A Geometry Day in Como (10.01.2020)

Kähler immersions into complex space forms: old and new results

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Aim of the talk: *provide an overview of the main results (mostly in the homogeneous and Kähler-Einstein case) on Kähler immersions of Kähler manifolds into finite or infinite dimensional complex space forms.*

Advertising for the book: L., M. Zedda, *Kähler immersions of Kähler manifolds into complex space forms.*, Lectures Notes of the Unione Matematica Italiana, Springer 2018.

0. Complex space forms and Kähler immersions

1. The work of Calabi

2. Related results

3. Homogeneous Kähler manifolds (h.K.m.) into complex space forms

4. The Kähler-Einstein case

5. The Ricci flat case

6. Idea of some proofs

Complex space forms

A complex space form (S, g_S) is a finite or infinite dimensional Kähler manifold with constant holomorphic sectional curvature $H(g_S)$.

Simply-connected and complete complex space forms

<u>Complex Euclidean space</u> $\mathbb{C}^{N\leq\infty} := (\mathbb{C}^{N\leq\infty}, g_0)$

$\mathbb{C}^{\infty} := \ell^2(\mathbb{C})$ ($z = \{z_j\} \in \ell^2(\mathbb{C})$ iff $\sum_{j=1}^{\infty} |z_j|^2 < \infty$)

$\omega_0 = \frac{i}{2}\partial\bar{\partial}|z|^2 = \frac{i}{2}\sum_{j=1}^{N} dz_j \wedge d\bar{z}_j$, $|z|^2 = |z_1|^2 + \cdots + |z_N|^2$.

<u>Complex hyperbolic space</u> $\mathbb{C}H^{N\leq\infty} := (\{z \in \mathbb{C}^N \mid |z|^2 < 1\}, g_{hyp})$

$\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial}\log(1 - |z|^2)$.

<u>Complex projective space</u> $\mathbb{C}P^{N\leq\infty} = (\mathbb{C}^{N+1} \setminus \{0\}/z \sim \lambda z, g_{FS})$

$\omega_{FS}|_{U_0} = \frac{i}{2}\partial\bar{\partial}\log(1+|z|^2)$, $z_j = \frac{Z_j}{Z_0}$, $j = 1, \ldots, N$, $U_0 = \{Z_0 \neq 0\}$.

Kähler immersions into complex space forms

Let (M, g) be a (finite dimensional) Kähler manifold.

A Kähler immersion $f : (M, g) \to (S, g_S)$ is a holomorphic map (i.e. $df \circ J = J_S \circ df$) which is isometric (i.e. $f^* g_S = g$).

Terminology: A Kähler metric g on a complex manifold M is projectively induced if (M, g) can be Kähler immersed into a finite or infinite dimensional complex projective space.

1. The work of E. Calabi (1953)

Three theorems of Calabi

Theorem (rigidity) *Given two Kähler immersions f_1 and f_2 from a Kähler manifold (M, g) into (S, g_S) there exists $\mathcal{U} \in$ Aut$(S) \cap$ Isom(S, g_S) such that $f_2 = \mathcal{U} \circ f_1$.*

Theorem *(Calabi's criterium)* Let (M, g) be a Kähler manifold and let $p \in M$. A neighbourhood U of $p \in M$ can be Kähler immersed into (S, g_S) iff g is real-analytic and a "certain $\infty \times \infty$ matrix" associated to g is positive semidefinite on U with at least $\lambda_1, \ldots, \lambda_N$ positive eigenvalues, $N = \dim S$.

Theorem *(extension)* A simply-connected Kähler manifold (M, g) admits a Kähler immersion into (S, g_S) iff there exists an open set $U \subset M$ such that $(U, g_{|U})$ can be Kähler immersed into (S, g_S).

When (M, g) is the complex Euclidean space

$\mathbb{C}^n \not\to \mathbb{C}H^{N\leq\infty}, \mathbb{C}P^{N<\infty}$

$\mathbb{C}^n \stackrel{tot.geod.}{\longrightarrow} \mathbb{C}^{N\leq\infty}, \ n \leq N$

Calabi's embedding:

$$\mathbb{C}^n \to \mathbb{C}P^\infty : z \mapsto (\dots, \sqrt{\frac{1}{j!}}\, z^j, \dots), \ |j| \geq 0$$

$z^j = z_1^{j_1} \cdots z_n^{j_n}$ $|j| = j_1 + \cdots + j_n$, $j! = j_1! \cdots j_n!$

When (M, g) is the complex hyperbolic space

Let $\mathbb{C}H^n_\lambda = (\mathbb{C}H^n, \lambda g_{hyp})$, $\lambda > 0$, $\mathbb{C}H^n := \mathbb{C}H^n_1 = (\mathbb{C}H^n, g_{hyp})$

$\mathbb{C}H^n_\lambda \nrightarrow \mathbb{C}^{N<\infty}, \mathbb{C}P^{N<\infty}$

$\mathbb{C}H^n_\lambda \to \mathbb{C}H^{N\leq\infty} \Leftrightarrow \lambda = 1,\ n \leq N$

Calabi's embeddings:

$$\mathbb{C}H^n_\lambda \to \ell^2(\mathbb{C}) : z \mapsto \sqrt{\lambda}(\dots, \sqrt{\frac{(|j|-1)!}{j!}}\ z^j, \dots),\ |j| \geq 1$$

$$\mathbb{C}H^n_\lambda \to \mathbb{C}P^\infty : z \mapsto (\dots, \sqrt{\frac{\lambda(\lambda+1)\cdots(\lambda-1+|j|)}{j!}}\ z^j, \dots),\ |j| \geq 0$$

$z^j = z_1^{j_1} \cdots z_n^{j_n}$, $|j| = j_1 + \cdots + j_n$, $j! = j_1! \cdots j_n!$

When (M, g) is the complex projective space

Let $\mathbb{C}P^n_\lambda = (\mathbb{C}P^n, \lambda g_{FS})$, $\lambda > 0$, $\mathbb{C}P^n := \mathbb{C}P^n_1 = (\mathbb{C}P^n, g_{FS})$

$$\boxed{\mathbb{C}P^n_\lambda \nrightarrow \mathbb{C}^{N\leq\infty}, \mathbb{C}H^{N\leq\infty}}$$

Let $k \in \mathbb{Z}^+$ *and* $N_k := \frac{(n+k)!}{n!k!} - 1$. *Then the map*

$$\mathbb{C}P^n_k \xrightarrow{V_k} \mathbb{C}P^{N_k} : [Z] \longmapsto [\dots, \sqrt{\frac{|j|!}{j!}}\ Z^j, \dots],$$

$Z^j = Z_0^{j_0} \cdots Z_n^{j_n}$, $|j| = j_0 + \cdots + j_n = k$, $j! = j_0! \cdots j_n!$ *is a Kähler embedding, i.e. it is a holomorphic embedding satisfying*

$$V_k^* g_{FS} = k g_{FS}$$

Some consequences of Calabi's work

1. Any abelian variety equipped the flat metric cannot be Kähler immersed into $\mathbb{C}P^{N<\infty}$.

2. Any compact Riemann surface of genus ≥ 2 *with the hyperbolic metric cannot be Kähler immersed into* $\mathbb{C}P^{N<\infty}$ *or into an abelian variety equipped with the flat metric.*

2. Related results

Codimension restrictions

Theorem *(B. O'Neill, 1965)* Let (M^n, g) and (S^N, g_S) be finite dimensional complex space forms (not necessarily complete and simply-connected). If (M, g) can be Kähler immersed into (S, g_S) and $N - n \leq \frac{n(n+1)}{2}$ then the immersion is totally geodesic.

Theorem (N. Mok, 2005) *Let (M^n, g) and (S^N, g_S) be compact and finite dimensional complex space forms (not necessarily simply-connected). If (M, g) can be Kähler immersed into (S, g_S) and $N - n \leq n - 1$ then the immersion is totally geodesic.*

The work of A. Ros

Theorem *(A. Ros, 1984)* Let (M, g) be a compact Kähler manifold which can be Kähler embedded into $\mathbb{C}P^{N<\infty}$. If $K > \frac{1}{2}$ then the embedding is totally geodesic.

Theorem *(A. Ros, 1985)* Let (M, g) be a compact Kähler manifold which can be Kähler immersed into $\mathbb{C}P^{N<\infty}$. If $H > 2$ then the immersion is totally geodesic.

Theorem *(A. Ros, 1985)* Let (M, g) be a compact Kähler manifold which can be Kähler embedded into $\mathbb{C}P^{N<\infty}$. Then $H \geq 2$ iff the second fundamental form is parallel.

Relatives Kähler manifolds

Two Kähler manifolds M_1 and M_2 are relatives (A. J. Di Scala, L., Ann. Scuola Norm. Sup. Pisa 2010) if there exists a Kähler manifold N, $\dim N > 0$, which is a Kähler submanifold of both M_1 and M_2.

Theorem (M. Umehara, 1987) Any two finite dimensional Kähler manifolds (M, g) and (N, G) with constant holomorphic sectional curvature of different sign are not relatives.

3. H.K.m. into complex space forms

H.K.m. into finite dimensional complex space forms

Theorem A (A. J. Di Scala, 2002) *If a h.K.m. admits a Kähler immersion into* $\mathbb{C}^{N<\infty}$ *then the immersion is totally geodesic.*

Theorem B (D. Alekseevsky, A. J. Di Scala, 2003) *If a h.K.m. admits a Kähler immersion into* $\mathbb{C}H^{N<\infty}$ *then the immersion is totally geodesic.*

Theorem C (M. Takeuchi, 1978) *Let* (M, g) *be a h.K.m. which can be Kähler immersed into* $\mathbb{C}P^{N<\infty}$*. Then* M *is compact,* ω *is integral,* $\pi_1(M) = 1$ *and the immersion is an embedding. Viceversa if* (M, g) *is a compact h.K.m such that* ω *is integral, and* $\pi_1(M) = 1$ *then* (M, g) *can be Kähler embedded into* $\mathbb{C}P^{N<\infty}$*.*

Homogeneous Kähler manifolds into $\ell^2(\mathbb{C}) = \mathbb{C}^\infty$

Theorem 1 (A. J. Di Scala, H. Hishi, L., 2012) *Let* (M, g) *be a* n*-dimensonal h.K.m. which can be Kähler immersed into* $\ell^2(\mathbb{C})$*. Then* $(M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$*. Moreover, the immersion is given, up to a unitary transformations, by*

$$f_0 \times f_1 \times \cdots \times f_l,$$

where f_0 *is the linear inclusion* $\mathbb{C}^k \stackrel{tot.geod.}{\longrightarrow} \ell^2(\mathbb{C})$ *and each* $f_r :$ $\mathbb{C}H^{n_r}_{\lambda_r} \longrightarrow \ell^2(\mathbb{C})$*,* $r = 1, \ldots, l$*, are Calabi's embeddings.*

Homogeneous Kähler manifolds into $\mathbb{C}H^\infty$

Theorem 2 (A. J. Di Scala, H. Hishi, L., 2012) *Let* (M, g) *be a* n*-dimensional h.K.m. which can be Kähler immersed into* $\mathbb{C}H^\infty$*. Then, up to a unitary transformations,* $(M, g) = \mathbb{C}H^n \stackrel{tot.geod.}{\longrightarrow} \mathbb{C}H^\infty$.

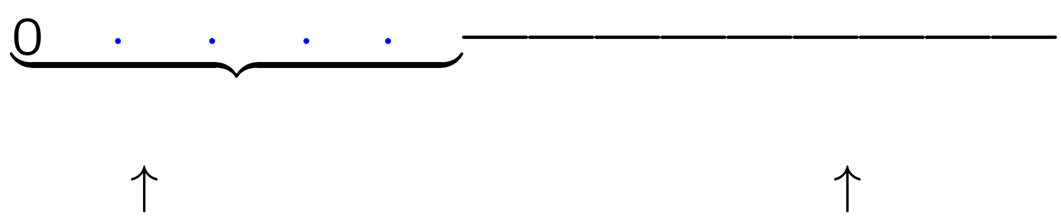
Homogeneous Kähler manifolds into $\mathbb{C}P^\infty$

Theorem 3 *(A. J. Di Scala, H. Hishi, L., 2012)* Let (M, g) be a h.K.m. which can be Kähler immersed into $\mathbb{C}P^\infty$. *Then ω is integral, $\pi_1(M) = 1$ and the immersion is an embedding.*

Theorem 4 (L., R. Mossa, 2014) *Let (M, g) be a simply-connected h.K.m. whose associated Kähler form ω is integral. Then there exist $m_0 > 0$ and a Kähler embedding $(M, m_0 g) \to \mathbb{C}P^\infty$.*

The Wallach set of a bounded symmetric domain

Let Ω be an irreducible bounded symmetric domain. The Wallach set* $W(\Omega) \subset \mathbb{R}^+$ which "looks like":



discrete part of $W(\Omega)$ *continuous part of* $W(\Omega)$

A property of the Wallach set: $W(\Omega) = \mathbb{R}^+$ (and hence the discrete part of $W(\Omega)$ is empty) if and only if $\Omega = \mathbb{C}H^n$.

*$W(\Omega)$ consists of all $\lambda \in \mathbb{R}^+$ such that there exists a Hilbert space $\mathcal{H}_\lambda$ whose reproducing kernel is $K^{\frac{\lambda}{\gamma}}$, where γ is the genus of Ω and K is the Bergman kernel of Ω.

The Wallach set and Kähler immersions into $\mathbb{C}P^\infty$

Theorem W (L., M. Zedda, 2010) *Let (Ω, g_B) be a irreducible bounded symmetric domain (g_B the Bergman metric). Then $(\Omega, \lambda g_B)$ can be Kähler immersed into $\mathbb{C}P^\infty$ if and only if $\lambda\gamma \in W(\Omega) \setminus \{0\}$, where γ denotes the genus of Ω.*

Consequence: Let $(\Omega, g_B) \neq \mathbb{C}H^n$ be a irreducible bounded symmetric domain. One can find $\lambda > 0$ such that $\lambda\gamma \notin W(\Omega)$:

$$0 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad * \quad \text{—————————————}$$

$$\uparrow$$
$$\lambda\gamma \notin W(\Omega)$$

By Theorem W, λg_B is not projectively induced and $\lambda \omega_B$ is integral (this shows the necessity of taking $m_0 > 1$ in Theorem 4).

Corollary of Theorem W: The complex hyperbolic space $\mathbb{C}H^n$ is the only irreducible bounded symmetric domain (Ω, g_B) where λg_B is projectively induced, for all $\lambda > 0$. Equivalently, $\mathbb{C}H^n$ is the only irreducible bounded symmetric domain which can be Kähler immersed into $\ell^2(\mathbb{C})$.

Lemma H (A. J. Di Scala, H. Hishi, L., 2012) *Let* (Ω, g_Ω) *be a homogeneous bounded domain. If* (Ω, g_Ω) *can be Kähler immersed into* $\ell^2(\mathbb{C})$*, then* $(\Omega, g_\Omega) = \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$.

4. The Kähler-Einstein case

Kähler-Einstein into $\mathbb{C}^{N<\infty}$ and $\mathbb{C}H^{N<\infty}$

Theorem (M. Umehara, 1987) *If a KE manifold admits a Kähler immersion into $\mathbb{C}H^{N<\infty}$ (resp. $\mathbb{C}^{N<\infty}$) then the immersion is totally geodesic.*

Conjecture A: *Let M be a complex manifold equipped either with an extremal Kähler metric $g^{\dagger}$ or with a Kähler Ricci soliton $(g, X)^{\ddagger}$. If (M, g) admits a Kähler immersion into $\mathbb{C}H^{N<\infty}$ (resp. $\mathbb{C}^{N<\infty}$) then (M, g) is KE* (and hence totally geodesic).

† The (1,0)-part of the Hamiltonian vector field associated to the scalar curvature of g is holomorphic.

‡ $Ric_g = \lambda\, g + L_X g$, where X is the real part of a holomorphic vector field.

Conjecture A cannot cannot be extended to $\mathbb{C}H^\infty$

A Kähler metric g on a complex manifold is radial if it admits a Kähler potential $\Phi : U \to \mathbb{R}$ which depends only on $|z_1|^2 + \cdots + |z_n|^2$ (U does not necessarily contains the origin).

§. radial if it admits a Kähler potential $\Phi : U \to \mathbb{R}$ which depends only on $|z_1|^2 + \cdots + |z_n|^2$ (U does not necessarily contains the origin).

Theorem(L., F. Zuddas, in preparation) There exist nonhomogeneous radial KE manifolds different from the hyperbolic metric which can be Kähler immersed into $\mathbb{C}H^\infty$ (and hence into $\ell^2(\mathbb{C})$).

$^\S Ric_g = \lambda\ g + L_X g$, where X is the real part of a holomorphic vector field.

Conjecture B: *Let* (D, g_B) *be a bounded domain* $D \subset \mathbb{C}^n$ *equipped with the Bergman metric* g_B. *If* g_B *is Einstein and* (D, g_B) *can be Kähler immersed into into* $\mathbb{C}H^\infty$ *(or into* $\ell^2(\mathbb{C})$*), then* $(D, g_B) = \mathbb{C}H^n$.

Kähler-Einstein into $\mathbb{C}P^{N<\infty}$

Theorem[¶] (B. Smyth, 1967) *A compact KE manifold of complex dimension n which admits a Kähler embedding into $\mathbb{C}P^{n+1}$ is totally geodesic or the complex quadric $Q = \{Z_0^2 + \cdots + Z_{n+1}^2 = 0\}$.*

Theorem (S. S. Chern, 1967) *A KE manifold of complex dimension n which admits a Kähler immersion into $\mathbb{C}P^{n+1}$ is totally geodesic or an open subset of the complex quadric.*

Theorem[‖] (J. Hano, 1975) *Let $M \subset \mathbb{C}P^{N<\infty}$ be a complete intersection. If the restriction of g_{FS} to M is Einstein then M is totally geodesic or the complex quadric.*

[¶] Extended to cscK by S. Kobayashi (1967) and by M. Kon (1975) to cscK and immersions.

[‖] Extended to Kähler-Ricci solitons by L. Bedulli and A. Gori (2014).

Theorem (K. Tsukada, 1986) *A KE manifold of complex dimension n which admits a Kähler immersion into $\mathbb{C}P^{n+2}$ is totally geodesic or an open subset of the complex quadric.*

Theorem (F. Salis, 2017) *A rotation invariant KE manifold of complex dimension n which admits a Kähler immersion into $\mathbb{C}P^{n+k}$, $k \geq 3$, is an open subset of one of the following: $\mathbb{C}P^n$, $\mathbb{C}P_2^2$, $\mathbb{C}P^1 \times \mathbb{C}P^1$.*

Theorem (D. Hulin, 2000) *A compact KE manifold which admits a Kähler embedding into $\mathbb{C}P^{N<\infty}$ has positive scalar curvature.*

Open problem: *Drop the embedding assumption.*

Conjecture C: *A KE manifold which admits a Kähler immersion into* $\mathbb{C}P^{N<\infty}$ *is an open subset of a compact and simply-connected h.K.m. and the immersion is an embedding.*

Conjecture D: *Let* M *be a complex manifold equipped either with an extremal Kähler metric* g *or with a Kähler Ricci soliton* (g, X)*. If* (M, g) *admits a Kähler immersion into into* $\mathbb{C}P^{N<\infty}$ *then* (M, g) *is KE.*

Conjecture C cannot cannot be extended to $\mathbb{C}P^{\infty}$

Let $\Omega \neq \mathbb{C}H^{d}$ be an irreducible bounded symmetric domain of complex dimension d, genus γ, volume $V(\Omega)$ and Bergman Kernel $K(z,z)$. Let $N_{\Omega}(z,z) = V(\Omega)K(z,z)$ and consider the Cartan-Hartogs domain

$$M_{\Omega} = \left\{ (z,w) \in \Omega \times \mathbb{C},\ |w|^2 < N_{\Omega}(z,z)^{\frac{\gamma}{d+1}} \right\}.$$

with the complete and nonhomogeneous KE metric g_{Ω} (A. Wang, W. Yin, L. Zhang, G. Roos, Science in China, 2006) whose associated Kähler form is

$$\omega_{\Omega} = -\frac{i}{2}\partial\bar{\partial}\log[N_{\Omega}(z,z)^{\frac{\gamma}{d+1}} - |w|^2].$$

Theorem *(L., M. Zedda, 2010)* (M_Ω, cg_Ω) can be Kähler embedded into $\mathbb{C}P^\infty$ for $c >> 1$.

Theorem *(Y. Hao, A. Wang, L. Zhang, 2015) Let Ω_1 and Ω_2 irreducible bounded symmetric domains and*

$$M_{\Omega_1\times\Omega_2} = \left\{(z_1, z_2, w) \in \Omega_1 \times \Omega_2 \times \mathbb{C},\ |w|^2 < N_{\Omega_1\times\Omega_2}(z_1, z_2)\right\}.$$

where

$$N_{\Omega_1\times\Omega_2}(z_1, z_2) = N_{\Omega_1}(z_1, z_1)^{\frac{\gamma_1}{d_1+d_2+1}} N_{\Omega_2}(z_2, z_2)^{\frac{\gamma_2}{d_1+d_2+1}}.$$

Then for $c >> 1$ the Kähler metric $cg_{\Omega_1\times\Omega_2}$ with

$$\omega_{\Omega_1\times\Omega_2} = -\frac{i}{2}\partial\bar\partial \log[N_{\Omega_1\times\Omega_2}(z_1, z_2) - |w|^2].$$

is KE complete, nonhomogeneous and projectively induced.

5. The Ricci flat case

Conjecture E: A Ricci flat projectively induced Kähler metric is flat.

The Taub-NUT metrics

The Taub-NUT metrics is the family of Kähler metrics g_m *on* $\mathbb{C}^2$ *whose associated Kähler forms are given by*

$$\omega_m = \frac{i}{2}\partial\bar{\partial}\left[u^2 + v^2 + m(u^4 + v^4)\right], \ m \geq 0,$$

where $|z_1| = e^{m(u^2-v^2)}u, \ |z_2| = e^{m(v^2-u^2)}v$. *For* $m = 0$, g_0 *is flat metric and for* $m \neq 0$, g_m *is Ricci flat (not flat) and complete (C. LeBrun, Proceedings of Symposia in Pure Mathematics, 1991).*

Theorem *(L., M. Zedda, F. Zuddas, 2012)* For $m > \frac{1}{2}$ the Kähler metric g_m on $\mathbb{C}^2$ is not projectively induced.

Open problem: Show that g_m is projectively induced iff $m = 0$.

Radial projectively induced Ricci flat Kähler metrics

A Kähler metric g is said to be stable-projectively induced if there exists $\epsilon > 0$ such that λg is projectively induced for all $\lambda \in (1-\epsilon, 1+\epsilon)$.

Theorem *(L., F. Salis, F. Zuddas, 2018)* The only Ricci-flat, stable-projectively induced and radial Kähler metric is the flat one.

Open problem: Drop the assumption on stability in the theorem.

Corollary: The Eguchi-Hanson metric namely the Ricci flat and complete Kähler metric g_{EH} on $\hat{\mathbb{C}}^2$ (the blow-up of $\mathbb{C}^2$ at the origin) given in $\hat{\mathbb{C}}^2 \setminus E = \mathbb{C}^2 \setminus \{0\}$ (E exceptional divisor) by the potential

$$\Phi(x) = \sqrt{x^2+1} + \log x - \log(1+\sqrt{x^2+1}), \;\; x = |z_1|^2 + |z_2|^2$$

is not projectively induced.

Calabi's Ricci flat metrics on the canonical bundle

Let (M, g) *be a compact KE manifold of complex dimension* $n-1$ *and with associated Kähler form* ω_g *and Einstein constant* $k_0 > 0$. *Let* $\pi : \Lambda^{n-1}M \to M$ *be the canonical line bundle over* M,

Calabi (1979) shows that there exists a smooth function $u : [0, +\infty) \to \mathbb{R}$ *(which can be written explicitely) such that if* $\omega_g = \frac{i}{2\pi}\partial\bar{\partial}\Phi$ *on* U, *then the function* $\Psi : \pi^{-1}(U) \to \mathbb{R}$ *defined by*

$$\Psi = \Phi \circ \pi + u\left(\det(g)^{-1}|\xi|^2\right)$$

is a Kähler potential on $\pi^{-1}(U)$ *for a Ricci flat and complete metric* g_C *on* $\Lambda^{n-1}M$.

Theorem *(L., M. Zedda, F. Zuddas, 2020)* The metric g_C is not projectively induced.

Corollary: For any $c > 0$ cg_{EH} is not projectively induced.

Remark: The metrics cg_{EH}, $c > 0$, on $\widehat{\mathbb{C}}^2$ are examples of Ricci flat and complete Kähler metrics which cannot be (locally) Kähler immersed into any finite or infinite dimensional complex space form for all $c > 0$. Other examples of such metrics were constructed by Calabi (1953).

Conjecture E cannot be weakened to scalar flat metrics

S. Simanca (1991) constructs a scalar flat Kähler complete (not Ricci-flat) metric g_S *on* $\hat{\mathbb{C}}^2$ *whose Kähler potential on* $\hat{\mathbb{C}}^2 \setminus E = \mathbb{C}^2 \setminus \{0\}$ *can be written as*

$$\Phi_S(|z|^2) = |z|^2 + \log |z|^2, |z|^2 = |z_1|^2 + |z_2|^2.$$

Theorem *(F. Cannas Aghedu, L., 2019)* $(\hat{\mathbb{C}}^2, g_S)$ can be Kähler embedded into $\mathbb{C}P^\infty$.

Conjecture F: Let M be a complex manifold equipped with an extremal Kähler metric g. If (M, g) admits a Kähler immersion into $\mathbb{C}P^\infty$ then (M, g) is cscK.

Conjecture G: Let M be a complex manifold equipped with a Kähler Ricci soliton (g, X). If (M, g) admits a Kähler immersion into $\mathbb{C}P^\infty$ then (M, g) is KE.

6. Sketch of the proofs of Theorem 1, 2, 3, 4

Sketch of the proof of Theorem 1

$(M, g) \xrightarrow{f} \ell^2(\mathbb{C})$ we want to prove that:

$(M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$ and $f = f_0 \times f_1 \times \cdots \times f_l$.

1. Theorem FC + Calabi's rigidity theorem+ max principle $\Rightarrow$

$$\mathcal{F} = \mathbb{C}^k \times \not{F} \times \not{\mathbb{C}} \overset{\text{Kähler}}{\hookrightarrow} \begin{array}{c} (M, g) \\ \pi\downarrow \\ (\Omega, g_\Omega) \end{array} \to \ell^2(\mathbb{C})$$

2. Riemannian geometry + homogeneity $\Rightarrow$

$$(M, g) \overset{\text{Kähler}}{=} \mathbb{C}^k \times (\Omega, g_\Omega) \Rightarrow (\Omega, \lambda g_\Omega) \to \ell^2(\mathbb{C}),\ \forall \lambda > 0.$$

3. S. Bochner (1947) $\Rightarrow (\Omega, \lambda g_\Omega) \to \mathbb{C}P^\infty, \ \forall \lambda > 0$.

4. Lemma H $\Rightarrow (\Omega, g_\Omega) = \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l} \Rightarrow$

$\Rightarrow (M, g) = \mathbb{C}^k \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l}$.

5. The fact that the immersion f is, up to a unitary transformation, of the form $f = f_0 \times f_1 \times \cdots \times f_l$ follows by the reducibility of a Kähler product into $\ell^2(\mathbb{C})$ and by Calabi's rigidity theorem. □

Sketch of the proof of Theorem 2 (based on Theorem 1)

If $(M, g) \to \mathbb{C}H^{\infty}$ we want to prove that

$$(M, g) = \mathbb{C}H^{n} \overset{tot.geod.}{\longrightarrow} \mathbb{C}H^{\infty}.$$

1. $(M, g) \to \mathbb{C}H^{\infty} \Rightarrow (M, g) \to \ell^{2}(\mathbb{C})$.

2. Theorem 1 $\Rightarrow (M, g) = \mathbb{C}^{k} \times \mathbb{C}H^{n_1}_{\lambda_1} \times \cdots \times \mathbb{C}H^{n_l}_{\lambda_l} \Rightarrow M = \mathbb{C}H^{n}$. $\Box$

Sketch of the proof of Theorem 3

Let $f : (M, g) \to \mathbb{C}P^{\infty}$ be a Kähler immersion.

The integrality of $\omega = f^{*}\omega_{FS}$ is immediate since ω_{FS} is integral.

Th. FC $\Rightarrow$ $\mathcal{F} = \mathbb{C}^{k} \times \not{F} \times C \overset{\text{Kähler}}{\hookrightarrow} \begin{matrix} (M, g) \\ \pi \downarrow \\ (\Omega, g_{\Omega}) \end{matrix} \to \mathbb{C}P^{\infty}$ $\Rightarrow M \overset{top}{=}$ $\Omega \times \mathbb{C}^{n} \times C$ is simply-connected.

Calabi's rigidity $\Rightarrow f \circ g = \mathcal{U}_{g} \circ f$, $\forall g \in G = \mathrm{Aut}(M) \cap \mathrm{Isom}(M, g)$ $\Rightarrow f(M)$ is a h.K.m. $\Rightarrow f(M) \subset \mathbb{C}P^{\infty}$ is simply-connected.

$f : M \to f(M)$ is a local isometry $\Rightarrow f$ is a covering map $\Rightarrow f$ is injective. $\square$

Sketch of the proof of Theorem 4

Let (M, g) be a simply-connected h.K.m. with ω integral we want to show that $(M, m_0 g) \to \mathbb{C}P^\infty$, for some $m_0 \in \mathbb{Z}$.

1. Let L be a holomorphic line bundle with $c_1(L) = [\omega]$ and consider the Hilbert space

$$\mathcal{H}_m = \{s \in H^0(L) \mid \int_M h_m(s,s)\frac{\omega^n}{n!} < \infty\}$$

where h_m is an Hermitian metric on L^m such that $\mathrm{Ric}(h_m)^{**} = m\omega$.

2. There exists $m_0 \in \mathbb{Z}$ such that $\mathcal{H}_{m_0} \neq \{0\}$ (J. Rosenberg, M. Vergne, 1984);

**Ric$(h_m) = -\frac{i}{2}\partial\bar{\partial}\log h_m(\sigma(x), \sigma(x))$, where $\sigma : U \to L^m$ is a trivialising holomorphic section of L^m.

3. Consider the smooth function on M given by:

$$\epsilon_{m_0}(x) = \sum_{j=0}^{\infty} h_{m_0}(s_j(x), s_j(x)),$$

where $\{s_0, \ldots, \ldots, \}$ is an orthonormal basis of $\mathcal{H}_{m_0}$.

Homogeneity $+ \pi_1(M) = 1 \Rightarrow \epsilon_{m_0}(x)$ is a positive constant.

4. Therefore the "Kodaira map"

$$\varphi_{m_0} : M \to \mathbb{C}P^{\infty}, x \mapsto [s_0(x), \ldots, s_{d_{m_0}}(x)]$$

is well-defined and it satisfies

$$\varphi_{m_0}^{*}\omega_{FS} = m_0\omega + \frac{i}{2}\partial\bar{\partial} \log \epsilon_{m_0} = m_0\omega.$$

□

Thank you for your attention!