \ldots, k$,

$$c^{(k)}(M_1, \omega_1; \alpha_1, \ldots, \alpha_k) \leq c^{(k)}(M_2, \omega_2; \varphi_*(\alpha_1), \ldots, \varphi_*(\alpha_k));$$

- (**conformality**) $c^{(k)}(M, \lambda\omega; \alpha_1, \ldots, \alpha_k) = |\lambda| c^{(k)}(M, \omega; \alpha_1, \ldots, \alpha_k)$, for every $\lambda \in \mathbb{R} \setminus \{0\}$ and all homology classes $\alpha_i \in H_*(M; \mathbb{Q}) \setminus \{0\}$, $i = 1, \ldots, k$;

Pseudo symplectic capacities

A map c^k from the set $\mathcal{C}(2n, k)$ of all tuples $(M, \omega; \alpha_1, \dots, \alpha_k)$ consisting of a $2n$-dimensional connected symplectic manifold (M, ω) and k nonzero homology classes $\alpha_i \in H_*(M; \mathbb{Q})$, $i = 1, \dots, k$ to $[0, +\infty]$ is called a *k-pseudo symplectic capacity* [G. Lu, Israel J. Math., 2006] if it satisfies the following properties:

- (**pseudo monotonicity**) if there exists a symplectic embedding $\varphi : (M, \omega_1) \to (M, \omega_2)$ then, for any $\alpha_i \in H_*(M_1; \mathbb{Q})$, $i = 1, \dots, k$,
$$c^{(k)}(M_1, \omega_1; \alpha_1, \dots, \alpha_k) \leq c^{(k)}(M_2, \omega_2; \varphi_*(\alpha_1), \dots, \varphi_*(\alpha_k));$$
- (**conformality**) $c^{(k)}(M, \lambda\omega; \alpha_1, \dots, \alpha_k) = |\lambda| c^{(k)}(M, \omega; \alpha_1, \dots, \alpha_k)$, for every $\lambda \in \mathbb{R} \setminus \{0\}$ and all homology classes $\alpha_i \in H_*(M; \mathbb{Q}) \setminus \{0\}$, $i = 1, \dots, k$;
- (**nontriviality**) $c^{(k)}(B^{2n}(1), \omega_0; pt, \dots, pt) = \pi = c^{(k)}(Z^{2n}(1), \omega_0; pt, \dots, pt)$, where pt denotes the homology class of a point. go back