EMS-RSME JOINT MATHEMATICAL WEEKEND

Bilbao, October 7-9, 2011

Global symplectic coordinates on Kähler manifolds

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SYMPLECTIC COORDINATES

Let (M^{2n}, ω) be a symplectic manifold and let $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$ be the standard symplectic form on $\mathbb{R}^{2n}$.

Theorem (Darboux): Given $p \in M$ there exist an open set $U_p \subset M$ and a diffeomorphism

$$\psi : U_p \to \psi_p(U_p) \subset \mathbb{R}^{2n}$$

such that

$$\psi^* \omega_0 = \omega_{|U_p}$$

Question: How large U_p can be taken?

Theorem (Gromov, Inv.Math. 1985): There exists a symplectic form ω on $\mathbb{R}^{2n}, n \geq 2$, such that $(\mathbb{R}^{2n}, \omega)$ cannot be symplectically embedded into $(\mathbb{R}^{2n}, \omega_0)$.

SYMPLECTIC COORDINATES: THE KÄHLER CASE

Theorem (D. McDuff, J. Diff. Geom. 1988): Let (M, ω) be a Kähler manifold. Assume that $\pi_1(M) = \{1\}$, M is complete and $K \leq 0$. Given $p \in M$ there exists a diffeomorphism

$$\psi_p : M \to \mathbb{R}^{2n}, \ \psi_p(p) = 0$$

satisfying $\psi_p^* \omega_0 = \omega$.

Theorem (E. Ciriza, Diff. Geom. Appl. 1993): Let $T \subset M$ be a complex and totally geodesic submanifold of M passing through p. Then, $\psi_p(T) = \mathbb{C}^k \subset \mathbb{C}^n \cong \mathbb{R}^{2n}, \dim_{\mathbb{C}} T = k$.

PROBLEMS

1. Find explicit global symplectic coordinates on Kähler manifolds satisfying the assumptions of McDuff's theorem.

We will analyze the case of **Hermitian symmetric spaces of noncompact type (HSSNT) and their compact duals.**

2. Find an example of complete Kähler manifold (M, ω) diffeomorphic to $\mathbb{R}^4$, such that K is positive somewhere and (M, ω) is globally symplectomorphic to $(\mathbb{R}^4, \omega_0)$.

We will show that $\mathbb{C}^2$ equipped with a **Taub–NUT** Kähler form $\omega_m, m > 0$, (which is an example of complete Ricci flat not flat Kähler metric on $\mathbb{C}^2$) is globally symplectomorphic to $(\mathbb{R}^4, \omega_0)$.

DEFINITION OF HSSNT

An HSSNT (M, ω) is a Kähler manifold, which is holomorphically isometric to a bounded symmetric domain $(M, 0)$ of $M \subset \mathbb{C}^n$ centered at the origin $0 \in \mathbb{C}^n$ equipped with a multiple of the Bergman metric ω_B such that for all $p \in M$ the geodesic symmetry:

$$s_p : \mathrm{exp}_p(v) \mapsto \mathrm{exp}_p(-v), \forall v \in T_pM$$

is a globally defined holomorphic isometry of M.

An HSSNT is a homogenous Kähler manifold (converse not true Pyateskii–Shapiro).

THE CASE OF THE UNIT DISK (1)

$\mathbb{C}H^1 = \{z \in \mathbb{C} \mid |z|^2 < 1\},\ \omega = \omega_{hyp} = \frac{i}{2}\frac{dz\wedge d\bar{z}}{(1-|z|^2)^2} = -\frac{i}{2}\partial\bar{\partial}\log(1-|z|^2) = \frac{i}{2}\frac{dz\wedge d\bar{z}}{(1-|z|^2)^2}$

We look for a map

$$\psi : \mathbb{C}H^1 \to \mathbb{R}^2, \psi(0) = 0$$

such that

$$\psi^*\omega_0 = \omega_{hyp},\ \omega_0 = dx \wedge dy = \frac{i}{2} dz \wedge d\bar{z}$$

Assume $\psi(z) = f(r)z,\ r = |z|^2$.

$\Rightarrow \psi^*\omega_0 = (2rf\frac{\partial f}{\partial r} + f^2)dx \wedge dy = \frac{\partial}{\partial r}(rf^2)dx \wedge dy = \omega_{hyp} = \frac{1}{(1-r)^2}dx \wedge dy$

$\Rightarrow \frac{\partial}{\partial r}(rf^2) = \frac{1}{(1-r)^2} \Rightarrow rf^2 = (1-r)^{-1} + C \Rightarrow C = -1,\ rf^2 = r(1-r)^{-1} \Rightarrow f(r) = (1-r)^{-\frac{1}{2}}$

Hence

$$\boxed{\psi(z) = \frac{z}{\sqrt{1-|z|^2}}}$$

THE CASE OF THE UNIT DISK (2)

Let $\mathbb{C}P^1$ be the one-dimensional complex projective space, (namely the compact dual of $\mathbb{C}H^1$) endowed with the Fubini–Study form ω_{FS}. Then

$$\mathbb{R}^2 \cong \mathbb{C} \cong U_0 = \{z_0 \neq 0\} \subset \mathbb{C}P^1$$

and

$$\omega_{FS}|_{U_0} = \frac{i}{2}\frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$$

and it is easily seen that $\psi : \mathbb{C}H^1 \subset \mathbb{C} \subset \mathbb{C}P^1 \to \mathbb{C}, z \mapsto \frac{z}{\sqrt{1-|z|^2}}$ satisfies:

$$\psi^*\omega_{FS} = \omega_0,$$

where ω_0 is the restriction of ω_0 to $\mathbb{C}H^1 \subset \mathbb{C}$.

Summarizing we have proved a "symplectic duality" between $(\mathbb{C}H^1, \omega_{hyp})$ and $(\mathbb{C}P^1, \omega_{FS})$, namely there exists a diffeomorphism

$$\psi : \mathbb{C}H^1 \to \mathbb{R}^2 \cong \mathbb{C} \cong U_0 \subset \mathbb{C}P^1$$

satisfying:

$$\boxed{\psi^*\omega_0 = \omega_{hyp}} \qquad \boxed{\psi^*\omega_{FS} = \omega_0}$$

THE FIRST CARTAN DOMAIN (1)

Let

$$D_I[n] = \{Z \in M_n(\mathbb{C}) \mid I_n - ZZ^* > 0\}$$

be the first Cartan domain equipped with the hyperbolic form

$$\omega_{hyp} = -\frac{i}{2}\partial\bar{\partial} \log \det(I_n - ZZ^*)$$

The compact dual of $D_I[n]$ is $\mathrm{Grass}_n(\mathbb{C}^{2n})$ endowed with the Fubini-Study form

$$\omega_{FS} = P^* \omega_{FS}$$

obtained as follows:

$$D_I[n] \subset M_n(\mathbb{C}) = \mathbb{C}^{n^2} \subset \mathrm{Grass}_n(\mathbb{C}^{2n}) \overset{P=Plucker}{\hookrightarrow} \mathbb{C}P^N,$$

$N = \left(\begin{array}{c} 2n \\ n \end{array} \right) - 1.$

THE FIRST CARTAN DOMAIN (2)

Theorem: The map

$$\Psi : D_I[n] \to M_n(\mathbb{C}) = \mathbb{C}^{n^2}$$

defined by

$$\Psi(Z) = (I_n - ZZ^*)^{-\frac{1}{2}} Z$$

is a diffeomorphism. Its inverse is given by

$$\Psi^{-1} : \mathbb{C}^{n^2} \to D_I[n], \; X \mapsto (I_n + XX^*)^{-\frac{1}{2}} X.$$

Moreover, Ψ is a *symplectic duality* namely,

$$\boxed{\Psi^* \omega_0 = \omega_{hyp}} \qquad \boxed{\Psi^* \omega_{FS} = \omega_0}$$

where

$$\boxed{\omega_0 = \frac{i}{2} \partial \bar{\partial} \, \mathrm{tr}(ZZ^*)}$$

and

$$\boxed{\omega_{FS} = \frac{i}{2} \partial \bar{\partial} \log \det(I_n + ZZ^*) \text{ on } \mathbb{C}^{n^2} \subset \mathrm{Grass}_n(\mathbb{C}^{2n})}$$

JORDAN TRIPLE SYSTEMS

A **Hermitian Jordan triple system** is a pair $(\mathcal{M}, \{,\ ,\})$, where $\mathcal{M}$ is a complex vector space and $\{,\ ,\}$ is a $\mathbb{R}$-trilinear map

$$\{,\ ,\} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}, (u, v, w) \mapsto \{u, v, w\}$$

$\mathbb{C}$-bilinear and simmetric in u and w and $\mathbb{C}$-antilinear in v and satisfying the **Jordan identity**:

$$\{x, y, \{u, v, w\}\} - \{u, v, \{x, y, w\}\} =$$

$$= \{\{x, y, u\}, v, w\} - \{u, \{v, x, y\}, w\}.$$

Let $u, v \in \mathcal{M}$, and let $D(u, v) : \mathcal{M} \to \mathcal{M}$ be the operator on $\mathcal{M}$ defined by

$$\boxed{\text{D(u, v)(w)=\{u, v, w\}}}$$

A HJTS is called **positive** and we write **HPJTS** if

$$(u, v) \mapsto \mathrm{tr}\, D(u, v)$$

is positive definite.

The *quadratic representation*

$$Q : \mathcal{M} \to End(\mathcal{M})$$

is defined by

$$2Q(u)(v) = \{u, v, u\}, \ \ u, v \in \mathcal{M}.$$

The **Bergman operator**

$$B(u, v) : \mathcal{M} \to \mathcal{M}$$

is given by the equation

$$\boxed{B(u, v) = Id_{\mathcal{M}} - D(u, v) + Q(u)Q(v)}$$

HPJTS $\longrightarrow$ HSSNT

$(\mathcal{M}, \{,\ ,\}) \longrightarrow (M, 0) = \{u \in \mathcal{M} \mid B(u,u) >> 0\}_0$, where "$>>$" means positive definite w.r.t. $(u,v) \mapsto \operatorname{tr} D(u,v)$.

The **Bergman form** ω_{Berg} of M is defined as:

$$\omega_{Berg} = -\frac{i}{2}\partial\bar{\partial} \log \det B.$$

We also define (in the irreducible case)

$$\boxed{\omega_{hyp} = -\tfrac{i}{2g}\partial\bar{\partial} \log \det B(z,z)}$$

HSSNT $\longrightarrow$ HPJTS

$(M, 0) \longrightarrow (\mathcal{M} = T_0M, \{,\ ,\})$, where

$$\boxed{\{u, v, w\} = -\tfrac{1}{2}\left(R_0(u, v)w + JR_0(u, Jv)w\right)}$$

(W. Bertram, Lectures Notes in Math. N. 1754)

THE FIRST CARTAN DOMAIN AS HPJTS

Let $\mathcal{M} = M_n(\mathbb{C})$ with the triple product

$$\{u, v, w\} = uv^*w + wv^*u, \;\; u, v, w \in M_n(\mathbb{C})$$

$$\mathrm{tr}\, D(u, u) = \mathrm{tr}(uu^*)$$

$$B(u, v)(w) = (I_n - uv^*)w(I_n - v^*u)$$

The HSSNT $(M, 0)$ associated to $(M_n(\mathbb{C}), \{,,\})$ is the first Cartan domain

$$D_I[n] = \{Z \in M_n(\mathbb{C}) \mid I_n - ZZ^* > 0\}$$

$$\omega_{hyp} = \frac{\omega_{Berg}}{2n} = -\frac{i}{2}\partial\bar{\partial} \log \det(I_n - ZZ^*)$$

COMPACTIFICATIONS OF HPJTS

Let (M, ω_{hyp}) be an HSSNT and let (M^*, ω_{FS}) be its compact dual equipped with the Fubini–Study form ω_{FS}.

More precisely, one has the following inclusions:

$$(M, 0) \stackrel{Harish-Chandra}{\subset} \mathcal{M} = T_0 M \cong \mathbb{C}^n \stackrel{Borel}{\subset} M^* \stackrel{BW}{\hookrightarrow} \mathbb{C}P^N$$

and we set

$$\omega_{FS} = \mathrm{BW}^* \omega_{FS}.$$

Remark: The local expression of ω_{FS} restricted to $\mathcal{M}$ is given (in the irreducible case) by

$$\boxed{\omega_{FS} = \frac{i}{2g} \partial\bar{\partial} \log \det B(z, -z)}$$

$$\boxed{\omega_{hyp} = -\frac{i}{2g} \partial\bar{\partial} \log \det B(z, z)}$$

Theorem (A. J. Di Scala – L., Adv. Math. 2008): Let (M, ω_{hyp}) be an HSSNT and (M^*, ω_{FS}) its compact dual. Then the map

$$\boxed{\Psi_M : M \to \mathcal{M} \cong \mathbb{C}^n \subset M^*,\ \ z \mapsto B(z,z)^{-\frac{1}{4}} z}$$

satisfies the following properties:

(D) Ψ_M is a **diffeomorphism** and its inverse is given by

$$\boxed{\Psi_M^{-1} : \mathcal{M} \cong \mathbb{C}^n \subset M^* \to M,\ \ z \mapsto B(z,-z)^{-\frac{1}{4}} z}$$

(S) Ψ_M is a **simplectic duality**, i.e.:

$$\boxed{\Psi_M^* \omega_0 = \omega_{hyp}} \qquad \boxed{\Psi_M^* \omega_{FS} = \omega_0}$$

where $\omega_0 = \frac{i}{2}\partial\bar{\partial}\,\mathrm{tr}\, D(u,u)$ is the flat Kähler form on $\mathcal{M}$.

(H) the map

$$\Psi : HSSNT \to \mathrm{Diff}_0(M, \mathcal{M}),\ M \mapsto \Psi_M$$

is **hereditary**, i.e.: for all $(T, 0) \stackrel{i}{\hookrightarrow} (M, 0)$ complete, complex and totally geodesic submanifold one has

$$\Psi_M\,|_T = \Psi_T,\ \Psi_M(T) = \mathcal{T} \subset \mathcal{M},$$

where $\mathcal{T}$ is the HPJTS associated to T.

Idea of the proof: show the result for classical Cartan's domains and then use Jordan algebras to extend the result to the exceptional domains.

Remark: One can give an alternative proof of the theorem (A. J. Di Scala, G. Roos, L., *The bisymplectomorphism group of a bounded symmetric domain*, Transformation group, 2008) using spectral decomposition tools of a HPJTS.